The Solution of the Diophantine Equation

\[ x^2 + 3y^2 = z^2 \]

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Abstract

In this paper we are interested to show how to solve the diophantine equation \( x^2 + 3y^2 = z^2 \) by using the arithmetic technical.

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1 Introduction

In [4] Ribet showed that the curve \( E_{\text{Frey}} : y^2 = x(x - a')(x + b') \) where \( a' + b' = c' \) cannot be modular. And Wiles showed in 1994 that all semistable elliptic curves over \( \mathbb{Q} \) are modular. According to theorems Ribet and wiles we deduce that the diophantine equation \( x^n + y^n = z^n \) with \( xyz = 0 \) has no solution in \( \mathbb{Z}^3 \). Many mathematicians are interested to study the diophantine equation; As Bennett studied the diophantine equation \( x^{2n} + y^{2n} = z^5 \) [2] in
2004, Frits Beukers studied the diophantine equation $Ax^p + By^q = Cz^r$ in 1998 [1] and Nils Bruin search the solution of the diophantine equation $x^9 + y^8 = z^3$ with $xyz = 0$ in 1999 [3].

In our paper we are interested to search the solution of the diophantine equation $x^2 + 3y^2 = z^2$.

### 2 Results and Discussion

#### 2.1 Diophantine equation $x^2 + 3y^2 = z^2$

In this section we show the following result which characterizes the solution of the diophantine equation $x^2 + 3y^2 = z^2$.

**Proposition 2.1** If $(x, y, z) \in \mathbb{Z}^3$ is a solution of the diophantine equation $x^2 + 3y^2 = z^2$ then $3 \wedge xz = 1$.

**Proof**

Assume the contrary. If 3 divides $xz$ then we have two cases:

The first case, if $x = 3x_1$ then the equation $x^2 + 3y^2 = z^2$ is equivalent to $(3x_1)^2 + 3y^2 = z^2$. Therefore $z = 3z_1$ so $3x_1^2 + y^2 = 3z_1^2$ and then $y = 3y_1$. We deduce that 3 divides $x \wedge y = 1$, which is a contradiction.

The second case, if $z = 3z_1$ then the equation $x^2 + 3y^2 = z^2$ is equivalent to $x^2 + 3y^2 = (3z_1)^2$. Therefore $x = 3x_1$ so $3x_1^2 + y^2 = 3z_1^2$ and then $y = 3y_1$. We deduce that 3 divides $x \wedge y = 1$, which is a contradiction.

**Theorem 2.1** Let $E : x^2 + 3y^2 = z^2$ diophantine equation and $(x, y, z) \in \mathbb{Z}^3$ with $x \wedge y = 1$, $y$ is even and $xz \wedge 3 = 1$. then the following properties are equivalent:

(i) $(x, y, z)$ is the solution of $E$.

(ii) $|z| = 3y_1^2 + y_2^2$, $|x| = 3y_1^2 - y_2^2$, $|y| = 2y_1y_2$ with $y_1 \wedge y_2 = 1$

**Proof**

(ii) $\implies$ (i)

We have:

\[
\begin{align*}
x^2 + 3y^2 &= (3y_1^2 - y_2^2)^2 + 3(2y_1y_2)^2 \\
&= 9y_1^4 + 6(y_1y_2)^2 + y_2^4 \\
&= (3y_1^2 + y_2^2)^2 \\
&= z^2
\end{align*}
\]

(i) $\implies$ (ii)

Since $y$ is even and $x \wedge y = 1$ then $x$ is odd. Therefore $z$ is also odd because $z^2 = x^2 + 3y^2$ which is implies that $\frac{z^2}{2}$ and $\frac{z^2}{2}$ are integers and $y = 2y_0$.

We have:
The solution of the Diophantine equation \( X^2 + 3Y^2 = Z^2 \)

Then \( y_0^2 = \frac{(z-x) (z+x)}{2} \). And since \( x \land y = 1 \) and \( x^2 + 3y^2 = z^2 \) then \( z \land x = 1 \).
Assume that \( d = \frac{(z-x)}{2} \land \frac{(z+x)}{2} \) then \( d \) divides \( \frac{(z-x)}{2} + \frac{(z+x)}{2} = z \) and \( d \) divides \( \frac{(z-x)}{2} - \frac{(z+x)}{2} = x \) So \( d \) divides \( x \land z = 1 \) therefore \( \frac{(z-x)}{2} \land \frac{(z+x)}{2} = 1 \).
Then we deduce that:

\[
\frac{y}{(z-x)} = 2y_1y_2, \quad \frac{y}{(z+x)} = 3y_1^2 \quad \frac{y}{2} = y_2^2
\]

Which is implies that:

\[
z = \frac{(z-x) + (z+x)}{2} = \frac{3y_1^2 + y_2^2}{2}
\]

and

\[
x = -\frac{(z-x) + (z+x)}{2} = -3y_1^2 + y_2^2
\]

**Theorem 2.2** Let \( E : x^2 + 3y^2 = z^2 \) diophantine equation and \( (x, y, z) \in \mathbb{Z}^3 \) with \( x \land y = 1 \), \( y \) is odd and \( xz \land 3 = 1 \). Then the following properties are equivalent:

(i) \( (x, y, z) \) is the solution of \( E \)

(ii) \( | z | = \frac{3y_1^2 + y_2^2}{2} \), \( | x | = \frac{3y_1^2 - y_2^2}{2} \), \( | y | = y_1y_2 \) with \( y_1 \land y_2 = 1 \)

**Proof**

(ii) \( \implies \) (i)
We have:

\[
x^2 + 3y^2 = \left( \frac{3y_1^2 - y_2^2}{2} \right)^2 + 3(y_1y_2)^2
\]

\[
= \frac{9y_1^4 - 6y_1y_2 + y_2^4}{4} + 3y_1^2y_2^2
\]

\[
= \frac{9y_1^4 + 4y_2^4}{4} \frac{1}{z^2}
\]

(i) \( \implies \) (ii)
We have:

\[
x^2 + 3y^2 = \frac{z^2}{3y^2} = \frac{z^2}{x^2}
\]

\[
= (z - x)(z + x)
\]

Implying that: \( y = y_1y_2 \), \( z - x = 3y_1^2 \) and \( z + x = y_2^2 \). Consequently \( x = -\frac{3y_1^2 + y_2^2}{2} \) and \( z = \frac{3y_1^2 + y_2^2}{2} \).

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References


[3] Nils Bruin. The Diophantine equation $x^2 \pm y^4 = \pm z^6$ and $x^9 + y^8 = z^3$, Compositio Math 118 1999 N 3, 305-321.


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