The Number of Fuzzy Subgroups of Rectangle Groups

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Abstract

In this paper, we construct the formula to determine the number of fuzzy subgroups of finite groups, which are their lattices have a pattern. We discuss for the special pattern, that is "rectangle". First, we determine for rectangle $2 \times n$, then we get for $3 \times 4$. We also explain the applications of that formula for some groups. We can see that the result of Bashir Humera and Zahid Raza in [1] about the number of fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$ is special case of this formula.

Keywords: Fuzzy Subgroups, Lattice, Lattice Method

1 Introduction

One of the most important problem of fuzzy group theory is to classify the fuzzy subgroup of a finite group. Many papers have treated the particular case of finite cyclic group. Laszlo [2] studied about the construction of fuzzy subgroup of group of order one to six. Zhang and Zou [3] have determined the number of fuzzy subgroup of cyclic groups of order $p^n$ where $p$ is a prime number. Murali and Makamba ([4], [5]) considered a similar problem. They have determined the number of fuzzy subgroups of $\mathbb{Z}_{p^n}, \mathbb{Z}_p \times \mathbb{Z}_q, \mathbb{Z}_{p^n} \times \mathbb{Z}_q$ and abelian groups of order $p^n q^m$ where $p$ and $q$ are different primes. Tarnauceanu and Bentea [6] have determined this number for finite abelian groups. Sulaiman and Abd Ghafur [7] have determined the number of fuzzy subgroups of finite cyclic groups. Sulaiman [8] has constructed fuzzy subgroups of Symmetric group $S_4$. 
The interesting result of Sulaiman [9] is a method to determine the number of fuzzy subgroup of finite groups. This method called "Lattice Method". By using this method Sulaiman [9] has counted the number of fuzzy subgroups of $p - group$ (Theorem 18), the number of fuzzy subgroups of $Z_p \times Z_q$ (Theorem 21) and the number of fuzzy subgroups of $Z_{p^n} \times Z_q$ (Theorem 23).

In this article, we use that method to construct some formulas to determine the number of fuzzy subgroups of finite groups which their lattices have a pattern. That pattern is "rectangle". We have determined a formulas for rectangle "$2 \times n$".

2 Preliminary

We recall some definitions and results that will be used later. Proving of some theorems in this section can be written in [9].

**Definition 2.1** Let $X$ be a nonempty set. A fuzzy subset of $X$ is a function from $X$ into $[0, 1]$.

**Definition 2.2** (Rosenfeld, see [10]) Let $G$ be a group. A fuzzy subset $\mu$ of $G$ is called a fuzzy subgroup of $G$ if

1. $\mu(xy) \geq \min\{\mu(x), \mu(y)\}, \forall x, y \in G$
2. $\mu(x^{-1}) \geq \mu(x), \forall x \in G$.

**Theorem 2.3** (Rosenfeld, see [10]) Let $e$ denote the identity element of group $G$. If $\mu$ is a fuzzy subgroup of $G$, then

1. $\mu(e) \geq \mu(x), \forall x \in G$
2. $\mu(x^{-1}) = \mu(x), \forall x \in G$.

**Theorem 2.4** (Sulaiman and Abdul Ghafur, see [11]) A fuzzy subset $\mu$ of $G$ is a fuzzy subgroup if and only if there is a chain of subgroups $G$, $P_1(\mu) \leq P_2(\mu) \leq \ldots \leq P_m(\mu) = G$ such that $\mu$ is in the form

$$\mu(x) = \begin{cases} \theta_1, x \in P_1(\mu) \\ \theta_2, x \in P_2(\mu) \\ \vdots \\ \theta_m, x \in P_m(\mu) \end{cases}$$

If there is no confusion, then $P_l(\mu)$ as in (1) can simply be written as $P_l$. We
define length (or order) of fuzzy subgroup $\mu$ in (1) as be $m$.

**Theorem 2.5 (Sulaiman, see [9])** Let $G_1 < G_2 < \cdots < G_{k-1} < G_k = G$ be the only maximal chain from $G_1$ to $G$ on the lattice subgroups of $G$. Then

\[
 n(F_{p_1=G_1}) = \begin{cases} 1, & k = 1, 2 \\ 2^{k-2}, & k > 2 \end{cases}.
\]

**Theorem 2.6 (Sulaiman, see [9])** Let $G_1 < G_2 < \cdots < G_{k-1} < G_k = G$ be the only maximal chain from $G_1$ to $G$ on the lattice subgroups of $G$. For $m, 1 \leq m \leq m - 2$ we have

\[
 n(F_{p_1=G_m}) = 2. n(F_{p_1=G_{m+1}}).
\]

**Theorem 2.7 (Sulaiman, see Theorem 11 in [9])** Let $H$ be a subgroup of $G$, and let the set of all subgroups of $G$ which contain $H$ (but are not equal to $H$) be $\{H_1, H_2, \ldots, H_k\}$. Then

\[
 n(F_{p_1=H}) = \sum_{i=1}^{k} n(F_{p_1=H_i}).
\]

**Corollary 2.8 (Sulaiman, see Corollary 14 in [8])** Let $G$ be a group. Then

\[
 n(F_G) = 2. n(F_{p_1=\{e\}}).
\]

### 3 Main Result

**Definition 3.1** Let $G$ be a group and the number of it's subgroups is finite. The diagram of lattice subgroups $G$ is called "rectangle" if satisfies these conditions: The subgroups of $G$ can be labeled by $K^i_j$ where $1 \leq i \leq m, 1 \leq j \leq s$ for some $m, s \in Z$ with $K^1_1 = G$, $K^m_s = \{e\}$ such that: (i) for fixed $i, 1 \leq i \leq m, K^i_k < K^i_{k-1}, \forall k, 2 \leq k \leq s$, (ii) for fixed $j, 1 \leq j \leq m, K^i_j < K^{i-1}_j, \forall t, 2 \leq t \leq m$. In this case the size of the diagram is $m \times s$. The diagram of "rectangle lattice" is shown in Figure 1.
Theorem 3.2 Let $G$ be a group that satisfied Definition 3.1 with $m = 2, s \in N$. We have:

i) $n(F_{P_1 = K^2_s = \{e\}}) = 2 \cdot n(F_{P_1 = K^2_{s-1}}) + n(F_{P_1 = K^1_s})$

ii) $n(F_{P_1 = K^2_s = \{e\}}) = 2^s + (s - 3)2^{s-2} = 2^{s-2}(s + 1)$.

Proof. Consider the diagram of $G$ (see Figure 2).
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i) According to the Theorem 2.7, we have

\[ n(F_{P_1 = K_t^2}) = \sum_{i=1}^{s-1} n(F_{P_1 = K_i^2}) + \sum_{j=1}^{s} n(F_{P_1 = K_j^3}) \]

\[ = \sum_{i=1}^{s-1} \left[ n(F_{P_1 = K_i^2}) + n(F_{P_1 = K_i^3}) \right] + n(F_{P_1 = K_1^3}) \]  

(2)

\[ n(F_{P_1 = K_{s-1}^2}) = \sum_{i=1}^{s-2} n(F_{P_1 = K_i^2}) + \sum_{j=1}^{s-1} n(F_{P_1 = K_j^3}) \]

\[ = \sum_{i=1}^{s-2} \left[ n(F_{P_1 = K_i^2}) + n(F_{P_1 = K_i^3}) \right] + n(F_{P_1 = K_{s-1}^3}) \]  

(3)

Equality (2) can be write as

\[ n(F_{P_1 = K_t^2}) = \sum_{i=1}^{s-1} \left[ n(F_{P_1 = K_i^2}) + n(F_{P_1 = K_i^3}) \right] + n(F_{P_1 = K_1^3}) \]

\[ = \sum_{i=1}^{s-2} \left[ n(F_{P_1 = K_i^2}) + n(F_{P_1 = K_i^3}) \right] + n(F_{P_1 = K_{s-1}^3}) + n(F_{P_1 = K_s^3}) \]  

(4)

Substitute (3) to (4) to get

\[ n(F_{P_1 = K_t^2}) = n(F_{P_1 = K_{s-1}^2}) + n(F_{P_1 = K_{s-1}^3}) + n(F_{P_1 = K_s^3}) \]

\[ = 2 \cdot n(F_{P_1 = K_{s-1}^2}) + n(F_{P_1 = K_s^3}) \]

ii) We proof by induction on \( s \). Using Theorem 2.6 and Theorem 2.7 we have \( n(F_{P_1 = K_t^2}) = 1 \). Therefore, the statement is true for \( s = 1 \). Assume that the statement is true for \( s = t \). It means \( n(F_{P_1 = K_{t+1}^2}) = 2^t \times (t - 3) \times 2^{s-2} \). For \( s = t + 1 \), the diagram of lattice subgroup of \( G \) is shown in Figure 3.

\[ \text{Figure 3: Lattice for } s = t + 1 \]
Using Theorem 3.2 i) and equality (4) we have
\[
n \left( F_{p_1=K_{t+1}^1 = (e)} \right) = 2 \cdot n \left( F_{p_1=K_{t}^2} \right) + n \left( F_{p_1=K_{t+1}^1} \right)
\]
\[
= 2 \left[ 2^t + (t - 3)2^{t-2} \right] + n \left( F_{p_1=K_{t+1}^1} \right).
\]
By using Theorem 2.4 we have
\[
n \left( F_{p_1=K_{t+1}^2 = (e)} \right) = 2^{t+1} + (t - 3)2^{t-1} + 2^{t-1} = 2^{t+1} + (t - 1)2^{t-1}.
\]
Analogue with this proving, we have the next theorem.

**Theorem 3.3** Let $G$ be a group that satisfied Definition 3.1 with $m \in N, s = 2$. We have:
\begin{itemize}
  \item[i)] $n(F_{p_1=K_{t+1}^2}) = 2 \cdot n(F_{p_1=K_{t}^2}) + n(F_{p_1=K_{t+1}^1})$
  \item[ii)] $n(F_{p_1=K_{t+1}^2}) = 2^m + (m - 3)2^{m-2} = 2^{m-2}(m + 1)$
\end{itemize}

**Theorem 3.4** Consider group $G = \mathbb{Z}_{p^n} \times \mathbb{Z}_q$. The diagram lattice subgroup of $G$ is shown as Figure 4. The diagram is satisfy Definition 3.1 with $m = n + 1$ and $s = 2$. By applying Theorem 3.2 and Corollary 2.8 we have the number of fuzzy subgroups of $G = \mathbb{Z}_{p^n} \times \mathbb{Z}_q$ is $2^n(n + 2)$.

![Figure: Lattice Subgroup of $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$]
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References


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