On Boolean Like Ring Extension of a Group

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Abstract

In this paper we introduce the notion of Boolean like ring extension (for short, BLR extension) of a group and obtain various properties. We show that any BLR extension of a group is a group. Further, scalar multiplication on any BLR extension of a group is defined and studied certain properties.

Keywords: Boolean like rings, Boolean like ring extension

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Introduction

The concept Boolean Like Rings is due to A.L Foster ([1]). Further studies has been made by Swaminathan V. ([5]). Moreover, Foster also introduced the concept of Boolean extension of a universal algebra ([2]). This paper is a study of BLR-extension of a group. This paper is divided in to 4 sections. In section 1, we give some preliminary results on Boolean Like rings and Boolean extension. In section 2, we introduce the notion of BLR extension of a group (Definition 2.1) by generalizing the notion of Boolean extension of a group of Foster A. L [2]. We also proved that the BLR-extension of a group is a group (See Theorem 2.4). Further, we substantiate that the notion of BLR-extension is a proper generalization of Boolean extension of Foster (See example 2.6). In section 3, given any group we identify the group in the BLR-extension. In section 4, we give scalar multiplication on any BLR

extension of a group and study certain properties. We also introduce norm of an element of a group.

1 Preliminaries: Boolean Like rings and its properties

In this section we recollect some definitions and examples concerning the notion of Boolean like rings of A.L.Foster ([1],[2]) , and Swaminathan. V. ([3])

Definition 1.1. *A Boolean like ring R is a commutative ring with unity in which for all elements $a, b, \in R$, $a + a = 0$, $ab(1 - a)(1 - b) = 0$.*

Example 1.2. *Every Boolean ring is a Boolean like ring but not conversely. For example, the ring $(H_4, +, \cdot)$, where $H_4 = \{0, 1, p, q\}$ and the binary operations '+' and ' $\cdot$ ' defined by*

+	0	1	p	q
0	0	1	p	q
1	1	0	q	p
p	p	q	0	1
q	q	p	1	0

$\cdot$	0	1	p	q
0	0	0	0	0
1	0	1	p	q
p	0	p	0	p
q	0	q	p	1

is a Boolean like ring but not a Boolean ring (since $p^2 = 0(\neq p)$).

Let R be a Boolean Like ring, then

Lemma 1.3. *For all $a \in R$, $a^4 = a^2$ (weak idempotent law)*

Corollary 1.4. *For all $a \in R$ and for all non negative integers n , $a^{n+4} = a^{n+2}$.*

Corollary 1.5. *From corollary 1.4, we have, $a^n = a$ or a^2 or a^3 for any non negative integer n .*

Lemma 1.6. *In a Boolean like ring R , $(\sum_{i=1}^n a_i)^2 = \sum_{i=1}^n a_i^2$ for all $a_i \in R$.*

Theorem 1.7. *Let J be a Boolean algebra and U be the Boolean extension of an algebra A . If $U = U_{[J]}$ is finite (that is if both the kernel A and the core J are finite), the order of U is n^k where n is the order of A and k is the number of atoms in J .*

In the rest of the section $R = (R, +, \cdot)$ is a Boolean Like ring and $G = (G, +)$ is a group.

2 Boolean Like Ring extension of a Group

Let $R = (R, +, \cdot)$ be a Boolean like ring and $G = (G, +)$ be a group.

Definition 2.1. By a BLR extension of a group, we mean the set V of all mappings $f : G \rightarrow R$ with the following properties:

- P1: $f(x) = 0$ except for a finite number of elements $x \in G$;
- P2: $f(x)f(y) = 0$ for $x, y \in G$ with $x \neq y$;
- P3: $\sum_{x \in G} (f(x))^2 = 1$

In this article V denotes the BLR extension of a group G and an immediate consequence of P2 (of the definition 2.1) of V is the following:

Lemma 2.2. If $f \in V$ and $f(x)$ is an invertible element of R for some $x \in G$, then $f(y) = 0$ for all $y(\neq x) \in R$.

Proof. Let $f \in V$ and $f(x)$ is an invertible element of R for some $x \in G$. Then, $\exists r \in R$ such that $rf(x) = 1$. But then, $f(y) = 1f(y) = rf(x)f(y) = 0$ (by P2) for all $y(\neq x) \in R$. $\square$

For $f, g \in V$, we define, $f + g$ and $-f$, respectively, as the mapping: $G \rightarrow R$ by:

- (i): $(f + g)(x) = \sum_{\alpha+\beta=x} f(\alpha)g(\beta)$
- (ii): $(-f)(x) = f(-x)$ for all $x \in G$

Lemma 2.3. If $f, g \in V$, then

1. $f + g \in V$
2. $-f \in V$

Proof. 1. Clearly $f + g$ is well-defined. Now we have to show that $f + g$ satisfies the properties P1, P2, P3. Since $f(x)$ and $g(x)$ vanishes for almost all $x \in G$ and $(f+g)(x)$ is defined by the means of equation (i), we have that $(f+g)(x)$ vanishes for almost all $x \in G$. So that $f + g$ satisfies P1. Let $x, y \in G$ and $x \neq y$, then $(f+g)(x)(f+g)(y) = (\sum_{\alpha+\beta=x} f(\alpha)g(\beta))(\sum_{\gamma+\delta=y} f(\gamma)g(\delta)) = \sum_{\alpha+\beta=x} \sum_{\gamma+\delta=y} f(\alpha)g(\beta)f(\gamma)g(\delta)$. Since $x \neq y$, at least one of the conditions $\alpha \neq \gamma$ or $\beta \neq \delta$ must hold. In any case it follows that, $f(\alpha)g(\beta)f(\gamma)g(\delta) = 0$. Thus, all the terms in the right hand side of the above equation are zeros. Hence, $(f + g)(x)(f + g)(y) = 0$ and $f + g$ satisfies P2. Now

consider, $\sum_{x \in G} ((f+g)(x))^2 = \sum_{x \in G} \left(\sum_{\alpha+\beta=x} f(\alpha)g(\beta) \right)^2 = \sum_{x \in G} \sum_{\alpha+\beta=x} (f(\alpha))^2 (g(\beta))^2$ (Since x runs through all the elements of G , α and β can be made run through all the elements of G . Now, fix α and let β run through all the elements of G), we have, $\sum_{\alpha \in G} (f(\alpha))^2 \left[\sum_{\beta \in G} (g(\beta))^2 \right] = \sum_{\alpha \in G} (f(\alpha))^2 = 1$ (by P3 and Lemma 1.6), showing that $f+g$ satisfies P3. Thus, $f+g \in V$.

2. It is easy to see that $-f$ is in V . $\square$

Theorem 2.4. *Any BLR extension V of a group is a group.*

Proof. It is easy to show that $'+'$ defined by (i) is associative. Moreover, The mapping $\tilde{0} : G \rightarrow R$ by $\tilde{0}(0) = 1$ and $\tilde{0}(x) = 0$ for all $x(\neq 0) \in G$, then $\tilde{0} \in V$ and is the zero for V . For any $f \in V$, $-f$ defined by (ii) is its inverse. $\square$

Remark 2.5. *The notion of BLR extension of a group G is a proper generalization to the notion of Boolean-extension of a group G (of Foster ([2])) and we show below by means of an example that this concept is indeed a proper generalization.*

Example 2.6. *Let $R = Z_2 \times H_4$, then R is a Boolean like ring. In fact, the elements of R are: $a = (0,0)$, $b = (0,1)$, $c = (0,p)$, $d = (0,q)$, $e = (1,0)$, $f = (1,1)$, $g = (1,p)$ and $h = (1,q)$. Let G be the group $\{0,1,2\}$ with addition modulo 3 and V be the BLR extension of G . Then any $f \in V$ can be specified by an ordered triple $\{\alpha, \beta, \gamma\}$ where $\alpha, \beta, \gamma \in R$ and $f(0) = \alpha$, $f(1) = \beta$ and $f(2) = \gamma$. Since α, β, γ have to satisfy the properties P1, P2, P3 of definition 2, we can verify by direct computation that the elements of V are the only following 18 ordered triplets: (a, a, f) , (a, f, a) , (f, a, a) , (a, b, e) , (a, e, b) , (b, a, e) , (b, e, a) , (e, b, a) , (e, a, b) , (a, h, a) , (a, a, h) , (h, a, a) , (a, d, e) , (a, e, d) , (d, e, a) , (d, a, e) , (e, a, d) and (e, d, a) .*

By theorem 1.7, if V is to be isomorphic with a non trivial Boolean-extension, then the equation $n^t = 18$ must be solvable in the set of positive integers n and $t(> 1)$. Thus, V cannot be isomorphic with any possible Boolean-extension of any group A .

3 Identification of G in the BLR extension of a group G

We begin with the following:

Lemma 3.1. *For any $g \in G$, Define the mapping $\tilde{g} : G \rightarrow R$ by putting $\tilde{g}(g) = 1$ and $\tilde{g}(x) = 0$ for all $x(\neq g) \in G$. Then $\tilde{g} \in V$.*

Now define $\tilde{G} = \{\tilde{g} : g \in G\}$.

Theorem 3.2. $\tilde{G} = \{\tilde{g} : g \in G\}$ is a subgroup of V .

Proof. Let $g, h \in G$. Then, $(\tilde{g} - \tilde{h})(g - h) = \sum_{\alpha + \beta = g - h} \tilde{g}(\alpha)(-\tilde{h})(\beta) = \sum_{\alpha + \beta = g - h} \tilde{g}(\alpha)(\tilde{h})(-\beta)$
 (by (i) and (ii)) $= \tilde{g}(g)\tilde{h}(h)$ (by lemma 3.1) $= 1$. Thus, $(\tilde{g} - \tilde{h})(x) = 0$ for all $x(\neq g - h) \in G$ (by lemma 2.2 and 3.1). Thus, $\tilde{g} - \tilde{h} = \widetilde{(g - h)}$ and $\tilde{g} - \tilde{h} \in \tilde{G}$. Hence, $\tilde{G}$ is a subgroup of V . $\square$

Theorem 3.3. The mapping $\phi : G \rightarrow \tilde{G}(g \rightarrow \tilde{g})$ is an isomorphism from G to $\tilde{G}$.

Proof. It is clear that the mapping $\phi : G \rightarrow \tilde{G}$ is well- defined and surjective from $G \rightarrow \tilde{G}$. Now, we show that the mapping is injective. Let $g, h \in G$, so that $\phi(g) = \phi(h)$. Then, $\tilde{g} = \tilde{h}$ and hence $\tilde{g}(h) = \tilde{h}(h) = 1$ (by definition of $\tilde{h}$), so that $g = h$; for other wise $\tilde{g}(h) = 0$. Thus, $\phi : G \rightarrow \tilde{G}$ is a one to one mapping. Further, we observe that $\phi(g + h) = \widetilde{(g + h)} = \tilde{g} + \tilde{h}$ (by theorem 3.2) $= \phi(g) + \phi(h)$. Hence, ϕ is an isomorphism. $\square$

Corollary 3.4. If V is abelian, then G is abelian.

4 Scalar multiplication in any BLR extension of a group and its properties.

In this section, we introduce a (scalar) multiplication of the elements of V by the elements of the R and study its properties. Further, we define norm of an element of V and studied its property. Let $a \in R$ and $f \in V$. Then, we define af as the mapping from G to R as follows:

(iii) $(af)(x) = a^2 f(x)$ if $x(\neq 0) \in G$ and $(af)(0) = 1 - a^2 + a^2 f(0)$

Lemma 4.1. For $a \in R$ and $f \in V, af \in V$

Proof. Since $f(x)$ vanishes for almost all x in G , then af also vanishes for almost all x in G . Thus, af satisfies P1. Let $x \neq y$. Then, if both $x, y \neq 0$, then $(af)(x)(af)(y) = (a^2 f(x))(a^2 f(y)) = a^2 f(x)f(y) = 0$ (by property P2 and Lemma 1.3). If one of them is 0, (say $x = 0$ and $y \neq 0$), then $(af)(0)(af)(y) = (1 - a^2 + a^2 f(0))(a^2 f(y)) = a^2 f(y) - a^2 f(y) - a^2 f(0)f(y) = 0$ (by lemma 1.3 and P2). Thus, af satisfies the property P2. Now consider, $\sum_{x \in G} ((af)(x))^2 = ((af)(0))^2 + \sum_{x \in G, x \neq 0} ((af)(x))^2 = (1 - a^2 + a^2 f(0))^2 + \sum_{x \in G, x \neq 0} a^2 (f(x))^2 = 1 - a^2 + a^2 (f(0))^2 + \sum_{x \in G, x \neq 0} a^2 (f(x))^2 =$

$1 - a^2 + a^2(\sum_{x \in G} (f(x))^2) = 1 - a^2 + a^2$ (by P3) = 1. Thus, af satisfies the property P3 and hence $af \in V$. $\square$

The mapping defined above is called scalar multiplication and have the following properties.

Theorem 4.2. For $a, b \in R$ and $f \in V$

1. $a(bf) = (ab)f$
2. $(a + b)f = af + bf$ if $ab = 0$
3. $a(f + g) = af + ag$

Proof. 1. If $x \neq 0$, we have $(a(bf))(x) = a^2(bf)(x) = a^2(b^2f(x)) = (a^2b^2)f(x) = (ab)^2f(x)$. If $x = 0$, we have $(a(bf))(0) = 1 - a^2 + a^2(bf(0)) = 1 - a^2 + a^2(1 - b^2 + b^2f(0)) = 1 - a^2b^2 + a^2b^2f(0) = ((ab)f)(0)$. Therefore, $a(bf) = (ab)f$.

2. Let $a, b \in R$ with $ab = 0$. If $x \neq 0$, then $((a + b)f)(x) = (a + b)^2f(x) = (a^2 + 2ab + b^2)f(x) = (a^2 + b^2)f(x) = a^2f(x) + b^2f(x)$. On the other hand, $(af + bf)(x) = \sum_{\alpha+\beta=x} (af)(\alpha)(bf)(\beta) = (af)(0)(bf)(x) + (af)(x)(bf)(0) + \sum_{\alpha+\beta=x, \alpha \neq \beta \neq 0} (af)(\alpha)(bf)(\beta) = (1 - a^2 + a^2f(0))(b^2f(x)) + (a^2f(x))(1 - b^2 + b^2f(0)) + \sum_{\alpha+\beta=x, \alpha \neq \beta \neq 0} a^2f(\alpha)(\alpha)b^2f(\beta) = b^2f(x) - a^2b^2f(x) + a^2b^2f(0)f(x) + a^2f(x) - a^2b^2f(x) + a^2b^2f(x)f(0) = a^2f(x) + b^2f(x)$ (By property (P2), we have $f(\alpha)f(\beta) = 0$ for every $\alpha \neq \beta$). Also, If $x = 0$, we have $((a + b)f)(0) = 1 - (a + b)^2 + (a + b)^2f(0) = 1 - a^2 - b^2 + a^2f(0) + b^2f(0)$. On the other hand, $(af + bf)(0) = \sum_{\alpha+\beta=0} (af)(\alpha)(bf)(\beta) = (af)(0)(bf)(0) + \sum_{\alpha+\beta=0, \alpha \neq \beta \neq 0} (af)(\alpha)(bf)(\beta) = (1 - a^2 + a^2f(0))(1 - b^2 + b^2f(0)) + \sum_{\alpha+\beta=0, \alpha \neq \beta \neq 0} a^2f(\alpha)b^2f(\beta) = (1 - a^2 + a^2f(0))(1 - b^2 + b^2f(0))$ (Since by (P2) $f(\alpha)f(\beta) = 0$ for $\alpha \neq \beta$) $= 1 - a^2 - b^2 + a^2f(0) + b^2f(0)$ (since $ab = 0$).

Hence, we have $(a + b)f = af + bf$ if $ab = 0$

3. Let $a \in R$ and $f \in V$. If $x \neq 0$, $(a(f + g))(x) = a^2(f + g)(x) = a^2 \sum_{\alpha+\beta=x} f(\alpha)g(\beta) = a^2[f(0)g(x) + f(x)g(0) + \sum_{\alpha+\beta=x, \alpha \neq \beta \neq 0} f(\alpha)g(\beta)]$. On the other hand, $(af + ag)(x) = \sum_{\alpha+\beta=x} (af)(\alpha)(bf)(\beta) = (af)(0)(bf)(x) + (af)(x)(bf)(0) + \sum_{\alpha+\beta=x, \alpha \neq \beta \neq 0} f(\alpha)g(\beta)$

$$\begin{aligned}
&= (1 - a^2 + a^2 f(0))a^2 g(x) + a^2 f(x)(1 - a^2 + a^2 g(0)) + \sum_{\alpha+\beta=x, \alpha \neq \beta \neq 0} a^4 f(\alpha)g(\beta) = \\
&a^2 g(x) - a^4 g(x) + a^4 f(0)g(x) + a^2 f(x) - a^4 f(x) + a^4 f(x)g(0) + \sum_{\alpha+\beta=x, \alpha \neq \beta \neq 0} a^4 f(\alpha)g(\beta) \\
&= a^2 f(0)g(x) + a^2 f(x)g(0) + \sum_{\alpha+\beta=x, \alpha \neq \beta \neq 0} a^2 f(\alpha)g(\beta) \text{ (since } a^2 = a^4) = a^2[f(0)g(x) + \\
&f(x)g(0)] + \sum_{\alpha+\beta=x, \alpha \neq \beta \neq 0} a^2 f(\alpha)g(\beta). \text{ If } x = 0, (a(f+g))(0) = 1 - a^2 + a^2(f+ \\
&g)(0) = 1 - a^2 + a^2[\sum_{\alpha+\beta=0, \alpha \neq \beta \neq 0} f(\alpha)g(\beta)]. \text{ On the other hand, } (af+ag)(0) = \\
&\sum_{\alpha+\beta=0} (af)(\alpha)(bf)(\beta) = (af)(0)(ag)(0) + \sum_{\alpha+\beta=0, \alpha \neq \beta \neq 0} (af)(\alpha)(bf)(\beta) = (1 - a^2 + a^2 f(0))(1 - \\
&a^2 + a^2 g(0)) + \sum_{\alpha+\beta=0, \alpha \neq \beta \neq 0} (af)(\alpha)(bf)(\beta) = 1 - a^2 + a^2 g(0) - a^2 + a^4 - a^4 g(0) + a^2 f(0) - \\
&a^4 f(0) + a^4 f(0)g(0) + \sum_{\alpha+\beta=0, \alpha \neq \beta \neq 0} (af)(\alpha)(bf)(\beta) = 1 - a^2 + a^2[\sum_{\alpha+\beta=0, \alpha \neq \beta \neq 0} (af)(\alpha)(bf)(\beta)]. \\
\end{aligned}$$

Thus, we have, $a(f+g) = af+ag$. $\square$

Corollary 4.3. 1. $a\tilde{0} = \tilde{0}$

2. $0f = \tilde{0}$ for all $f \in V$

3. $1f = f$ for all $f \in V$

Proof. 1. Let $a \in R$. If $x \neq 0$, then $(a\tilde{0})(x) = a^2\tilde{0}(0) = a^2 \cdot 0 = 0$ and If $x = 0$, then $(a\tilde{0})(0) = 1 - a^2 + a^2\tilde{0}(0) = 1 - a^2 + a^2 = 1$. Thus, $a\tilde{0} = \tilde{0}$.

2. Let $f \in V$. If $x \neq 0$, then $(0f)(x) = 0^2 f(x) = 0 \cdot f(x) = 0$. If $x = 0$, then $(0f)(0) = 1 - 0^2 + 0^2 f(0) = 1$. Thus, $0f = \tilde{0}$.

3. Let $f \in V$. If $x \neq 0$, then $(1f)(x) = 1^2 \cdot f(x) = 1 \cdot f(x) = f(x)$. If $x = 0$, then $(1f)(0) = 1 - 1^2 + 1^2 f(0) = 1 - 1 + f(0) = f(0)$. Thus, $1f = f$. $\square$

Theorem 4.4. Let V be the BLR extension of the a group G . Then the mapping $\|\cdot\| : V \rightarrow R (f \rightarrow \|f\|)$ defined by $\|f\| = 1 - (f(0))$ for any $f \in V$ have the following properties.

1. $\|f\| = 0$ if and only if $f = \tilde{0}$.

2. For any $a \in R$, $\|af\| = a^2 \|f\|$

Proof. 1. Suppose $\|f\| = 0$. Then, $1 - (f(0)) = 0$. Hence, $f(0)$ is a unit in R . So, $f(x) = 0$ if $x (\neq 0) \in G$ (by lemma 2.2). Consequently, $f = \tilde{0}$. Conversely, let $f = \tilde{0}$. Then, $\|f\| = 1 - (\tilde{0}(0)) = 1 - 1 = 0$ (By (i)).

2. Let $a \in R$. Then, $\|af\| = 1 - (af)(0) = 1 - (1 - a^2 + a^2 f(0)) = a^2 - a^2 f(0) = a^2(1 - f(0)) = a^2 \|f\|$. $\square$

Let $\tilde{G}^*$ denote the set of all non- zero elements of the subgroup $\tilde{G}$. Then we have the following.

Theorem 4.5. *If $\tilde{g}_1, \tilde{g}_2, \dots, \tilde{g}_n \in \tilde{G}^*$, $a_1, a_2, \dots, a_n \in R$ with $a_1^2 a_2^2 \dots a_n^2 \neq 0$ and $a_1 \tilde{g}_1 + a_2 \tilde{g}_2 + \dots + a_n \tilde{g}_n = \tilde{0}$ then $\tilde{g}_1 + \tilde{g}_2 + \dots + \tilde{g}_n = \tilde{0}$.*

Proof. If possible let $\tilde{g}_1 + \tilde{g}_2 + \dots + \tilde{g}_n \neq \tilde{0}$. Then, since $g_1 + g_2 + \dots + g_n = 0$ implies $\tilde{g}_1 + \tilde{g}_2 + \dots + \tilde{g}_n = \tilde{0}$ (by lemma 3.1) and hence $g_1 + g_2 + \dots + g_n \neq 0$, so that $g_1 + g_2 + \dots + g_n = g$ for some $g (\neq 0) \in G$. Since $a_1 \tilde{g}_1 + a_2 \tilde{g}_2 + \dots + a_n \tilde{g}_n = \tilde{0}$ and $g (\neq 0) \in G$, it follows that, $0 = \tilde{0}(g) = (a_1 \tilde{g}_1 + a_2 \tilde{g}_2 + \dots + a_n \tilde{g}_n)(g) = \sum_{\alpha_1 + \alpha_2 + \dots + \alpha_n = 0} (a_1 \tilde{g}_1)(\alpha_1)(a_2 \tilde{g}_2)(\alpha_2) \dots (a_n \tilde{g}_n)(\alpha_n)$. Now, if $(\alpha_1, \alpha_2, \dots, \alpha_n)$ be any other ordered n-tuple other than $(g_1, g_2, \dots, g_n)$ such that $\alpha_1 + \alpha_2 + \dots + \alpha_n = g = g_1 + g_2 + \dots + g_n$, then, $\alpha_i \neq g_i$ for some i with $\alpha_i \neq 0$, so that $(a_i \tilde{g}_i)(\alpha_i) = a_i^2 g_i(\alpha_i) = 0$. This implies that $(a_1 \tilde{g}_1)(\alpha_1)(a_2 \tilde{g}_2)(\alpha_2) \dots (a_n \tilde{g}_n)(\alpha_n) = 0$. Since, $\tilde{g}_1, \tilde{g}_2, \dots, \tilde{g}_n \in \tilde{G}^*$ and $\tilde{G}^*$ consists of the non-zero elements of $\tilde{G}$, it follows that $g_1, g_2, \dots, g_n$ are all different from 0 in G . And hence it follows that, $0 = \tilde{0}(0) = (a_1 \tilde{g}_1)(g_1)(a_2 \tilde{g}_2)(g_2) \dots (a_n \tilde{g}_n)(g_n) = a_1^2 a_2^2 \dots a_n^2$, which is a contradiction to the hypothesis. Consequently, $\tilde{g}_1 + \tilde{g}_2 + \dots + \tilde{g}_n = \tilde{0}$. $\square$

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