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A Note on Direct Product of Fuzzy Modules over Fuzzy Rings

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Abstract

In this paper, a new kind of direct product of fuzzy modules over fuzzy rings is studied by using the definition of new type of fuzzy modules over fuzzy rings which is introduced in [3]. Some properties of this new structure is investigated analogues to ordinary module theory.

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1. Introduction

The concept of fuzzy group is first intoduced by Rosenfeld in [1]. In this definition, the set G is already asumed to be a group and fuzzy subgroup is the fuzzy subset of this group G with nonfuzzy binary operations. Since then many researchers have studied about fuzzy structures in many different types as you can see in [2], [7], [6]. The construction of fuzzy groups based on fuzzy binary operation which is different from Rosenfeld's is given by Yuan and Lee [8] and fuzzy ring is introduced in [5]. In these approach, while the set is assumed to be nonfuzzy, the binary operation is fuzzy. By the use of these fuzzy group and fuzzy ring structures, the concept of fuzzy module over fuzzy ring is presented in [3].

In this work, by using the definition of fuzzy modules over fuzzy rings [3], a new kind of direct product of fuzzy is introduced and studied. Some properties of this new structure is investigated analogues of ordinary module theory.

For general background and terminology, the reader may consult [6], [3], [8].

2. Preliminary Notes

In this section the preliminary definitions that are required in this paper are summerized. All contents and further of this section are contained in [3].

Let G be a fuzzy binary operation on an ordinary nonempty set M. Hence we have a mapping

$$G : F(M) \times F(M) \to F(M) \ ,$$

$$(A,B) \mapsto G(A,B) \ ,$$

where $F(M) = \{A \mid A : M \to [0,1] \text{ is a mapping}\}$ and

$$G(A,B)(c) = \bigvee_{a,b\in M} (A(a) \wedge B(b) \wedge G(a,b,c)) \ . \tag{2.1}$$

Let $A = \{a\}$ ve $B = \{b\}$ and let $G(A,B)$ be denoted as $a \circ b$, then

$$(a \circ b)(c) = G(a,b,c), \quad \forall c \in M \ , \tag{2.2}$$

$$((a \circ b) \circ c)(z) = \bigvee_{d\in M} (G(a,b,d) \wedge G(d,c,z)) \ , \tag{2.3}$$

$$(a \circ (b \circ c))(z) = \bigvee_{d\in M} (G(b,c,d) \wedge G(a,d,z)) \ . \tag{2.4}$$

Using the notations in Eqs. (1)-(4), the definitions of the concept of fuzzy group, fuzzy ring and fuzzy module is given in, [5], [3], [8].

Definition 1. [8] *Let M be a nonempty set and G be a fuzzy binary operation on M. (M,G) is called a fuzzy group if the following conditions are true:*

G1: $\forall a,b,c,z_1,z_2 \in M$, $((a \circ b) \circ c)(z_1) > \theta$ and $(a \circ (b \circ c))(z_2) > \theta$ implies $z_1 = z_2$;
G2: $\exists e_o \in M$ such that $(e_o \circ a)(a) > \theta$ and $(a \circ e_o)(a) > \theta$, for any $a \in M$;
G3: $\forall a \in M$, $\exists b \in M$ such that $(a \circ b)(e_o) > \theta$ and $(b \circ a)(e_o) > \theta$.

Definition 2. [5] *Let R be a nonempty set and G and H be two fuzzy binary operations on R, and let $G(a,b)$ and $H(a,b)$are denoted as $a \circ b$ and $a * b$, respectively with the following operations*

$$(a * b)(c) = H(a,b,c) \ , \forall c \in R, \tag{2.5}$$

$$(a * (b \circ c))(z) = \bigvee_{d\in R} G(b,c,d) \wedge H(a,d,z)) \ , \tag{2.6}$$

$$((a * b) \circ (a * c))(z) = \bigvee_{d\in R} (H(a,b,d) \wedge H(a,c,e) \wedge G(d,e,z)) \ . \tag{2.7}$$

Then (R,G,H) is called fuzzy ring if the following conditions hold.

R1: (R,G) is an abelian fuzzy group;
R2: $\forall a,b,c,z_1,z_2 \in R$, $((a * b) * c)(z_1) > \theta$ and $(a * (b * c))(z_2) > \theta$ implies $z_1 = z_2$;
R3: $\forall a,b,c,z_1,z_2 \in R$, $((a \circ b) * c)(z_1) > \theta$ and $((a * c) \circ (b * c))(z_2) > \theta$ implies $z_1 = z_2$; $(a * (b \circ c))(z_1) > \theta$ and $((a * b) \circ (a * c))(z_2) > \theta$ implies $z_1 = z_2$.

Definition 3. [3] *Let (R,G,H) be a fuzzy ring and (M,J) be an abelian fuzzy group and let p be fuzzy function $R \times M$ into M. Then we have a mapping*

$$p : F(R) \times F(M) \to F(M) \ ,$$

$$(A,N) \mapsto p(A,N) \ ,$$

$$p(A,N)(x) = \bigvee_{(r,n)\in A\times N} (A(r) \wedge N(n) \wedge p(r,n,x)) \ , \tag{2.8}$$

where $F(R) = \{A \mid A : R \to [0,1] \text{ is a mapping}\}$ and $F(M) = \{N \mid N : M \to [0,1] \text{ is a mapping}\}$.

Let $A = \{r\}$ and $N = \{m\}$, and let $p(A,N)$ and $J(a,b)$ are denoted as $r \odot m$ and $a \oplus b$,respectively. Then

$$(r \odot m)(x) = p(r,m,x), \quad \forall x \in M \ , \tag{2.9}$$

$$(r \odot (m_1 \oplus m_2))(x) = \bigvee_{m\in M} (J(m_1,m_2,m) \wedge p(r,m,x)) \ , \tag{2.10}$$

$$((r_1 \circ r_2) \odot m)(x) = \bigvee_{r \in R} (G(r_1, r_2, r) \wedge p(r, m, x)) \ , \tag{2.11}$$

$$((r_1 * r_2) \odot m)(x) = \bigvee_{r \in R} (H(r_1, r_2, r) \wedge p(r, m, x)) \ , \tag{2.12}$$

$$(r_1 \odot (r_2 \odot m)(x) = \bigvee_{m_1 \in M} (p(r_2, m, m_1) \wedge p(r_1, m_1, x)) \ , \tag{2.13}$$

$$((r \odot m_1) \oplus (r \odot m_2))(x) = \bigvee_{x_1, x_2 \in M} (p(r, m_1, x_1) \wedge p(r, m_2, x_2) \wedge J(x_1, x_2, x)) \ . \tag{2.14}$$

Let (R, G, H) be a fuzzy ring and (M, J) be an abelian fuzzy group. M is called a (left) fuzzy module over R or (left) R-fmodule together with a fuzzy function $p : R \times M \to M$ if the following conditions hold. For all $r, r_1, r_2 \in R$ and for all $m, m_1, m_2 \in M$,

M1: $(r \odot (m_1 \oplus m_2))(x) > \theta$ and $((r \odot m_1) \oplus (r \odot m_2))(y) > \theta$ implies $x = y$;

M2: $((r_1 \circ r_2) \odot m)(x) > \theta$ and $((r_1 \odot m) \oplus (r_2 \odot m))(y) > \theta$ implies $x = y$;

M3: $((r_1 * r_2) \odot m)(x) > \theta$ and $(r_1 \odot (r_2 \odot m))(y) > \theta$ implies $x = y$.

3. Direct Product of Fuzzy Modules over Fuzzy Rings

Let (R_1, G_1, H_1) and (R_2, G_2, H_2) be two fuzzy rings, (M, J_M) and (N, J_N) be two fuzzy modules over R_1 and R_2, respectively.

Definition 4. *Let J be a fuzzy binary operation on $M \times N$. Then we have a mapping*

$$J : F(M \times N) \times F(M \times N) \to F(M \times N),$$

$$(A_1 \times B_1, A_2 \times B_2) \mapsto J(A_1 \times B_1, A_2 \times B_2),$$

where $F(M \times N) = \{A \times B \mid A \times B : M \times N \to [0, 1]$ is a mapping, A and B are fuzzy subsets of M and N, respectively$\}$ and

$$J(A_1 \times B_1, A_2 \times B_2)(x, y) = \bigvee_{\substack{x_1, x_2 \in M, \\ y_1, y_2 \in N}} \begin{pmatrix} A_1(x_1) \wedge A_2(x_2) \wedge B_1(y_1) \wedge B_2(y_2) \\ \wedge J_M(x_1, x_2, x) \wedge J_N(y_1, y_2, y) \end{pmatrix}.$$

Let $A_1 \times B_1 = \{(x_1, y_1)\}$ and $A_2 \times B_2 = \{(x_2, y_2)\}$, and $J(A_1 \times B_1, A_2 \times B_2)$ be denoted as $(x_1, y_1) \boxplus (x_2, y_2)$, then

$$\begin{aligned}
((x_1, y_1) \boxplus (x_2, y_2))(x, y) &= J_M(x_1, x_2, x) \wedge J_N(y_1, y_2, y), \\
(((x_1, y_1) \boxplus (x_2, y_2)) \boxplus (x_3, y_3))(x, y) &= \bigvee_{u \in M,\ v \in N} \begin{pmatrix} J_M(x_1, x_2, u) \wedge J_N(y_1, y_2, v) \\ \wedge J_M(u, x_3, x) \wedge J_N(v, y_3, y) \end{pmatrix}, \\
((x_1, y_1) \boxplus ((x_2, y_2) \boxplus (x_3, y_3)))(x, y) &= \bigvee_{u \in M,\ v \in N} \begin{pmatrix} J_M(x_2, x_3, u) \wedge J_N(y_2, y_3, v) \\ \wedge J_M(x_1, u, x) \wedge J_N(y_1, v, y) \end{pmatrix}.
\end{aligned}$$

Similarly we can define fuzzy product as follows:

$$p : F(R_1 \times R_2) \times F(M \times N) \to F(M \times N),$$

$$(K \times L, A \times B) \mapsto p(K \times L, A \times B),$$

where $F(R_1 \times R_2) = \{K \times L \mid K \times L : R_1 \times R_2 \to [0,1]$ is a mapping, K and L are fuzzy subsets of R_1 and R_2, respectively$\}$ and $F(M \times N) = \{A \times B \mid A \times B : M \times N \to [0,1]$ is a mapping, A and B are fuzzy subsets of M and N, respectively$\}$ and

$$p(K \times L, A \times B)(x,y) = \bigvee_{\substack{m \in M, n \in N \\ r_1 \in R_1, r_2 \in R_2}} \left(A(m) \wedge B(n) \wedge K(r_1) \wedge L(r_2) \wedge p(r_1,m,x) \wedge p(r_2,n,y)\right).$$

Let $A \times B = \{(m,n)\}$ and $K \times L = \{(r_1,r_2)\}$, and $p(K \times L, A \times B)$ be denoted as $(r_1,r_2) \boxdot (m,n)$, then

$$\begin{aligned}
((r_1,r_2) \boxdot (m,n))(x,y) &= p(r_1,m,x) \wedge p(r_2,n,y), \\
((r_1,r_2) \boxdot ((m_1,n_1) \boxplus (m_2,n_2)))(x,y) &= \bigvee_{m \in M,\, n \in N} \begin{pmatrix} J_M(m_1,m_2,m) \wedge J_N(n_1,n_2,n) \\ \wedge p(r_1,m,x) \wedge p(r_2,n,y) \end{pmatrix}, \\
(((r_1,r_2) \boxplus (r_3,r_4)) \boxdot (m,n))(x,y) &= \bigvee_{r \in R_1, s \in R_2} \begin{pmatrix} G_1(r_1,r_3,r) \wedge G_2(r_2,r_4,s) \\ \wedge p(r,m,x) \wedge p(s,n,y) \end{pmatrix}, \\
(((r_1,r_2) \boxtimes (r_3,r_4)) \boxdot (m,n))(x,y) &= \bigvee_{r \in R_1, s \in R_2} \begin{pmatrix} H_1(r_1,r_3,r) \wedge H_2(r_2,r_4,s) \\ \wedge p(r,m,x) \wedge p(s,n,y) \end{pmatrix} \\
((r_1,r_2) \boxdot ((r_3,r_4) \boxdot (m,n)))(x,y) &= \bigvee_{x_1 \in M,\, y_1 \in N} \begin{pmatrix} p(r_3,m,x_1) \wedge p(r_4,n,y_1) \\ \wedge p(r_1,x_1,x) \wedge p(r_2,y_1,y) \end{pmatrix}. \\
(((r_1,r_2) \boxdot (m_1,n_1)) \boxplus ((r_1,r_2)) \boxdot (m_2,n_2)))(x,y) &= \bigvee_{\substack{x_1,x_2 \in M \\ y_1,y_2 \in N}} \begin{pmatrix} p(r_1,m_1,x_1) \wedge p(r_2,n_1,y_1) \\ \wedge p(r_1,m_2,x_2) \wedge p(r_2,n_2,y_2) \\ \wedge J_M(x_1,x_2,x) \wedge J_N(y_1,y_2,y) \end{pmatrix}
\end{aligned}$$

Theorem 1. *$(M \times N, J)$ is a fuzzy module over fuzzy ring $(R_1 \times R_2, S_1, S_2)$.*

Proof. $(M \times N, J)$ is an abelian fuzzy group from Theorem 3.2 in [4].

Let $(r_1,r_2) \in R_1 \times R_2$ and $(m_1,n_1),(m_2,n_2),(x_1,y_1),(x_2,y_2) \in M \times N$ such that

$$\begin{aligned}
((r_1,r_2) \boxdot ((m_1,n_1) \boxplus (m_2,n_2)))(x_1,y_1) &> \theta, \\
(((r_1,r_2) \boxdot (m_1,n_1)) \boxplus ((r_1,r_2) \boxdot (m_2,n_2)))(x_2,y_2) &> \theta.
\end{aligned}$$

Hence there exists $m, u_1, u_2 \in M$, $n, v_1, v_2 \in N$ such that

$$\begin{aligned}
J_M(m_1,m_2,m) \wedge J_N(n_1,n_2,n) \wedge p(r_1,m,x_1) \wedge p(r_2,n,y_1) &> \theta, \\
p(r_1,m_1,u_1) \wedge p(r_2,n_1,v_1) \wedge p(r_1,m_2,u_2) \wedge p(r_2,n_2,v_2) \wedge J_M(u_1,u_2,x_2) \wedge J_N(v_1,v_2,y_2) &> \theta.
\end{aligned}$$

Since M and N are fuzzy modules, we have $x_1 = x_2$, $y_1 = y_2$, and consequently $(x_1,y_1) = (x_2,y_2)$.

Let $(r_1,r_2),(r_3,r_4) \in R_1 \times R_2$, $(m,n),(x_1,y_1),(x_2,y_2) \in M \times N$ such that

$$\begin{aligned}
(((r_1,r_2) \boxplus (r_3,r_4)) \boxdot (m,n))(x_1,y_1) &> \theta, \\
(((r_1,r_2) \boxdot (m,n)) \boxplus ((r_3,r_4) \boxdot (m,n)))(x_2,y_2) &> \theta.
\end{aligned}$$

Then there exists $s_1 \in R_1$, $s_2 \in R_2$, $m, u_1, u_2 \in M$ and $n, v_1, v_2 \in N$ such that

$$\begin{aligned}
G_1(r_1,r_3,s_1) \wedge G_2(r_2,r_4,s_2) \wedge p(s_1,m,x_1) \wedge p(s_2,n,y_1) &> \theta, \\
p(r_1,m,u_1) \wedge p(r_2,n,v_1) \wedge p(r_3,m,u_2) \wedge p(r_4,n,v_2) \wedge J_M(u_1,u_2,x_2) \wedge J_N(v_1,v_2,y_2) &> \theta.
\end{aligned}$$

Since M and N are fuzzy modules, similarly we have $x_1 = x_2$, $y_1 = y_2$, and consequently $(x_1, y_1) = (x_2, y_2)$.

Let $(r_1, r_2), (r_3, r_4) \in R_1 \times R_2$ and $(m, n), (x_1, y_1), (x_2, y_2) \in M \times N$ such that

$$\begin{aligned} (((r_1, r_2) \boxtimes (r_3, r_4)) \boxdot (m, n))(x_1, y_1) &> \theta, \\ ((r_1, r_2) \boxdot ((r_3, r_4) \boxdot (m, n)))(x_2, y_2) &> \theta. \end{aligned}$$

Then there exists $s_1 \in R_1$, $s_2 \in R_2$, $u \in M$ and $v \in N$ such that

$$\begin{aligned} H_1(r_1, r_3, s_1) \wedge H_2(r_2, r_4, s_2) \wedge p(s_1, m, x_1) \wedge p(s_2, n, y_1) &> \theta, \\ p(r_3, m, u) \wedge p(r_4, n, v) \wedge p(r_1, u, x_2) \wedge p(r_2, v, y_2) &> \theta. \end{aligned}$$

Since M and N are fuzzy modules, similarly we have $x_1 = x_2$, $y_1 = y_2$, and consequently $(x_1, y_1) = (x_2, y_2)$. Consequently, according to Definition 3 ,$M \times N$ is obtained as a fuzzy module over fuzzy ring $R_1 \times R_2$. □

Theorem 2. *Let M_1 and M_2 be two fuzzy modules over fuzzy rings (R_1, G_1, H_1) and (R_2, G_2, H_2), respectively, and let N_1 and N_2 be two nonempty subsets of M_1 and M_2, respectively. Then $N_1 \times N_2$ is a fuzzy submodule of $M_1 \times M_2$ if and only if both N_1 and N_2 are fuzzy submodules of M_1 and M_2, respectively.*

Proof. Suppose that $N_1 \times N_2$ is a fuzzy submodule of $M_1 \times M_2$. Then $N_1 \times N_2$ is a fuzzy subgroup of $M_1 \times M_2$. By Theorem 3.3 in [4], N_1 and N_2 are fuzzy subgroups of M_1 and M_2. Let $r_1 \in R_1$, $r_2 \in R_2$, $n_1 \in N_1$, $n_2 \in N_2, x \in M_1$ and $y \in M_2$ such that $p(r_1, n_1, x) > \theta$ and $p(r_2, n_2, y) > \theta$. By

$$((r_1, r_2) \boxdot (n_1, n_2))(x, y) > p(r_1, n_1, x) \wedge p(r_2, n_2, y) > \theta$$

we have $(x, y) \in N_1 \times N_2$, and so $x \in N_1$ and $y \in N_2$.

Conversely, if N_1 and N_2 are fuzzy submodules of M_1 and M_2, respectively, then $N_1 \times N_2$ is a fuzzy submodule of $M_1 \times M_2$ from Theorem 1. □

Corollary 1. *Let $N_1, N_2, ..., N_n$ be nonempty subsets of fuzzy modules $M_1, M_2, ..., M_n$, respectively. $N_1 \times N_2 \times ... \times N_n$ is a fuzzy submodule of $M_1 \times M_2 \times ... \times M_n$ if and only if for all $i \in \{1, 2, ..., n\}$, N_i is a fuzzy submodule of M_i.*

Proof. One can easily verify this corollary by using mathematical induction method. □

Theorem 3. *Let $f : M_1 \times M_2 \to N_1 \times N_2$ be a fuzzy module homomorphism. Then*

1. If $A \times B$ is a fuzzy submodule of $M_1 \times M_2$ then $f(A \times B)$ is a fuzzy submodule of $N_1 \times N_2$,
2. If $C \times D$ is a fuzzy submodule of $N_1 \times N_2$ then $f^{-1}(C \times D)$ is a fuzzy submodule of $M_1 \times M_2$.

Proof. (1) Let $A \times B$ be a fuzzy submodule of $M_1 \times M_2$. Then $A \times B$ is an abelian fuzzy subgroup of $M_1 \times M_2$. From Theorem 3.7 in [4], $f(A \times B)$ is an abelian fuzzy subgroup of $N_1 \times N_2$.For any $(r_1, r_2) \in R_1 \times R_2$ and $(n_1, n_2) \in f(A \times B)$, there exists $m_1 \in M_1$, $m_2 \in M_2$ such that $f(m_1, m_2) = (n_1, n_2)$. Let $((r_1, r_2) \boxdot (m_1, m_2))(x, y) > \theta$. Since f is a fuzzy homomorphism, we have $((r_1, r_2) \boxdot (n_1, n_2))(f(x, y)) > \theta$ and $f(x, y) \in N_1 \times N_2$.

(2) One can prove in a similar manner. □

Corollary 2. *Let $f : M_1 \times M_2 \times ... \times M_n \to N_1 \times N_2 \times ... \times N_n$ be a fuzzy module homomorphism. Then*

1. If $A_1 \times A_2 \times ... \times A_n$ is a fuzzy submodule of $M_1 \times M_2 \times ... \times M_n$ then $f(A_1 \times A_2 \times ... \times A_n)$ is fuzzy submodule of $N_1 \times N_2 \times ... \times N_n$,

2. If $B_1 \times B_2 \times ... \times B_n$ is a fuzzy submodule of $N_1 \times N_2 \times ... \times N_n$ then $f^{-1}(B_1 \times B_2 \times ... \times B_n)$ is fuzzy submodule of $M_1 \times M_2 \times ... \times M_n$.

Proof. It can be proved easily by mathematical induction method. □

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