Relative Hopf Modules in the Braided Monoidal Category $\mathcal{LM}$\(^1\)

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Abstract

Let $L$ be a quasitriangular weak Hopf algebra with bijective antipode and $A$ a weak Hopf algebra in the braided monoidal category $\mathcal{LM}$. $B$ is a right $A$–comodule algebra in $\mathcal{LM}$. The definition of $(A, B)$–Hopf module and the fundamental structure theorem of $(A, B)$–Hopf module in $\mathcal{LM}$ are given.

Mathematics Subject Classification: 16W30, 16S37

Keywords: weak Hopf algebra, quasitriangular weak Hopf algebra, relative Hopf module

1 Introduction

Yanmin Yin and Mingchuan Zhang discussed Hopf modules in the braided monoidal category $\mathcal{LM}$ in [2]. Y. Doi. introduced relative Hopf modules in [4]. In this paper, we will discuss relative Hopf modules in category $\mathcal{LM}$.

Throughout the paper, $k$ is a field, and all algebras and coalgebras are based on it. All Hopf algebras and weak Hopf algebras are finite dimensional. We recall some basic definitions and propositions from [2].

\(^1\)This work is supported by National Foundation of China(No.11271319) and Natural Science Foundation of Zhejiang(No.LY14A010006)
Definition 1.1. ([2]) A quasitriangular weak Hopf algebra is a pair \((L, R)\), where \(L\) is a weak Hopf algebra and \(R \in \Delta^{\text{op}}(1)(L \otimes L)\Delta(1)\) (called the \(R\)-matrix) satisfying the following conditions:

\[
\Delta^{\text{op}}(l)R = R\Delta(l)
\]

for all \(l \in L\), where \(\Delta^{\text{op}}\) denotes the conditions opposite to \(\Delta\),

\[
(id \otimes \Delta)(R) = R_{13}R_{12},
\]

\[
(\Delta \otimes id)(R) = R_{13}R_{23},
\]

where \(R_{12} = R \otimes 1, R_{23} = 1 \otimes R\), etc. as usual, and such that there exists \(\overline{R} \in \Delta(1)(L \otimes L)\Delta^{\text{op}}(1)\) with

\[
\overline{R}\overline{R} = \Delta^{\text{op}}(1), \quad \overline{R}R = \Delta(1).
\]

We write \(R = R^1 \otimes R^2\), then \(R_{21} = R^2 \otimes R^1\).

Let \(L\) be a quasitriangular weak Hopf algebra with a bijective antipode, then the category \(_L\mathcal{M}\) is a braided monoidal category whose objects are left \(L\)-modules. The braiding \(\tau_{V,W} : V \otimes W \to W \otimes V\) is defined by

\[
\tau_{V,W}(v \otimes w) = (R^2 \cdot w) \otimes (R^1 \cdot v), \quad \text{for all } v \in V, w \in W.
\]

Definition 1.2. ([2]) Let \(L\) be a weak Hopf algebra. An algebra \(A\) is called a left \(L\)-module algebra if \(A\) is a left \(L\)-module via \(l \otimes x \mapsto l \cdot x\) such that

1. \(l \cdot (xy) = (l_1 \cdot x)(l_2 \cdot y)\),
2. \(l \cdot 1 = \varepsilon(l) \cdot 1\), for all \(x, y \in A, l \in L\).

An coalgebra \(A\) is called a left \(L\)-module coalgebra if \(A\) is a left \(L\)-module via \(l \otimes x \mapsto l \cdot x\) such that

3. \(\Delta(l \cdot x) = l_1 \cdot x_1 \otimes l_2 \cdot x_2\),
4. \(\varepsilon_s(l) \cdot x = x_1\varepsilon(l \cdot x_2)\), for all \(l \in L, x \in A\).

Definition 1.3. ([2]) Let \((L, R)\) be a quasitriangular weak Hopf algebra. An object \(A \in _L\mathcal{M}\) is called a weak Hopf algebra in this category if it is both an algebra and a coalgebra satisfying the following conditions:

1. \(\Delta\) and \(\varepsilon\) are not necessarily unit-preserving, such that

\[
\Delta(xy) = x_1(R^2 \cdot y_1) \otimes (R^1 \cdot x_2)y_2,
\]

\[
\varepsilon(xy) = \varepsilon(x_1y_1)\varepsilon(y_2z) = \varepsilon(x(R^2 \cdot y_2))\varepsilon((R^1 \cdot y_1)z),
\]

\[
\Delta^2(1) = 1_1 \otimes 1_2 1'_1 \otimes 1'_2 = 1_1 \otimes (R^2 \cdot 1'_1)(R^1 \cdot 1_2) \otimes 1'_2,
\]

where \(\Delta(1) = 1_1 \otimes 1_2 = 1'_1 \otimes 1'_2\).
2. \(A\) is both a left \(L\)-module algebra and \(L\)-module coalgebra.
(3) There exists an antipode $S : A \rightarrow A$ (here $S$ is a morphism in the category of $LM$) satisfying

$$
x_1 S(x_2) = \varepsilon((R^2 \cdot 1_1)(R^1 \cdot x))1_2,
S(x_1) x_2 = \varepsilon((R^2 \cdot x)(R^1 \cdot 1_2))1_1,
S(x_1) x_2 S(x_3) = S(x), \text{ for all } x \in A.
$$

Similar to the notation of weak Hopf algebra, we note

$$
\varepsilon_l(x) = \varepsilon((R^2 \cdot 1_1)(R^1 \cdot x))1_2, \quad \varepsilon_s(x) = \varepsilon((R^2 \cdot x)(R^1 \cdot 1_2))1_1.
$$

As $S$ is left $L$–linear, we can easily check that $\varepsilon_l$ and $\varepsilon_s$ are also left $L$–linear. Moreover it is both an anti-algebra map and an anti-coalgebra map, that is

$$
S(xy) = (R^2 \cdot S(y))(R^1 \cdot S(x)), \quad \Delta(S(x)) = R^2 \cdot S(x_2) \otimes R^1 \cdot S(x_1), x, y \in A.
$$

**Proposition 1.4.** ([2]) Suppose $A$ is a weak Hopf algebra in $LM$. For all $x \in A$, we have the identities

$$
x_1 \otimes \varepsilon_s(x_2) = x_1 \otimes S(1_2), \quad \varepsilon_l(x_1) \otimes x_2 = S(1_1) \otimes 1_2 x.
$$

## 2 ($A, B$)–Hopf modules in category $LM$

**Definition 2.1.** Let $B$ be an algebra in $LM$ and $A$ a weak Hopf algebra in $LM$. Then we call $B$ a right $A$–comodule algebra in $LM$, if the following conditions hold:

1. $B$ is a right $A$–comodule;
2. $B$ is a left $L$–module algebra;
3. $(bc)_0 \otimes (bc)_1 = b_0(R^2 \cdot c_0) \otimes (R^1 \cdot b_1)c_1$;
4. $\rho_B(l \cdot b) = l_1 \cdot b_0 \otimes l_2 \cdot b_1$.

**Example 2.2.** A weak Hopf algebra $A$ in $LM$ is a right $A$–comodule algebra in $LM$.

**Definition 2.3.** Let $A$ be a weak Hopf algebra in $LM$. A right $(A, B)$–Hopf module $M$ in $LM$ is an object $M \in LM$ such that it is both a right $A$–comodule via $\rho_M : M \rightarrow M \otimes H$, $\rho_M(m) = m_0 \otimes m_1$ and a right $B$–module and for $l \in L, m \in M, b \in B$, the following conditions hold:

1. $\rho_M(mb) = m_0(R^2 \cdot b_0) \otimes (R^1 \cdot m_1)b_1$,
2. $\rho_M(l \cdot m) = l_1 \cdot m \otimes l_2 \cdot m$,
3. $l \cdot (mb) = (l_1 \cdot m)(l_2 \cdot b)$.

**Example 2.4.** A right $A$–comodule algebra $B$ in $LM$ is a right $(A, B)$–Hopf module in $LM$. 
**Theorem 2.5.** Let $A$ be a weak Hopf algebra in $LM$ and $B$ is a right $A$–comodule algebra. If there exists a right $A$–comodule map $\varphi : A \rightarrow B$ satisfying $\varphi(1_A) = 1_B$, then every right $(A,B)$–Hopf module is injective as an $A$–comodule in category $LM$.

**Proof.** Suppose $M$ is a right $(A,B)$–Hopf module in $LM$. $M \otimes A$ is a right $A$–comodule whose structure is given by $I \otimes \Delta_A$, then the comodule structure map $\rho_M : M \rightarrow M \otimes A$ is an $A$–comodule map. We show that there is an $A$–comodule map $\lambda : M \otimes A \rightarrow M$ with $\lambda \rho_M = I$. Thus $M$ is injective since it is isomorphic to a direct summand of $M \otimes A$, an injective $A$–comodule.

\[
\rho_B(\varphi(1_A)) = (\varphi \otimes I)\Delta(1_A) = \varphi(1_A) \otimes 1_A^2 = \rho_B(1_B) = 1^0_B \otimes 1^1_B.
\]
\[
\rho_M(m) = \rho_M(m1_B) = m_0(R^2 \cdot 1^0_B) \otimes (R^1 \cdot m_1)1^1_B.
\]

Define a map $\lambda : M \otimes A \rightarrow M$ by $\lambda(m \otimes a) = m_0\varphi(S(m_1)a)$ for $m \in M, a \in A$. Then

\[
\lambda \rho_M(m) = \lambda(m_0 \otimes m_1) = m_0\varphi(S(m_1)m_2) = m_0\varphi(\varepsilon_s(m_1)) = (m_0\varphi(\varepsilon_s(m_1)))_0 \varepsilon((m_0\varphi(\varepsilon_s(m_1)))_1) = m_0(R^2 \cdot \varphi(1_A)) \otimes \varepsilon((R^1 \cdot m_1)1_2 \varepsilon_s(m_2)) = m_0\varepsilon(m_1 \varepsilon_s(m_2)) = m_0\varepsilon(m_11_1S(1_2)) = m.
\]

Hence $\lambda \rho_M = I$. Next we will prove $\lambda$ is a right $A$–comodule map.

\[
\rho_M \lambda(m \otimes a) = \rho_M(m_0\varphi(S(m_1)a)) = m_0 R^2 \cdot (\varphi(S(m_1)a))_0 \otimes (R^1 \cdot m_1)(\varphi(S(m_1)a))_1 = m_0 R^2 \cdot \varphi(S(m_2)_1 r^2 \cdot a_1) \otimes (R^1 \cdot m_1)(r^1 \cdot S(m_2))_2 a_2 = m_0 R^2 \cdot \varphi(r^{2r} \cdot S(m_3)_2 r^2 \cdot a_1) \otimes (R^1 \cdot m_1)(r^1 r^{1r} \cdot S(m_2))_2 = m_0 R^2 \cdot \varphi((S(m_3)a_1)_0 \otimes (R^1 \cdot m_1)(r^1 \cdot S(m_1))_2 = m_0 R^2 \cdot \varphi(S(m_3)a_1) \otimes (R^1 \cdot (m_1 S(m_2)))_2 = m_0 R^2 \cdot \varphi(S(m_2)_1 a_1) \otimes (R^1 \cdot (\varepsilon_s(m_1)))_2 = m_0 \varphi(r^2 \cdot S(m_2)_1 R^2 \cdot a_1) \otimes (R^1 \cdot (S^{-1} \varepsilon_s(r^1 S(m_1))))_2 = m_0 \varphi(S(m_1)_1 R^2 \cdot a_1) \otimes (R^1 \cdot (S^{-1} \varepsilon_s(S(m_1)_2)))_2 = m_0 \varphi(S(m_1)_1 1_1 R^2 \cdot a_1) \otimes (R^1 \cdot 1_2 a_2 = m_0 \varphi(S(m_1)_1 a_1) \otimes a_2 = (\lambda \otimes I)(I \otimes \Delta_A)(m \otimes a).
\]

So $\rho_M \lambda = (\lambda \otimes I)(I \otimes \lambda)$, that is to say $\lambda$ is a right $A$–comodule map. Hence every right $(A,B)$–Hopf module is injective as an $A$–comodule in category $LM$. \qed

Let $A$ be a weak Hopf algebra in $LM$ and $B$ a right $A$–comodule algebra in $LM$. Define the $A$–invariant subspace of $B$ to be the set

\[
B_0 = \{b \in B | \rho_B(b) = b \otimes 1_A\}.
\]
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It is clear that $B_0$ is a subalgebra of $B$. Let $M$ be a right $(A, B)$--Hopf module in $L_M$. Define the set

$$M_0 = \{ m \in M | \rho_M(m) = m1_B^0 \otimes 1_B^1 \}.$$

For any $m \in M$ and $b_0 \in B_0$, we have $mb_0 \in M_0$. Thus $M_0$ is a right $B_0$--module. And further we will show $M_0$ is an object of $L_M$. Then it is easy to know $M_0 \otimes_{B_0} B$ is a right $(A, B)$--Hopf module in $L_M$.

**Theorem 2.6.** Let $A$ be a weak Hopf algebra and $B$ a right $A$--comodule algebra in category $L_M$. $M$ is a right $(A, B)$--Hopf module. If there is a right $A$--comodule map $\varphi : A \to B$, which is an algebra map, then

1. $M_0 \in L_M$;
2. $P(m) = m_0 \varphi(S(m_1)) \in M_0$;
3. The map $F : M_0 \otimes_{B_0} B \to M$, $F(n \otimes_{B_0} b) = nb, n \in M_0, b \in B$, is an isomorphism of $(A, B)$--Hopf modules. The inverse map is given by $G(m) = P(m_0)m_1$.

**Proof.** (1) For any $l \in L, m \in M_0$ we have

$$\rho_M(l \cdot m) = l_1 \cdot (m_1) \otimes l_2 \cdot 1_1 = (l_1 \cdot m)(l_2 \cdot 1_0) \otimes l_3 \cdot 1_1$$
$$= (l_1 \cdot m \otimes 1)\rho_B(l_2 \cdot 1_B) = (l_1 \cdot m \otimes 1)\rho_B(\varepsilon(l_2) \cdot 1_B)$$
$$= ((1(1)l \cdot m) \otimes 1)\rho_B(1(2) \cdot 1_B)$$
$$= (1(1) \cdot (l \cdot m))(1(2)1(1') \cdot 1_0) \otimes 1(2') \cdot 1_1$$
$$= (l \cdot m)(1(1) \cdot 1_0) \otimes 1(2) \cdot 1_1 = (l \cdot m)1_0 \otimes 1_1.$$

Hence $l \cdot m \in M_0$. So $M_0$ is a left $L$--module.

(2) We just need to prove the equation $\rho_M(P(m)) = P(m)1_0 \otimes 1_1$.

$$\rho_M(P(m)) = \rho_M(m_0 \varphi(S(m_1))) = m_0 R^2 \cdot \varphi(S(m_2)) \otimes (R^1 \cdot m_1)\varphi(S(m_2))_1$$
$$= m_0 R^2 \cdot \varphi(S(m_3)) \otimes (R^1 \cdot m_1)(r^1 \cdot S(m_2))$$
$$= m_0 R^2 \cdot \varphi(S(m_3)) \otimes R^1 \cdot (m_1 \cdot S(m_2))$$
$$= m_0 R^2 \cdot \varphi(S(m_2)) \otimes R^1 \cdot \varepsilon_S(m_1)$$
$$= m_0 R^2 \cdot \varphi(S(m_2)) \otimes R^1 \cdot (S^{-1}\varepsilon_S(m_1))$$
$$= m_0 \varphi(S(m_1)1_1) \otimes S^{-1}\varepsilon_S(S(m_1))$$
$$= m_0 \varphi(S(m_1)1_0) \otimes 1_1 = m_0 \varphi(S(m_1)1_0) \otimes 1_1$$
$$= P(m)1_0 \otimes 1_1.$$

(3) For any $n \in M_0, b, c \in B$, we have

$$\rho_M F(n \otimes b) = n1_B^0 R^2 \cdot b_0 \otimes (R^1 \cdot 1_B^1)b_1 = nb_0 \otimes b_1 = (F \otimes id)\rho(n \otimes b).$$
Hence $F$ is a right $A$–comodule map. It’s easy to know $F$ is a right $B$–module map and a left $A$–module map. For any $m \in M, b \in B, n \in M_0$, we have

$$GF(n \otimes_{B_0} b) = P((nb)_0) \otimes_{B_0} \varphi((nb)_1)$$

$$= P(n_0R^2 \cdot b_0) \otimes_{B_0} \varphi((R^1 \cdot n_1)b_1)$$

$$= P(n^1_0R^2 \cdot b_0) \otimes_{B_0} \varphi(R^1 \cdot 1^1_B b_1)$$

$$= P(nb_0) \otimes_{B_0} \varphi(b_1) = nb_0 \varphi(S(b_1)) \otimes_{B_0} \varphi(b_2)$$

$$= n \otimes_{B_0} b_0 \varphi(S(b_1)b_2) = n \otimes_{B_0} b.$$

From the proof of therom 2.5, we know

$$FG(m) = P(m_0)\varphi(m_1) = m_0\varphi(S(m_1))\varphi(m_2)$$

$$= m_0\varphi(\varepsilon_\varphi(m_1)) = m.$$

$\square$

References


Received: September 1, 2014