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Subperiodic Rings with Commutative Jacobson Radical

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Abstract

Let R be a ring with nilpotents N and center C and with Jacobson radical J. Let P be the set of potent elements x for which $x^n = x$, $n > 1$, $n = n(x, y)$ is an integer. R is called subperiodic if $R \backslash (J \cup C) \subseteq N + P$. The commutativity behavior of these rings is considered in the case where J is commutative.

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Throughout, R is a ring, N is the set of nilpotents, C is the center, J is the Jacobson radical of R. For x, y in R, $[x, y] = xy - yx$ is the usual commutator while $[x, y]_k$ is defined inductively as follows: $[x, y]_1 = xy - yx$, $[x, y]_k = [[x, y]_{k-1}, y]$ for all $k > 1$. An element x is called *potent* if $x^n = x$ for some integer $n > 1$.

Definition 1. A ring R is called *subperiodic* if, for any $x \in R \setminus (J \cup C)$,

$$x = a + b,\ a \text{ nilpotent, } b \text{ potent } (b^k = b, k > 1). \tag{1}$$

(The set of potent elements is denoted by P.)
In preparation for the proofs of the main theorems, we state the following lemmas.

Lemma 1. *Suppose R is any ring such that $[x, y]$ commutes with x. Then, for all positive integers n,*

$$[x^n, y] = nx^{n-1}[x, y]. \tag{2}$$

This lemma is readily proved by induction.

Lemma 2 ([1])**.** *Suppose R is a ring in which every element is central or potent. Then R is commutative.*

Lemma 3 ([2])**.** *Let R be a ring such that both J and R/J are commutative. Then N is an ideal and R/N is commutative.*

Lemma 4 ([3])**.** *Suppose R is a ring such that for every element x in R, there exists an integer $n > 1$ such that $x - x^n$ is in the center of R. Then R is commutative.*

Theorem 1. *Suppose R is a ring such that, for all $x, y \in R$, there exists an integer,*

$$k = k(x, y) \geq 1 \text{ such that } [xy, yx]_k \text{ is potent (in } P). \tag{3}$$

Then all idempotents are central.

Proof. Let $e^2 = e \in R$, $x \in R$, $a = ex - exe, f = e + a$. So $f^2 = f$, $ef = f$, $fe = e$. An argument shows that $[ef, fe]_k = (-1)^k a$ for all positive integers k. Hence, by (3), for some $k \geq 1$, $(-1)^k a$ is potent, and thus $a = 0$ (since $a^2 = 0$). So $ex = exe$. Similarly, $xe = exe$, and the theorem is proved. □

Theorem 2. *Suppose R is a subperiodic ring such that (3) above is true. Then the commutator ideal $C(R)$ is contained in J, and thus R/J is commutative.*

Proof. As is well known,

$$R/J \cong \text{a subdirect sum of primitive rings } R_i \, (i \in \Gamma). \tag{4}$$

By Definition 1, we see that

$$\text{For all } \textit{noncentral } x_i \text{ of } R_i,\ x_i = a_i + b_i,\ a_i \text{ nilpotent, and } b_i \text{ potent.} \tag{5}$$

Moreover, by (3), we have

$$\text{For all } x_i, y_i \in R_i,\ [x_i y_i, y_i x_i]_k \text{ is potent for some } k \geq 1. \tag{6}$$

Case 1. R_i is a division ring. Then every element of R_i is central or potent (by (5)), and hence R_i is commutative by Lemma 2. *Case* 2. R_i is a primitive ring which is *not* a division ring. Then R_i is isomorphic to a complete matrix ring D_n over a division ring D with $n > 1$. So D_n satisfies (6) above. This however, is false, as a consideration of

$$x_i = E_{11},\ y_i = E_{11} + E_{12} \text{ (all } E_{ij} \text{ in } D_n),$$

shows. Indeed, $[x_iy_i, y_ix_i]_k = (-1)^k E_{12}$ is potent for some $k \geq 1$ (see (6)) forcing the contradiction $E_{12} = 0$. This contradiction proves that $R/J \cong$ a subdirect sum of division rings R_i, and hence R/J is commutative (See *Case* 1), which implies that $C(R) \subseteq J$. □

Theorem 3. *Suppose R is a subperiodic ring which satisfies (3) above. Suppose, further, that J is commutative. Then N is an ideal and R/N is commutative.*

Proof. By Theorem 2, R/J is commutative, and the theorem follows, by Lemma 3. □

The following two lemmas will be needed for the proofs of the main theorems on commutativity.

Lemma 5. *Suppose R is a subperiodic ring, and suppose $\sigma : R \to R_i$ is a homomorphism of R onto a ring R_i. Suppose, further, that the set N of nilpotents of R is an ideal. Then the set N_i of nilpotents of R_i is contained in $\sigma(J) \cup C_i$, where C_i is the center of R_i.*

Proof. By contradiction. Suppose $d_i \in N_i, d_i \notin \sigma(J), d_i \notin C_i$. Let $\sigma(d) = d_i$, $(d \in R)$. Then, $d \notin (J \cup C)$, and hence $d = a + b$, $a \in N$, b potent with $b^k = b$ for some integer $k > 1$. Hence,

$$d - a = b = b^k = (d - a)^k.$$

Since N is an ideal and $a \in N$, we conclude that $d - d^k \in N$, hence

$$\text{For all } d \in R \setminus (J \cup C),\ d - d^k \in N,\ k > 1. \tag{7}$$

Let $d_i^n = 0$ (since $d_i \in N_i$). Since $d \notin (J \cup C)$, therefore by (7), $d - d^k \in N$, $k > 1$. Since N is an ideal,

$$(d - d^k) + d^{k-1}(d - d^k) + \cdots + (d^{k-1})^{n-1}(d - d^k) \in N,$$

which implies (as is readily verified)

$$d - d^{n+1}d^{n(k-2)} \in N. \tag{8}$$

Therefore, by (8),

$$d_i - d_i^{n+1}d_i^{n(k-2)} \in \sigma(N). \tag{9}$$

Since $d_i^n = 0$, (9) implies that $d_i \in \sigma(N) \subseteq \sigma(J)$ ($N \subseteq J$ follows from the fact that N is an ideal). So $d_i \in \sigma(J)$, contradiction. This proves the lemma. □

Lemma 6. *Suppose R is a subperiodic ring with central idempotents, and suppose N is an ideal. Let R_i be a subdirectly irreducible ring, and let $\sigma : R \to R_i$ be an onto homomorphism. Then R_i is of one of the following two types:*

Type 1: $R_i = \sigma(J) \cup C_i$, $C_i =$ center of R_i, or

Type 2: $R_i = \sigma(J) \cup C_i \cup U_i$, $U_i =$ the set of units of R_i, $1 \in R_i$.

Proof. Write R as a subdirect sum of subdirectly irreducible rings R_i. By (7), we see that for all $d \in R$, $d \in J$ or $d \in C$ or $d - d^k \in N, k > 1$. Moreover, if $d - d^k \in N$, say $(d - d^k)^q = 0$, then $d^q = d^{q+1}g(d)$ for some $g(x) \in \mathbb{Z}[x]$, and hence

$$d^q = d^{q+1}g(d) = d^q(dg(d)) = d^q(dg(d))^2 = \ldots = d^q e,$$

where $e = (dg(d))^q$ is an idempotent, and thus by (7) we see that

$$\text{For all } d \in R,\ d \in J \text{ or } d \in C \text{ or } d^q = d^q e,\ e^2 = e \in d\mathbb{Z}[d],\ e \in C. \qquad (10)$$

In view of the homomorphism $\sigma : R \to R_i$, let $x_i \in R_i$ and let $\sigma(x) = x_i$, $(x \in R)$. By hypothesis, any idempotent $e \in R$ is central in R, and hence $e_i = \sigma(e)$ is central in R_i. By (10), we conclude that for all

$$d_i \in R_i,\ d_i \in \sigma(J) \text{ or } d_i \in C_i \text{ or } d_i^q = d_i^q e_i,\ e_i^2 = e_i \in d_i\mathbb{Z}[d_i], e_i \in C_i. \qquad (11)$$

Since e_i is *central* idempotent in the subdirectly irreducible ring R_i, $e_i = 0$ or $e_i = 1$. If R_i does *not* have an identity, then by (11) and Lemma 5, R_i is as described in type 1 (since $e_i = 0$). On the other hand, if $1 \in R_i$, then R_i is as described in type 2, since $e_i \in d_i\mathbb{Z}[d_i]$. This proves the lemma, since in the latter case, $e_i = (d_i g(d_i))^q = 1$ implies that d_i is indeed a unit in R_i. □

We are now in a position to prove the main commutativity theorems.

Theorem 4. *Suppose that R is a subperiodic ring with identity, and suppose that J is commutative. Suppose, further, that*

(i) For all x, y in R, $[xy, yx]_k$ is potent for some $k \geq 1$. Then R is commutative.

Proof. By Theorem 3, N is an ideal. Moreover, by Theorem 1, all idempotents are central. As is well known, $R \cong$ a subdirect sum of subdirectly irreducible rings R_i. Let $\sigma : R \to R_i$ be the natural homomorphism of R onto R_i. In view of the above facts, it follows by Lemma 6 that R_i is of one of the following two types:

Type 1: $R_i = \sigma(J) \cup C_i$, $C_i =$ *center of* R_i, *or*

Type 2: $R_i = \sigma(J) \cup C_i \cup U_i$, $U_i =$ *the set of units of* R_i.

Since J is commutative, R_i is commutative if R_i is of the type 1. Hence, we may assume that

$$R_i = \sigma(J) \cup C_i \cup U_i,\ U_i = \textit{the set of units of } R_i. \qquad (12)$$

Our next goal is to prove that for all $a \in J$, $x \in R$,

$$[(1+a)x, x(1+a)]_k = [a,x]_{k+1} \text{ for all positive integers } k. \tag{13}$$

To begin with, note that, since J is commutative and $a \in J$,

$$[(1+a)x, x(1+a)]_1 = [x+ax, x+xa] = [[a,x],x] = [a,x]_2.$$

So (13) holds for $k=1$. Now, suppose (13) is true for $k=\gamma$. Then, since $a \in J$ and J is commutative,

$$[(1+a)x, x(1+a)]_{\gamma+1} = [[a,x]_{\gamma+1}, x+xa] = [a,x]_{\gamma+2},$$

which proves (13). Note, too, that the commutator $[a,x]_{k+1}$ is nilpotent, by Theorem 3. But, by hypothesis (i), $[(1+a)x, x(1+a)]_k$ is potent for some $k \geq 1$. The net result (see (13)) is that $[a,x]_{k+1}$ is both potent and nilpotent, and hence it must be zero, which proves the following:

For all $a \in J$, $x \in R$, $[a,x]_n = 0$ for some positive integer n. This reflects in R_i as follows:

$$\text{For all } a_i \in \sigma(J),\ x_i \in R_i,\ [a_i,x_i]_n = 0 \text{ for some positive integer n.} \tag{14}$$

Our next goal is to prove that

$$\text{The set } N_i \text{ of nilpotents of } R_i \text{ is an ideal.} \tag{15}$$

To prove this, let $a_i \in N_i$, $x_i \in R$. Then by, (12), $a_i x_i \in \sigma(J) \cup C_i \cup U_i$. If $a_i x_i \in C_i$, then $a_i x_i \in N_i$. Also, if $a_i x_i \in \sigma(J)$, then $(a_i x_i)^2 \in (\sigma(J))^2 \subseteq C_i$ (since $\sigma(J)$ is a commutative ideal), which implies that, for some $y_i \in R_i$, $((a_i x_i)^2)^k = (a_i x_i)^2 (a_i x_i)^2 \cdots (a_i x_i)^2 = a_i^k y_i$, for all positive integers k. Since $a_i \in N_i$, let $a_i^k = 0$ in the above equation to obtain $(a_i x_i)^{2k} = a_i^k y_i = 0$, which implies $a_i x_i \in N_i$. Finally, suppose $a_i x_i = u_i$, where u_i is a unit in R_i, and let $a_i^\delta = 0$, δ *minimal.* If $\delta = 1$, then $a_i x_i = 0 \in N_i$. Suppose then that $\delta > 1$. Note that

$$0 = a_i^\delta x_i = a_i^{\delta-1}(a_i x_i) = a_i^{\delta-1} u_i, (u_i \text{ is a } \textit{unit}),$$

and hence $a_i^{\delta-1} = 0$, contradicting the minimality of δ. This contradiction shows that $a_i x_i \in N_i$. Similarly $x_i a_i \in N_i$. Moreover, by Lemma 5 and Theorem 3, $N_i \subseteq \sigma(J) \cup C_i$ and hence N_i is commutative (since J is commutative) and (15) is proved.

Recall that $\sigma : R \to R_i$ is the natural homomorphism of the ground ring R onto the subdirectly irreducible ring R_i. Our next object is to prove that

$$\sigma(J) \text{ is contained in the center of } R_i. \tag{16}$$

Suppose not. Then,

$$[a_i, b_i] \neq 0, \text{ for some } a_i \in \sigma(J),\ b_i \in R_i. \tag{17}$$

Since $[a_i, b_i] \neq 0$, not both $2b_i$ and $3b_i$ commute with a_i. Assume, without loss of generality, that

$$[a_i, 2b_i] \neq 0,\ (a_i \in \sigma(J)\, b_i \in R_i). \tag{18}$$

In view of (12), we see that (17) and (18) imply that

$$b_i \text{ and } 2b_i \text{ are both } \textit{units} \text{ in } R_i, \text{ since } \sigma(J) \text{ is commutative.} \tag{19}$$

Since N is an ideal, by Theorem 3, all the hypotheses of Lemma 5 hold in R, and hence by (7) in the proof of *Lemma*5,

$$\text{For all } d \in R,\ d \in J \text{ or } d \in C \text{ or } d - d^k \in N,\ k > 1. \tag{20}$$

Note that, since R_i inherits (20) from R, we have:

$$\text{For all } d_i \in R_i,\ d_i \in \sigma(J) \text{ or } d_i \in C_i \text{ or } d_i - d_i^k \in N_i,\ k > 1. \tag{21}$$

Moreover, since $\sigma(J)$ is commutative, (17) and (18) imply

$$b_i \notin \sigma(J) \cup C_i \text{ and } 2b_i \notin \sigma(J) \cup C_i. \tag{22}$$

By (21) and (22), we conclude that for some positive integers k_i, l_i

$$b_i - b_i^{k_i} \in N_i \text{ and } (2b_i) - (2b_i)^{l_i} \in N_i,\ k_i > 1,\ l_i > 1. \tag{23}$$

By (15), N_i is an ideal of R_i. Let $\bar{x}_i = x_i + N_i$, for any $x_i \in R_i$. Then by (23),

$$(\bar{b}_i)^{k_i} = \bar{b}_i \text{ and } (\bar{2b_i})^{l_i} = \bar{2b_i},\ k_i > 1,\ l_i > 1. \tag{24}$$

Note that by (24),

$$(\bar{b}_i)^{(k_i-1)(l_i-1)+1} = \bar{b}_i \text{ and } (\bar{2b_i})^{(k_i-1)(l_i-1)+1} = \bar{2b_i}.$$

Thus,

$$2^{(k_i-1)(l_i-1)+1}\bar{b_i} = 2^{(k_i-1)(l_i-1)+1}(\bar{b_i})^{(k_i-1)(l_i-1)+1} = \bar{2b_i}.$$

Hence,

$$(2^{(k_i-1)(l_i-1)+1} - 2)\bar{b}_i = \bar{0}.$$

Since b_i is a unit (see (19)), $\bar{b}_i$ a unit also. Thus,

$$(2^{(k_i-1)(l_i-1)+1} - 2) \cdot \bar{1} = \bar{0}\ (k_i > 1,\ l_i > 1),$$

which implies that

$$(2^{(k_i-1)(l_i-1)+1} - 2) \cdot 1 \text{ is nilpotent,}$$

and hence R_i is not of zero characteristic. Since R_i is subdirectly irreducible, we conclude that

$$\text{Characteristic of } R_i = p^k,\ p \text{ prime.} \tag{25}$$

In view of (19), (24), and (25), it follows that the subring $\langle \bar{b}_i \rangle$ generated by the unit $\bar{b}_i$ is a finite commutative ring with identity which has no nonzero nilpotents, and hence

$$\langle \bar{b}_i \rangle = \bigoplus_{j=1}^{t} GF(p^{k_j}), t \text{ finite.} \tag{26}$$

Let $\alpha = k_1 k_2 \dots k_t$. Then, by (26) and (25), $\bar{b}_i^{p^{k\alpha}} = \bar{b}_i$, and thus by Lemma 5 and Theorem 3,

$$b_i^{p^{k\alpha}} - b_i \in N_i \subseteq \sigma(J) \cup C_i. \tag{27}$$

Returning to (14), let β be the least positive integer such that

$$[a_i, b_i]_\beta = 0 \ (\beta \textit{ minimal}),\ a_i \in \sigma(J). \tag{28}$$

We claim that

$$\beta \leq 2. \tag{29}$$

Suppose not. Then $\beta > 2$, and hence by (27) and the fact that $\sigma(J)$ is commutative,

$$[b_i^{p^{k\alpha}} - b_i, [a_i, b_i]_{\beta-2}] = 0,\ a_i \in \sigma(J),\ b_i \in R_i, \tag{30}$$

which implies

$$[[a_i, b_i]_{\beta-2}, b_i^{p^{k\alpha}}] = [[a_i, b_i]_{\beta-2}, b_i] \tag{31}$$

By (28), we see that $[[a_i, b_i]_{\beta-2}, b_i]$ commutes with b_i, and hence by (31) and Lemma 1,

$$p^{k\alpha} b_i^{p^{k\alpha}-1} [[a_i, b_i]_{\beta-2}, b_i] = [[a_i, b_i]_{\beta-2}, b_i],$$

which implies that $0 = [[a_i, b_i]_{\beta-2}], b_i]$, by (25). Thus, $[a_i, b_i]_{\beta-1} = 0$, which contradicts the *minimality* of β (see (28)). This contradiction proves that $\beta \leq 2$, and thus

$$[a_i, b_i] \text{ commutes with } b_i \tag{32}$$

By (17), $a_i \in \sigma(J)$, and hence by (27) and the fact that $\sigma(J)$ is commutative, we conclude that

$$[a_i, b_i^{p^{k\alpha}} - b_i] = 0,$$

which implies that

$$[a_i, b_i^{p^{k\alpha}}] = [a_i, b_i]. \tag{33}$$

Combining (33), (32), (25), and Lemma 1, we obtain $[a_i, b_i] = 0$, which contradicts (17). This contradiction proves (16).

To complete the proof, note that (21), we have:

$$\text{For all } d_i \in R_i,\ d_i \in \sigma(J) \text{ or } d_i \in C_i \text{ or } d_i - d_i^k \in N_i,\ k > 1. \tag{34}$$

Moreover, since $\sigma : R \to R_i$ is an onto homomorphism, and since N is an ideal (Theorem 3), it follows by Lemma 5 that $N_i \subseteq \sigma(J) \cup C_i \subseteq C_j$, by (16). Combining this with (16), (34), and Lemma 4, we see that R_i is commutative, and the ground ring R is commutative. This proves the theorem. □

In our next theorem, we remove the hypothesis that R has an identity.

Theorem 5. *Suppose that R is any subperiodic ring (not necessarily with identity), and suppose J is commutative. Suppose further, that*

(i) For all x, y in R, $[xy, yx]_k$ is potent for some $k \geq 1$.

Then R is commutative.

Proof. We begin with noting that $R \cong$ a subdirect sum of subdirectly irreducible rings R_i. Let $\sigma : R \to R_i$ be the natural homomorphism of R onto R_i. By Theorem 1, the idempotents of R are central. Moreover, by Theorem 3, N is an ideal. Hence, by Lemma 6, R_i is of one of the following two types: *Type* 1: $R_i = \sigma(J) \cup C_i$, $C_i =$ center of R_i, or *Type* 2: $R_i = \sigma(J) \cup C_i \cup U_i$, $U_i =$ set of all *units* in R_i, $1 \in R_i$. Since J is commutative, R_i is commutative if R_i is of *Type* 1. So we may assume that

$$R_i = \sigma(J) \cup C_i \cup U_i \tag{35}$$

We claim that

$$U_i \subseteq C_i. \tag{36}$$

Suppose not. Let $u_i \in U_i$ be such that

$$[u_i, x_i] \neq 0,\ x_i \in R_i,\ (u_i \text{ a unit.}) \tag{37}$$

Let $d \in R$, $\sigma(d) = u_i$. Then $d \notin C$. Also, $d \notin J$ (since $d \in J$ implies $\sigma(d) \in \sigma(J)$, and hence $u_i \in \sigma(J)$, which implies $\sigma(J) = R_i$, forcing the contradiction that R_i is commutative). By (10) in the proof of Lemma 6, we see $d^q = d^q e$, $e^2 = e \in C$, and hence $(\sigma(d))^q = (\sigma(d))^q \cdot \sigma(e)$, which implies $u_i^q = u_i^q(\sigma(e))$. Since u_i is a *unit,* we conclude that

$$\sigma(e) = 1,\ (e^2 = e \in C, 1 \in R_i). \tag{38}$$

Since e is a nonzero central idempotent in R, we see that

$$eR \text{ is a ring } \textit{with identity } e. \tag{39}$$

It is readily verified that eR is a ring (with identity) which satisfies all the hypotheses imposed on R in Theorem 4. (In verifying this, recall that $J(eR) \subseteq eJ(R)$.) Hence, by Theorem 4,

$$eR \text{ is commutative.} \tag{40}$$

Let $x_i, y_i \in R_i$, and suppose $x, y \in R$ are such that $\sigma(x) = x_i$, $\sigma(y) = y_i$. (Recall the *onto* homomorphism $\sigma : R \to R_i$.) By (40) we have $(ex)(ey) = (ey)(ex)$, and hence $\sigma(ex)\sigma(ey) = \sigma(ey)\sigma(ex)$, with $\sigma(e) = 1$ (see (38)), which implies $\sigma(x)\sigma(y) = \sigma(y)\sigma(x)$; that is, $x_i y_i = y_i x_i$. So R_i is commutative, which contradicts (37). This contradiction proves (36), namely

$$U_i \subseteq C_i. \tag{41}$$

Combining (41) and (35), we conclude that $R_i = \sigma(J) \cup C_i$, (see *Type* 1), and hence again R_i is commutative, and the theorem is proved. □

We conclude with the following corollaries.

Corollary 1. *Any subperiodic ring with commutative J for which $[xy, yx]_k = 0$, where $k = k(x, y) \geq 1$, is commutative.*

Corollary 2. *Any Subperiodic ring with commutative J for which $[xy, yx]_k$ is central, where $k = k(x, y) \geq 1$, is commutative (since $[xy, yx]_k \in C$ implies that $[xy, yx]_{k+1} = 0$).*

Our final corollary is the following well known theorem of Jacobson [4; p.217].

Corollary 3. *Suppose R is a ring with the property that for every x in R, there exists an integer $n(x) > 1$ such that $x = x^{n(x)}$. Then R is commutative. (Since in this case $J = \{0\}$).*

Related work appears in [5].

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