On the Weakly $\Gamma$-Division Rings

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Abstract

Weakly $\Gamma$-rings are a generalization of rings and their class contain as a subclass the class of $\Gamma$-rings of Nabusawa and $\Gamma$-rings. In this paper we introduce the notion of weakly $\Gamma$-division ring, which is similar with the notion of $\Gamma$-semigroup. We give some necessary and sufficient conditions when a weakly $\Gamma$-ring is a weakly $\Gamma$-division ring. Further, we also give some characterizations of weakly $\Gamma$-division rings in terms of left ideals, right ideals, quasi-ideals and their minimal respective structures. In this paper we also discuss the connection between the weakly $\Gamma$-division rings with $\Gamma$-division rings.

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1 Introduction and preliminaries

Nabusawa introduced in [4] the notion of a $\Gamma$-ring, which is called a $\Gamma$-ring of Nabusawa. The respective definition is:

Definition 1.1 [4] Let $M$ be an additive group whose elements are denoted by $a, b, c, \ldots$, and $\Gamma$ another additive group whose elements are $\gamma, \beta, \alpha, \ldots$. Suppose that $a\gamma b$ is defined to be an element of $M$ and $\gamma a \beta$ is defined to be an element of $\Gamma$ for every $a, b, \gamma$ and $\beta$. If the products satisfy the following conditions:
1. 

\[(a_1 + a_2)\gamma b = a_1\gamma b + a_2\gamma b, \quad a(\gamma_1 + \gamma_2)b = a \gamma_1 b + a \gamma_2 b, \quad a \gamma(b_1 + b_2) = a \gamma b_1 + a \gamma b_2.\]

2. 

\[(a\gamma b)\beta c = a\gamma (b \beta c) = a(\gamma b)\beta c.\]

3. If \(a \gamma b = 0\) for any \(a\) and \(b\) in \(M\) then \(\gamma = 0\), then \(M\) is called a \(\Gamma\)-ring.

Barnes has weakened some of the conditions of Definition 1.1 and has defined in [1] those that now are called \(\Gamma\)-rings.

Firstly, we define a \(\Gamma\)-multiplication in a set \(M\) to give a more modernizes form of the Barne’s definition for \(\Gamma\)-rings. Let \(M\) and \(\Gamma\) be two non-empty sets. A map from \(M \times \Gamma \times M\) to \(M\) will be called a \(\Gamma\)-multiplication in \(M\) and is denoted by \((\cdot)_\Gamma\). The result of this \(\Gamma\)-multiplication for every two elements \(a, b\) of \(M\) and every element \(\gamma \in \Gamma\) is denoted by \(a \gamma b\).

**Definition 1.2** [1] A \(\Gamma\)-ring is called every five tuple \((M, \Gamma, +, \bigoplus, (\cdot)_\Gamma)\) where \(M, \Gamma\) are sets, \(+\) is the addition in \(M\), \(\bigoplus\) is the addition in \(\Gamma\), \((\cdot)_\Gamma\) is a \(\Gamma\)-multiplication in \(M\), such that:

1. \((M, +)\) is an abelian group.
2. \((\Gamma, \bigoplus)\) is an abelian group.
3. \(\forall (a, b, c, \alpha, \beta) \in M^3 \times \Gamma^2, \quad (a \alpha b)\beta c = a a \alpha (b\beta c).\)
4. \(\forall (a, b, c, \gamma) \in M^3 \times \Gamma, \quad (a + b)\gamma c = a \gamma c + b \gamma c.\)
5. \(\forall (a, b, c, \gamma) \in M^3 \times \Gamma, \quad a \gamma(b + c) = a \gamma b + a \gamma c.\)
6. \(\forall (a, b, \alpha, \beta) \in M^2 \times \Gamma^2, a(\alpha \bigoplus \beta)b = a\alpha + a\beta b.\)

Based on the definition of Nabusawa’s \(\Gamma\)-rings, Sen in [7] defined \(\Gamma\)-semigroups that are called \(\Gamma\)-semigroups of Sen. In [8] Sen and Saha defined a generalization of \(\Gamma\)-semigroup of Sen that is called a \(\Gamma\)-semigroup. Formally the concept of \(\Gamma\)-semigroup is obtained from the concept of \(\Gamma\)-ring if we avoid its additions. More precisely we have the following definition.

**Definition 1.3** [8] A \(\Gamma\)-semigroup is called any pair \((S, (\cdot)_\Gamma)\), where \(S\) and \(\Gamma\) are two non empty sets and \((\cdot)_\Gamma\) is a \(\Gamma\)-multiplication in \(S\), which satisfies the following condition:

\[\forall (a, b, c, \alpha, \beta) \in S^3 \times \Gamma^2, \quad (a \alpha b)\beta c = a a \alpha (b\beta c).\]

Weakening some of the conditions in the definition of \(\Gamma\)-ring, we have defined in [5] the weakly \(\Gamma\)-ring as follows:
**Definition 1.4** A weakly \(\Gamma\)-ring will be called any triple \((M, +, (\cdot)_\Gamma)\) where \(M\), \(\Gamma\) are two nonempty sets, \(+\) is an addition in \(M\) and \((\cdot)_\Gamma\) is a \(\Gamma\)-multiplication in \(M\), such that:

1. \((M, +)\) is an abelian group.
2. \((M, (\cdot)_\Gamma)\) is a \(\Gamma\)-semigroup.
3. \(\forall (a, b, c, \gamma) \in M^3 \times \Gamma\), \((a + b)c = ac + bc\) and \(a\gamma(b + c) = a\gamma b + a\gamma c\).

The definitions 1.3 and 1.4 guarantee the existence of the connection between weakly \(\Gamma\)-rings and \(\Gamma\)-semigroups, which mimic the connection between plain semigroups and plain rings. We notice that plain rings, \(\Gamma\)-rings of Nambu-sawa and \(\Gamma\)-rings are weakly \(\Gamma\)-rings but the converse is not true.

Let \(S\) be a \(\Gamma\)-semigroup and \(\gamma\) a fixed element of \(\Gamma\). As in [8] we define the binary operation \(\circ_\gamma\) in \(S\) by the equality \(a \circ_\gamma b = a\gamma b\) for all elements \(a, b\) of \(S\). It is easy to see that \(\circ_\gamma\) is an associative binary operation on \(S\) and consequently \((S, \circ_\gamma)\) is a plain semigroup. We denote this semigroup by \(S_{\gamma}\).

The zero element of \(\Gamma\)-semigroup \(S\) is an element of \(S\), which is denoted by 0 such that for every \(a \in S\) and for every \(\gamma \in \Gamma\), we have \(0\gamma a = a\gamma 0 = 0\).

**Proposition 1.1** [8] Let \(S\) be a \(\Gamma\)-semigroup. If \(S_{\gamma}\) is a group for some \(\gamma \in \Gamma\), then \(S_{\gamma}\) is a group for all \(\gamma \in \Gamma\).

**Definition 1.5** [8] A \(\Gamma\)-semigroup \(S\) is called \(\Gamma\)-group if \(S_{\gamma}\) is a group for some (hence for all) \(\gamma \in \Gamma\).

Let \(M\) be a weakly \(\Gamma\)-ring and \(A, B\) two nonempty subsets of \(M\). We write:

\[
\{\sum_{i=1}^{n} a_i\gamma_i b_i \in M | a_i \in A, b_i \in B, \gamma_i \in \Gamma, n \in \mathbb{N}\}
\]

Also for every \(a \in M\) and for every \(\gamma \in \Gamma\) we write:

\[
M\gamma a = \{m\gamma a \in M | m \in M\},
\]

\[
a\gamma M = \{a\gamma m \in M | m \in M\}.
\]

**Definition 1.6** A right [left] ideal of a weakly \(\Gamma\)-ring \((M, +, (\cdot)_\Gamma)\) is called every subgroup \(R [L]\) of the group \((M, +)\) such that \(R\Gamma M \subseteq R [M\Gamma L \subseteq L]\).

The intersection of all right [left] ideals of \(M\) which contain the element \(a \in M\), is a right [left] ideal of \(M\) which is denoted by \(\langle a \rangle_r [\langle a \rangle_l]\). The right [left] ideal \(\langle a \rangle_r [\langle a \rangle_l]\) is called the principal right [left] ideal of \(M\) generated by \(a\). It is easy to prove the following proposition:
Proposition 1.2 For every element \( m \) of a weakly \( \Gamma \)-ring \( M \) and for every \( \gamma \in \Gamma \), the set \( m\gamma M \ [M\gamma m] \) is a right [left] ideal of \( M \).

Proposition 1.3 [9] A ring has no proper left [right] ideals, that is different from zero and the ring itself, if and only if it is either a division ring or a zero ring of prime order.

Definition 1.7 Let \((M, +, \cdot)\) be a weakly \( \Gamma \)-ring and \( \gamma \) an element of \( \Gamma \). The element \( a \) of \( M \) is called a \( \gamma \)-cancellable element if for every two elements \( b, c \) of \( M \) we have:

\[ a\gamma b = a\gamma c \Rightarrow b = c \text{ and } b\gamma a = c\gamma a \Rightarrow b = c. \]

Definition 1.8 [3] Let \( M \) be a \( \Gamma \)-ring. \( M \) is called a \( \Gamma \)-division ring if \( M' = M\setminus \{0\} \) is a \( \Gamma' \)-group where, \( \Gamma' = \Gamma \setminus \{0\} \).

In this paper, we introduce the notion of weakly \( \Gamma \)-division ring in analogy to the notions of \( \Gamma \)-group and like a natural generalization of division rings. Some equivalent definitions of weakly \( \Gamma \)-ring, which are similar with those of plain division rings are obtained by giving necessary and sufficient conditions, which show when a weakly \( \Gamma \)-ring is a weakly \( \Gamma \)-division ring. We also discuss the relation of the Definition 1.8 of \( \Gamma \)-division ring with the definition that we will give for weakly \( \Gamma \)-division ring. We give some characterizations of weakly \( \Gamma \)-division rings by left ideals, right ideals and quasi-ideals. Also we give some other characterization of weakly \( \Gamma \)-division ring by minimal quasi-ideals, minimal principal ideals and minimal right ideals.

In this paper we raise two open problems.

2 The definition of weakly \( \Gamma \)-division ring

Let \((M, +, \cdot)\) be a weakly \( \Gamma \)-ring and \( \gamma \) a fixed element of \( \Gamma \). We define the binary operation \( \circ \gamma \) in \( M \) by the equality:

\[ a \circ \gamma b = a\gamma b, \]

for every two elements \( a, b \) of \( M \). It is evident that \((M, +, \circ \gamma)\) is a ring. We denote this ring by \( M_\gamma \). Generally, if for any \( \gamma \in \Gamma \) the ring \((M, +, \circ \gamma)\) is a division ring, then the ring \((M, +, \circ \gamma)\) is not a division ring for every \( \gamma \in \Gamma \). To show this we give the following example:

Example 2.1 Let \((H, +)\) be the additive group of quaternion and \( \Gamma = H \). It is easy to see that if \( \cdot \) is a \( \Gamma \)-multiplication in \( H \) such that for every two elements \( a, b \) of \( H \) and for every \( \gamma \in H\setminus \{1\} \), \( a\gamma b \) is the usual multiplication of quaternions \( a, \gamma, b \), but for \( \gamma = 1 \) \( a\gamma b = 0 \), then \((H, +, \cdot)\) is a weakly \( \Gamma \)-ring.
It is evident that for every $\gamma \in \Gamma \setminus \{0, 1\}$ the ring $H_\gamma$ is a division ring but for $\gamma = 0$ and $\gamma = 1$ the rings $H_0$ and $H_1$ are not division rings.

**Definition 2.1** Let $(M, +, (\cdot)_\Gamma)$ be a weakly $\Gamma$-ring. The element $\gamma \in \Gamma$ is called a $\Gamma$-zero if for every two elements $a, b$ of $M$ we have $a\gamma b = 0$.

The set of elements of $\Gamma$, which are not $\Gamma$-zero of the weakly $\Gamma$-ring $M$ we will denote by $\Gamma_0$. For the weakly $\Gamma$-ring $(H, +, (\cdot)_\Gamma)$ of Example 2.1 we have $\Gamma_0 = \Gamma \setminus \{0, 1\}$.

Example 2.1 shows that there exist weakly $\Gamma$-rings, which have a $\Gamma$-zero element and further more they can have more than one $\Gamma$-zero element. If the weakly $\Gamma$-ring $M$ is a $\Gamma$-ring of Nabusawa, then from the condition 3 of the Definition 1.1 follows that $M$ has only one $\Gamma$-zero element exactly the zero of abelian group $\Gamma$.

Barnes has avoid the condition 3 from the Definition 1.1 that satisfies the $\Gamma$-ring of Nabusawa. The following example shows that in general in a $\Gamma$-ring we may have more than one $\Gamma$-zero, even an infinite number of $\Gamma$-zeros.

**Example 2.2** We note:

$$M = \left\{ \left( \begin{array}{cc} a & 0 \\ a & 0 \end{array} \right) \mid a \in \mathbb{R} \right\},$$

$$\Gamma = \left\{ \left( \begin{array}{cc} 0 & b \\ 0 & c \end{array} \right) \mid b, c \in \mathbb{R} \right\}.$$

These sets are closed under the usual matrix multiplication and form abelian groups under the usual matrix addition. By suitable computation we show that if $(\cdot)_\Gamma$ is a $\Gamma$-multiplication in $M$ such that for every two elements $A, B$ of $M$ and for every element $\gamma \in \Gamma$, $A\gamma B$ is the ordinary multiplication of matrices $A, \gamma, B$, then $(M, +, (\cdot)_\Gamma)$ is a $\Gamma$-ring and for this ring we have:

$$\Gamma_0 = \Gamma \setminus \left\{ \left( \begin{array}{cc} 0 & 0 \\ 0 & c \end{array} \right) \mid c \in \mathbb{R} \right\}.$$  

So, the $\Gamma$-ring $M$ has an infinite number of $\Gamma$-zeros.

**Definition 2.2** A weakly $\Gamma$-ring $(M, +, (\cdot)_\Gamma)$ is called a weakly $\Gamma$-division ring if $\Gamma_0 \neq \Phi$ and for every $\gamma \in \Gamma_0$, the ring $(M, +, \circ_\gamma)$ is a division ring.

For every division ring $(M, +, \cdot)$ and for every set $\Gamma$ of elements of $M$, which are different from zero, we obtain a weakly $\Gamma$-division ring. Indeed, if we define a $\Gamma$-multiplication in $M$ such that for every $\gamma \in \Gamma$ and every two elements $a, b$ of $M$, $a\gamma b$ is the multiplication of elements $a, \gamma, b$ of the division ring $(M, +, \cdot)$.

It is easy to see that $(M, +, (\cdot)_\Gamma)$, where $+$ is the addition of the division ring
\((M, +, \cdot)\), is a weakly \(\Gamma\)-ring and the ring \((M, +, \circ_{\gamma})\) is a division ring for every \(\gamma \in \Gamma\). It is clear that the \(\Gamma\)-ring of Example 2.1 \((H, +, (\cdot)_H)\) is a weakly \(\Gamma\)-division ring.

The \(\Gamma\)-ring of Example 2.2 \((M, +, (\cdot)_\Gamma)\) is a weakly \(\Gamma\)-division ring, because for every \(\gamma = \left(\begin{array}{cc} 0 & c \\ 0 & d \end{array}\right)\) that does not belong to \(\Gamma_0\), i.e. \(c \neq 0\), the ring \((M, +, \circ_{\gamma})\) is a division ring since \(I_2 = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)\) is the identity element of this ring, while \(\left(\begin{array}{cc} 1 & ac^2 \\ 0 & 1 \end{array}\right)\) is the inverse of the matrix \(A = \left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right)\) \(\neq 0\).

Further more, since \(A\gamma B = B\gamma A\) for every two matrices \(A, B\) of \(M\) and for every matrix \(\gamma \in \Gamma\), the \(\Gamma\)-ring \((M, +, (\cdot)_\Gamma)\) is a weakly \(\Gamma\)-field according to the following definition:

**Definition 2.3** A weakly \(\Gamma\)-ring \((M, +, (\cdot)_\Gamma)\) is called a weakly \(\Gamma\)-field if it is a weakly \(\Gamma\)-division ring and it is commutative, i.e.

\[\forall (a, \gamma, b) \in M \times \Gamma \times M, a\gamma b = b\gamma a.\]

Utilizing Definition 2.1 and Proposition 1.3 it is easy to prove the following two propositions:

**Proposition 2.1** Necessary and sufficient condition for a weakly \(\Gamma\)-ring \((M, +, (\cdot)_\Gamma)\), which has at least two elements and \(\Gamma_0 \neq \Phi\), to be a weakly \(\Gamma\)-division ring is

\[\forall (m, \gamma) \in M^* \times \Gamma_0, m\gamma M = M,\]

where \(M^* = M \setminus 0\).

**Proposition 2.2** Necessary and sufficient condition for a weakly \(\Gamma\)-ring \((M, +, (\cdot)_\Gamma)\), which has at least two elements and \(\Gamma_0 \neq \Phi\), to be a weakly \(\Gamma\)-division ring is

\[\forall (m, \gamma) \in M^* \times \Gamma_0, M\gamma m = M,\]

where \(M^* = M \setminus 0\).

From the above two propositions we get two equivalent definitions of the weakly \(\Gamma\)-division rings. Precisely:

**Definition 2.4** A weakly \(\Gamma\)-ring \((M, +, (\cdot)_\Gamma)\) is called a weakly \(\Gamma\)-division ring if it has at least two elements, \(\Gamma_0 \neq \Phi\) and for every element \(m \in M^* = M \setminus 0\) and for every \(\gamma \in \Gamma_0\), \(m\gamma M = M\) \([M\gamma m = M]\).

Let \((M, (\cdot)_\Gamma)\) be a \(\Gamma\)-group, \(M_1\) a subset of \(M\) and \(\Gamma_1\) a subset of \(\Gamma\). We denote

\[M_1\Gamma_1M_1 = \{a\gamma b \in M | a \in M_1, b \in M_1, \gamma \in \Gamma_1\}.\]
If $M_1 \Gamma_1 M_1 \subseteq M_1$, then we can define a $\Gamma_1$-multiplication in $M_1$ by assigning to each triple $(a_1, \gamma_1, b_1)$ the element $a_1 \gamma_1 b_1$ of $M_1$, which we find by the $\Gamma$-
multiplication on $M$. This $\Gamma_1$-multiplication in $M_1$ is called $\Gamma_1$-multiplication induced in $M_1$ by the $\Gamma$-multiplication $\cdot_{\Gamma}$ in $M$ according to $\Gamma_1$.

Now, we can find a natural connection that exists between $\Gamma$-groups and weakly $\Gamma$-division ring, which mimics the connection between plain division rings and plain groups. Precisely, it is not difficult to prove the following proposition, which is a necessary and sufficient condition for a weakly $\Gamma$-ring to be a weakly $\Gamma$-division ring and from this follows another equivalent definition with that of weakly $\Gamma$-division rings.

**Proposition 2.3** A weakly $\Gamma$-ring $(M, +, (\cdot)_{\Gamma})$ is a weakly $\Gamma$-division ring if and only if it has at least two elements, $\Gamma_0 \neq \Phi$ and it is such that $M^* \Gamma_0 M^* \subseteq M^*$, $M^* = M \setminus \{0\}$, and $(M^*, (\cdot)_{\Gamma_0})$, where $(\cdot)_{\Gamma_0}$ is a $\Gamma_0$-multiplication induced in $M^*$ by $\Gamma$-multiplication in $M$ according to $\Gamma_0$, is a $\Gamma_0$-group.

We see that when a weakly $\Gamma$-ring is a $\Gamma$-ring, then Proposition 2.3 of weakly $\Gamma$-division ring coincides with Definition 1.8 of $\Gamma$-division ring only if $\Gamma_0 = \Gamma \setminus \{0\}$. So, the $\Gamma$-ring of Example 2.2 is a weakly $\Gamma$-ring but it is not a $\Gamma$-division ring because we have $\Gamma_0 \neq \Gamma \setminus \{0\}$. Hence, the concept of weakly $\Gamma$-division is larger than the concept of a $\Gamma$-division ring. Since the $\Gamma$-ring of Example 2.2 is abelian, then the concept of weakly $\Gamma$-field given by us in Definition 2.3 is larger then the concept of $\Gamma$-field given in [2], [3]. It is clear that, when a weakly $\Gamma$-division ring is a $\Gamma$-ring of Nabusawa, then the concept of weakly $\Gamma$-division ring coincides with the concept of $\Gamma$-division ring. Proposition 2.3 helps us to find all weakly $\Gamma$-rings such that for any one of them to have a proposition analogue with Proposition 1.1. We have exactly the following proposition:

**Proposition 2.4** Let $(M, +, (\cdot)_{\Gamma})$ be a weakly $\Gamma$-ring which has at least two elements and $\Gamma_0 \neq \Phi$. Then $M^* \Gamma_0 M^* \subseteq M^*$, where $M^* = M \setminus \{0\}$ if and only if the following conditions are equivalent:

1. There exists one $\gamma \in \Gamma_0$ such that $(M, +, \circ_{\gamma})$ is a division ring.

2. For every $\gamma \in \Gamma_0$, the ring $(M, +, \circ_{\gamma})$ is a division ring.

**Proof.** Assume that conditions 1) and 2) are equivalent. Suppose that there exists a $\gamma \in \Gamma_0$ such that for every two elements $a, b$ of $M^* = M \setminus \{0\}$ we have $a\gamma b \notin M^*$. Then the ring $(M, +, \circ_{\gamma})$ is not a division ring and so we have a contradiction. Hence, $M^* \Gamma_0 M^* \subseteq M^*$.

Conversely, assume that $M^*$ is such that $M^* \Gamma_0 M^* \subseteq M^*$. If the condition 1) is true, then $(M^*, \circ_{\gamma})$ is a group because $(M, +, \circ_{\gamma})$ is a division ring. Hence $(M^*, (\cdot)_{\Gamma_0})$ is a $\Gamma_0$-group. According to Proposition 2.3 $(M, +, (\cdot)_{\Gamma})$ is a weakly
Γ-division ring and therefore the condition 2) is true. If the condition 2) is true, then it is evident that the condition 1) is true. ■

From this proposition we obtain immediately the following corollary:

\textbf{Corollary 2.1} Necessary and sufficient condition for a weakly Γ-ring \( M \) in which \( \Gamma_0 \neq \Phi \) to be a weakly Γ-division ring is that there exists one \( \gamma \in \Gamma_0 \) such that \((M, +, \circ_\gamma)\) is a division ring and \( M^*\Gamma_0M^* \subseteq M^* \).

\section{Some characterization of weakly Γ-division ring}

In this section we will characterize the weakly Γ-division rings by left ideals, right ideals, quasi-ideals, minimal quasi-ideals, minimal left ideals and minimal right ideals.

\textbf{Theorem 3.1} A weakly Γ-ring \( M \) in which \( \Gamma_0 \neq \Phi \) is a weakly Γ-division ring if and only if

1. It has at least two elements.

2. It has no proper left ideals [right ideals], i.e. left [right] ideals different from zero and \( M \).

3. \( \forall (m, \gamma) \in M^* \times \Gamma_0, M\gamma m \neq 0 \). [\( \forall (m, \gamma) \in M^* \times \Gamma_0, m\gamma M \neq 0 \).]

\textbf{Proof.} We prove the theorem only for left ideals. The proof for right ideals is similar.

Suppose that \( M \) is a weakly Γ-division ring. It is evident that \( M \) has at least two elements and \( \Gamma_0 \neq \Phi \). Let \( L \) be a non-zero left ideal of \( M \) and \( m \) a non-zero element of \( L \). From Proposition 2.2 for every \( \gamma \in \Gamma_0 \) and \( m \in M^* = M \setminus 0 \) we have \( M\gamma m \neq 0 \).

Conversely, assume that the weakly Γ-ring has at least two elements and it has no proper left ideals. Since for every \( \gamma \in \Gamma_0 \) and for every \( m \in M^* \) we have that \( M\gamma m \) is different from zero, then from Proposition 1.2 we have that \( M\gamma m \) is a non-zero left ideal and consequently \( M\gamma m = M \). Now from Proposition 2.2, we have that \( M \) is a weakly Γ-division ring. ■

A weakly Γ-division ring \( M \) is said to be a weakly Γ-ring without divisors of zero if

\[ \forall (a, \gamma, b) \in M \times \Gamma \times M, (a\gamma b = 0) \Rightarrow (a = 0 \lor b = 0 \lor \gamma \in \Gamma_0). \]

From Theorem 3.1 we have immediately the following corollary:
Corollary 3.1 A weakly $\Gamma$-ring $M$ in which $\Gamma_0 \neq \Phi$ is a weakly $\Gamma$-division ring if and only if it has at least two elements, it has no proper left[right] ideals and it has no divisors of zero.

Theorem 3.2 A weakly $\Gamma$-ring $M$ in which $\Gamma_0 \neq \Phi$ is a weakly $\Gamma$-division ring if and only if

1. It has at least two elements.
2. It has no proper left ideals and proper right ideals.
3. $M\gamma M \neq 0$ for every $\gamma \in \Gamma_0$.

Proof. Suppose that $M$ is a weakly $\Gamma$-division ring. It is clear that $M$ has at least two elements. Let $L$ be a non-zero left ideal of $M$ and $m$ a non-zero element of $L$. From Proposition 2.2, for every element $\gamma \in \Gamma_0$, we have:

$$M = M\gamma m \subseteq M\Gamma L \subseteq L.$$ 

So, $L = M$ and for every $\gamma \in \Gamma_0$ and $m \in M^*$ we have $M\gamma M \neq 0$. In the same way we prove that $M$ has no proper right ideals.

Conversely, assume that the weakly $\Gamma$-ring $M$ has at least two elements, has no proper left ideals and proper right ideals and for every element $\gamma \in \Gamma_0$, $M\gamma M \neq 0$. Therefore, there exists a non-zero element $m$ of $M$ such that $m\gamma M \neq 0$ and consequently $m\gamma M = M$. Therefore, for every $\gamma \in \Gamma_0$ there is an element $e \in M$ such that $m\gamma e = m$. From this, we have the equality $m\gamma e = m\gamma e$, which we rewrite

$$m\gamma(e\gamma e - e) = 0.$$

We denote

$$M_1 = \{m_1 \in M \mid m\gamma m_1 = 0\}.$$ 

It is easy to see that $M_1$ is a non-zero right ideal of $M$. So, $M_1 = 0$ because otherwise we would have $m\gamma M = 0$ in contradiction with the equality $m\gamma M = M$. Since $e\gamma e - e = m_1$ is an element such that $m\gamma m_1 = 0$, we have $m_1 = 0$ and consequently $e\gamma e = e$. Since for very $\gamma \neq 0$, $e\gamma M$ is a right ideal different from zero, we have $M = e\gamma M$. From this follows that for every $\gamma \in \Gamma_0$ the element $e \in M$ is a left identity of the ring $(M, +, \circ_\gamma)$. In the same way, by using the absence of the proper left ideals, we prove the existence of right identity $e_1$ of the ring $(M, +, \circ_\gamma)$. Now, it is clear that $e = e_1$ and the element $e$ is the identity element of the ring $(M, +, \circ_\gamma)$. For every $m \neq 0$, we have $m = m\gamma e$ and consequently $m\gamma M \neq 0$. Since $m\gamma M$ is a right ideal and $m\gamma M \neq 0$, then $m\gamma M = M$. From this, it follows that for every $\gamma \in \Gamma_0$ and
every element \( m \in M \setminus \{0\} \) there is an element \( m' \in M \) such that \( m\gamma m' = e \). So, for every \( \gamma \in \Gamma_0 \) the ring \((M, +, \circ_\gamma)\) is a division ring and consequently \((M, +, (\cdot)_\Gamma)\) is a weakly \( \Gamma \)-division ring.

The first conditions of two theorems 3.1 and 3.2 are the same, the second conditions of Theorem 3.1 is satisfied when the second conditions of the Theorem 3.2 is satisfied, while the third condition of Theorem 3.2 is satisfied when the third condition of Theorem 3.1 is satisfied. Naturally it arises the question if an analogue of Theorem 3.1 and Theorem 3.2 hold true which use the condition 2 of Theorem 3.1 and the condition 3 of Theorem 3.2. So, we raise the following open problem:

**Problem 3.1** Does the following proposition hold true: A weakly \( \Gamma \)-ring is a weakly \( \Gamma \)-division ring if and only if it has at least two elements, \( \Gamma_0 \neq \Phi \), and it doesn’t have proper left [right] ideals and for every \( \gamma \in \Gamma_0 \), \( M\gamma M \neq 0 \).

**Definition 3.1** A quasi-ideal of a weakly \( \Gamma \)-ring \((M, +, (\cdot)_\Gamma)\) is called every subgroup \( Q \) of group \((M, +)\) such that

\[
Q\Gamma M \cap M\Gamma Q \subseteq Q.
\]

The following theorem gives another characterization of a weakly \( \Gamma \)-division ring by quasi-ideals.

**Theorem 3.3** A weakly \( \Gamma \)-ring \( M \) in which \( \Gamma_0 \neq \Phi \) is a weakly \( \Gamma \)-division ring if and only if it has at least two elements and has no proper quasi-ideals, i.e. quasi-ideals different from zero and \( M \), and

\[
\forall (m, \gamma) \in M^* \times \Gamma_0, M\gamma m \neq 0.
\]

**Proof.** Assume that \( M \) is a weakly \( \Gamma \)-division ring. It is evident that \( M \) has at least two elements. Let \( Q \) be a non-zero quasi-ideal of \( M \) and \( m \) a non-zero element of \( Q \) and \( \gamma \) an element of \( \Gamma_0 \). According to Proposition 2.1 and Proposition 2.2 we have:

\[
M = M\gamma m = m\gamma M = M\gamma m \cap m\gamma M \subseteq M\Gamma Q \cap Q\Gamma M \subseteq Q.
\]

Hence \( Q = M \).

Conversely, assume that a weakly \( \Gamma \)-ring \( M \) has at least two elements, has no proper quasi-ideals and for every \( m \in M^* \) and \( \gamma \in \Gamma_0 \) we have \( M\gamma m \neq 0 \). Since every left ideal of \( M \) is a quasi-ideal of \( M \), then \( M \) has no proper left ideals and according to Theorem 3.1 we have that the weakly \( \Gamma \)-ring \( M \) is a weakly \( \Gamma \)-division ring.

In the same way we can prove the analogue theorem of Theorem 3.3, which in place of the condition \( M\gamma m \neq 0 \) it has the condition \( m\gamma M \neq 0 \).

From Theorem 3.3 we have the following corollary:
Corollary 3.2 A weakly $\Gamma$-ring $M$ is a weakly $\Gamma$-division ring if and only if it has at least two elements, $\Gamma_0 \neq \Phi$ and has no proper quasi-ideals and has no divisors of zero.

A left ideal [right ideal, quasi-ideals] $L[R, Q]$ of a weakly $\Gamma$-ring $M$ is called minimal if $L [R, Q]$ has no proper left ideals [right ideals, quasi-ideals].

Theorem 3.4 The weakly $\Gamma$-ring $M$ is a weakly $\Gamma$-division ring if and only if it has at least two elements, $\Gamma_0 \neq \Phi$ and for every $\gamma \in \Gamma_0$ there exists a $\gamma$-cancelable element which is contained in a minimal quasi-ideal of $M$.

Proof. Suppose that the weakly $\Gamma$-ring $M$ is a weakly $\Gamma$-division ring. It is evident that $M$ has at least two elements. From Theorem 3.3, we have that $M$ is a minimal quasi-ideal. The ring $(M, +, \circ_\gamma)$ is a division ring for every $\gamma \in \Gamma_0$ and its identity element $e_\gamma$ is a $\gamma$-cancelable element, which is included in the minimal quasi-ideal $M$.

Conversely, suppose that the weakly $\Gamma$-ring $M$ has at least two elements and for every $\gamma \in \Gamma_0$, $m$ is a $\gamma$-cancelable element, which is included in a minimal quasi-ideal $Q$ of $M$. Since, $M\gamma m \cap m\gamma M$ is a quasi-ideal of $M$, which is included in $Q$ and it contains the element $m\gamma m \neq 0$ of $Q$, then from minimality of $Q$ we have $Q = M\gamma m \cap m\gamma M$. So, there exist the elements $e_\gamma$ and $e'_\gamma$ such that $m = e_\gamma m = m\gamma e'_\gamma$. From this, we have that for every element $x \in M$

$$x\gamma m = x\gamma e_\gamma \gamma m \text{ and } m\gamma x = m\gamma e'_\gamma \gamma x,$$

and since $m$ is a $\gamma$-cancelable element the following equalities hold true

$$x = x\gamma e_\gamma = e'_\gamma \gamma x,$$

which shows that the element $e_\gamma$ is the identity element of the ring $(M, +, \circ_\gamma)$. In the same way as above, we show that:

$$Q = M\gamma m\gamma m \cap m\gamma m\gamma M.$$

So, exist the elements $a, b$ of $M$ such that

$$e_\gamma \gamma m = m\gamma e_\gamma = m = a\gamma m\gamma m = m\gamma m\gamma b,$$

and since $m$ is $\gamma$-cancelable, we have

$$e_\gamma = a\gamma m = m\gamma b \in Q.$$

Now, for every $c \in M$, $c = e_\gamma \gamma c = c\gamma e_\gamma \in Q$ and consequently $Q = M$. So, the weakly $\Gamma$-ring $M$ satisfies the conditions of Theorem 3.3 and consequently is a weakly $\Gamma$-division ring. ◼

We have to prove the following proposition, in order to be able to give a characterization of weakly $\Gamma$-division rings in terms of minimal left ideals and minimal right ideals.
Proposition 3.1 The intersection of every minimal left ideal $L$ with every minimal right ideal $R$ of a weakly $\Gamma$-ring $M$ is zero or a minimal quasi-ideal of $M$.

Proof. It is easy to prove that the intersection $L \cap R = Q$ is a quasi-ideal of $M$. Suppose that $Q \neq 0$ and we will prove that $Q$ is a minimal quasi-ideal of $M$. Let $Q_1$ be a non-zero quasi-ideal of $M$, which is included in $Q$ and consequently it is included in $L$. So, the set $M\Gamma Q_1$ is a left ideal of $M$, which is included in $L$. Since $L$ is a minimal ideal of $M$, then $M\Gamma Q_1 = 0$ or $M\Gamma Q_1 = L$. If $M\Gamma Q_1 = 0$, then $Q_1$ is a left ideal of $M$ different from zero and $Q_1 \subseteq L$. We have $Q_1 = L$ by the minimality of $L$ and consequently $Q_1 = Q$. If $M\Gamma Q_1 = L$, then we can prove in the same way that $Q_1 \Gamma M = 0$ or $Q_1 \Gamma M = R$. In the case when $Q_1 \Gamma M = 0$, $Q_1$ is a right ideal of $M$, which is included in $R$ and we have $Q_1 = R$ by the minimality of $R$, and consequently $Q_1 = Q$. In the case when $Q_1 \Gamma M = R$ we have:

$$Q = L \cap R = Q_1 \Gamma M \cap M\Gamma Q_1 \subseteq Q_1,$$

and so $Q_1 = Q$. Hence $Q = R \cap L$ is a minimal quasi-ideal of weakly $\Gamma$-ring $M$. ■

Now we will give the following characterization of a weakly $\Gamma$-division ring by the minimal left ideals and minimal right ideals.

Theorem 3.5 A weakly $\Gamma$-ring is a weakly $\Gamma$-division ring if and only if it has at least two elements, $\Gamma_0 \neq \Phi$ and for every $\gamma \in \Gamma_0$ there exists a $\gamma$-cancelable element $m_\gamma$ such that the principal left and right ideals $(m_\gamma)_l$ and $(m_\gamma)_r$ are minimal.

Proof. Assume that a weakly $\Gamma$-ring $M$ has at least two elements, $\Gamma_0 \neq \Phi$ and for every $\gamma \in \Gamma_0$ there exists a $\gamma$-cancelable element $m_\gamma$ such that the principal left ideal $(m_\gamma)_l$ and the principal right ideal $(m_\gamma)_r$ generated by $m_\gamma$ are minimal. From Proposition 3.1 the intersection $(m_\gamma)_l \cap (m_\gamma)_r = Q$ is a minimal quasi-ideal. For every $\gamma \in \Gamma_0$ the minimal quasi ideal $Q$ contains the $\gamma$-cancelable element $m_\gamma$ and from Theorem 3.4 we have that $M$ is a weakly $\Gamma$-division ring.

Conversely, assume that $M$ is a weakly $\Gamma$-division ring. Then it has at least two elements, $\Gamma_0 \neq \Phi$ and from Theorem 3.4 it has no proper left ideals and proper right ideals and consequently for every $\gamma \in \Gamma_0$, for the identity element $e_\gamma$ of the division ring $(M, +, \circ_\gamma)$, which is a $\gamma$-cancelable element we have that $(e_\gamma)_l$ and $(e_\gamma)_r$ are respectively minimal left ideal and minimal right ideal. ■

In connection with the characterizations of weakly $\Gamma$-division rings, which are given from Theorem 3.4 and Theorem 3.5 we raise the following problem:

Problem 3.2 Do the Theorem 3.4 and Theorem 3.5 remain true if we ask only for the existence of a $\gamma$-cancelable element?
On the weakly \( \Gamma \)-division rings

References


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