\beta_1 Near-Rings

G. Sugantha

Department of Mathematics
Pope’s College, Sawyerpuram-628 251
Tamil Nadu, India

R. Balakrishnan

PG & Research Department of Mathematics
V.O. Chidambaram College, Tuticorin-628 008
Tamil Nadu, India

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Abstract

In this paper we introduce the notion of \beta_1 near-rings and study some of their properties. We furnish a complete characterization and also a structure theorem for such near-rings.

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1 Introduction

A right near-ring is a non-empty set \( N \) together with two binary operations \( \cdot \) and \( + \) such that (1) \((N, +)\) is a group, (2) \((N, \cdot)\) is a semi-group and (3) \((n_1 + n_2)n_3 = n_1n_3 + n_2n_3\) for all \( n_1, n_2, n_3 \in N \).

Throughout this paper \( N \) stands for a right near-ring \((N, +, \cdot)\) with at least two elements and \('0'\) denotes the identity element of the group \((N, +)\). Obviously, \(0n = 0\) for all \( n \) in \( N \). \( N \) is said to be zero-symmetric if \( n0 = 0 \) for all \( n \) in \( N \). As in [2], a subgroup of \((M, +)\) of \((N, +)\) is called an
N-subgroup of $N$ if $NM \subset M$ and an invariant $N$ subgroup of $N$ if, in addition, $MN \subset M$. In [5], $N$ is defined to be Pseudo commutative if $xyz = yzx$ for all $x, y, z$ in $N$. The concept of a mate function in $N$ has been introduced in [4] with a view to handling the regularity structure with considerable ease. A map $f'$ from $N$ into $N$ is called (i) a mate function for $N$ if $x = xf(x)x$, (ii) a $P_3$ mate function, if, in addition, $xf(x) = f(x)x$ for all $x$ in $N$. By identity 1 of $N$, we mean only the multiplicative identity of $N$. Basic concepts and terms used but left undefined in this paper can be found in [2].

2 Notations

(i) $E$ denotes the set of all idempotents of $N$.

(e in $N$ is called an idempotent if $e^2 = e$)

(ii) $L$ denotes the set of all nilpotents of $N$.

(a in $N$ is nilpotent if $a^k = 0$ for some positive integer $k$.)

(iii) $N_0 = \{n \in N / n0 = 0\}$ - zero-symmetric part of $N$.

(iv) $N_d = \{n \in N / n(x+y) = nx+ny$ for all $x, y$ in $N\}$ - set of all distributive element of $N$.

(v) $C(N) = \{n \in N / nx = xn$ for all $x$ in $N\}$ - centre of $N$.

3 Preliminary Results

We freely make use of the following results and designate them as $R(1)$, $R(2)$,... etc.

$R(1)$ $N$ is subdirectly irreducible if and only if the intersection of any family of non-zero ideals of $N$ is again non-zero (Theorem 1.60, p.25 of [2]).

$R(2)$ $N$ has no non-zero nilpotent elements if and only if $x^2 = 0 \Rightarrow x = 0$ for all $x$ in $N$ (Problem 14, p.9 of [3]).

$R(3)$ If $f$ is a mate function for $N$, then for every $x$ in $N$, $xf(x)$, $f(x)x \in E$ and $N x = N f(x)x$, $x N = x f(x)N$ (Lemma 3.2 of [4]).

$R(4)$ If $L = \{0\}$ and $N = N_0$, then (i) $xy = 0 \Rightarrow yx = 0$ for all $x, y$ in $N$.

(ii) $N$ has Insertion of factors property- IFP for short- i.e for $x, y$ in $N$, $xy = 0 \Rightarrow xny = 0$ for all $n$ in $N$. If $N$ satisfies (i) and (ii) then $N$ is said to have $(*, IFP)$ (Lemma 2.3 of [4]).

$R(5)$ Any Pseudo commutative near-ring with a right identity is weak commutative (i.e. $xyz = yzx$ for all $x, y, z$ in $N$ [2]) (Proposition 2.9 of [5]).

$R(6)$ A zero-symmetric near-ring $N$ is a near-field if $N_d \neq \{0\}$ and for all $n \in N-\{0\}$, $Nn = N$ (Theorem 8.3, p.249 of [2]).
4 Definition of $\beta_1$ Near-Rings and Examples

In this section we define $\beta_1$ near-rings and give certain examples of this new concept.

Definition 4.1 Let $N$ be a right near-ring. If for every $x, y$ in $N$, $xNy = Nxy$ then we say $N$ is a $\beta_1$ near-ring.

Example 4.2 (a) The near-ring $(N, +, \cdot)$ defined on the Klein's four group $N = \{0, a, b, c\}$ where multiplication is defined as per scheme 4, p.408, Pilz[2].

\[
\begin{array}{c|cccc}
  & 0 & a & b & c \\
\hline
  0 & 0 & 0 & 0 & 0 \\
  a & 0 & 0 & a & a \\
  b & 0 & a & c & b \\
  c & 0 & a & b & c \\
\end{array}
\]

is a $\beta_1$ near-ring. It is worth noting that this near-ring does not admit mate functions.

(b) The near-ring $(N, +, \cdot)$ where $(N, +)$ is the group of integers modulo 5 and multiplication defined as per scheme 6, p.408, Pilz [2] is not a $\beta_1$ near-ring.

\[
\begin{array}{c|cccc}
  & 0 & 1 & 2 & 3 & 4 \\
\hline
  0 & 0 & 0 & 0 & 0 & 0 \\
  1 & 0 & 0 & 4 & 1 & 0 \\
  2 & 0 & 0 & 3 & 2 & 0 \\
  3 & 0 & 0 & 2 & 3 & 0 \\
  4 & 0 & 0 & 1 & 4 & 0 \\
\end{array}
\]

since $2N2 \neq N22$.

(c) The near-ring $(N, +, \cdot)$ where $(N, +)$ is the group of integers modulo 6 and multiplication defined as per scheme 36, p.409, Pilz [2].

\[
\begin{array}{c|cccc}
  & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
  0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  1 & 0 & 4 & 2 & 0 & 4 & 2 \\
  2 & 0 & 2 & 4 & 0 & 2 & 4 \\
  3 & 0 & 0 & 0 & 0 & 0 & 0 \\
  4 & 0 & 4 & 2 & 0 & 4 & 2 \\
  5 & 0 & 2 & 4 & 0 & 2 & 4 \\
\end{array}
\]
is a zero-symmetric $\beta_1$ near-ring with no identity.

5 Properties of $\beta_1$ Near-Rings

In this section, we study some of the important properties of $\beta_1$ near-rings and give a complete characterization of such near-rings. We also obtain a structure theorem for $\beta_1$ near-rings.

**Proposition 5.1** Let $N$ be a $\beta_1$ near-ring. If $N$ has identity 1, then $N$ is zero-symmetric.

**Proof** Let $N$ be a $\beta_1$ near-ring. Then for all $x, y$ in $N$, $xNy = Nxy$. Putting $y = 1$, we get $xN1 = Nx1$ for all $x$ in $N$. When $x = 0$, $0N = N0 = \{0\}$. It follows that $N$ is zero-symmetric.

**Remark 5.2** The converse of Proposition 5.1 is not valid. For example, the near-ring cited in Example 4.2 (c) is a zero-symmetric $\beta_1$ near-ring, but it has no identity.

**Proposition 5.3** If $N$ is a $\beta_1$ near-ring then $xNx = Nx^2$ for all $x$ in $N$.

**Proof** When $N$ is a $\beta_1$ near-ring, by definition, for all $x, y$ in $N$, $xNy = Nxy$ .... (1). The result follows by replacing $y$ by $x$ in (1).

**Remark 5.4** The converse of Proposition 5.3 is not true. For example, we consider the near-ring $(N, +, \cdot)$ where $(N, +)$ is the Klein’s four group $\{0, a, b, c\}$ and $\cdot$ is defined as per scheme 8 p.408, Pilz[2].

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satisfies the condition $xNx = Nx^2$ for all $x$ in $N$. But it is not a $\beta_1$ near-ring [since $bNc \neq Nbc$].

**Proposition 5.5** Every Pseudo commutative near-ring with identity is a $\beta_1$ near-ring.

**Proof** Let $N$ be a Pseudo commutative near-ring. ....(1). Let $x, y \in N$. 
If \( a \in xNy \), then there exists \( n \in N \) such that \( a = xny = ynx \) \( \text{[by (1)]} \). Therefore \( a \in Nxy \). Thus \( xNy \subseteq Nxy \) ........(2).

On the other hand, if \( b \in Nxy \), then for some \( n' \in N \), \( b = n'xy = n'yx \) \( \text{[by R(5)]} \). Consequently, \( Nxy \subseteq xNy \) .......(3).

Combining (2) and (3), we get \( N \) is a \( \beta_1 \) near-ring.

**Proposition 5.6** Homomorphic image of a \( \beta_1 \) near-ring is also a \( \beta_1 \) near-ring.

**Proof** Straight forward.

**Theorem 5.7** Every \( \beta_1 \) near-ring \( N \) is isomorphic to a subdirect product of subdirectly irreducible \( \beta_1 \) near-rings.

**Proof** By Theorem 1.62, p.26 of Pilz [2], \( N \) is isomorphic to a subdirect product of subdirectly irreducible near-rings \( N_i \)'s and each \( N_i \) is a homomorphic image of \( N \) under the projection map \( \pi_i \). The rest of the proof is taken care of by Proposition 5.6.

We furnish below a necessary and sufficient condition for a \( \beta_1 \) near-ring to admit mate functions.

**Lemma 5.8** Let \( N \) be a \( \beta_1 \) near-ring. Then \( N \) admits mate functions if and only if \( x \in Nx^2 \) for all \( x \) in \( N \).

**Proof** We first observe from Proposition 5.3 that, since \( N \) is \( \beta_1 \), \( xNx = Nx^2 \) for all \( x \) in \( N \) .......(1). For the 'only if' part, let \( f \) be a mate function for \( N \). Then for all \( x \) in \( N \), \( x = xf(x)x \in xNx \). It follows that \( x \in Nx^2 \). For the 'if' part, let \( x \in Nx^2 \) for all \( x \) in \( N \). Appealing to (1) we get, \( x = xnx \) for some \( n \) in \( N \). By setting \( n = f(x) \), we see that \( f \) is a mate function for \( N \).

In the following results we assume that \( N \) has a mate function.

**Theorem 5.9** Let \( N \) be a zero-symmetric \( \beta_1 \) near-ring with a mate function 'f'. Then we have,
(i) \( L = \{0\} \).
(ii) \( N \) has \( (*, IFP) \)
(iii) \( E \subseteq C(N) \).

**Proof** (i) Since \( f \) is a mate function for \( N \), Lemma 5.6 demands that
$x \in Nx^2$ for all $x$ in $N$. Therefore $x = nx^2$ for some $n$ in $N$. Suppose $x^2 = 0$.
Clearly, then $x = 0$. Now, $R(2)$ guarantees that $L = \{0\}$.
(ii) By (i) $L = \{0\}$. Now, $R(4)$ guarantees that $N$ has $(*, \text{IFP})$.
(iii) Let $e \in E$. Since $N$ is $\beta_1$, $e Ne = Ne.e = Ne$. Therefore for any $n$ in $N$,
e ne = ue \text{ and } ne = eve \text{ for some } u, v \text{ in } N. \text{ Now, } en = e(ue)$ and
e(ne) = eve. Thus $ene = ne$ for all $n$ in $N$ ...(1). We also have, $(ene - en)e = 0$
$\Rightarrow e(ene - en) = 0 \Rightarrow en(ene - en) = 0 \Rightarrow en(ene - en) = 0$ [by (ii)].
Consequently, $(ene - en)^2 = 0$ and (i) guarantees $ene - en = 0$. Therefore $ene = en$ for all $n$ in $N$ ...(2). From (1) and (2) we get, $en = ne$ for all $n$ in $N$.
Thus $E \subset C(N)$.

We furnish below a characterization theorem for $\beta_1$ near-ring.

**Theorem 5.10** Let $N$ be a zero-symmetric near-ring with a mate function '$f$'. Then $N$ is $\beta_1$ if and only if $xN = Nx^2$ for all $x$ in $N$ and $E \subset C(N)$.

**Proof** For the ’only if’ part, first we observe that ”$E \subset C(N)$” ......(1).
follows from Theorem 5.8 (iii). Now, for any $x$ in $N$, if $a \in Nx^2$, then
[since $N$ is $\beta_1$] $a \in xNx \subset xN$. Therefore $Nx^2 \subset xN$ ......(2).
On the other hand, if $b \in xN$, then for some $n$ in $N$, $b = xn = xf(x)xn = (nf(x))x$
[by (1)] $\in xNx = Nx^2$ [since $N$ is $\beta_1$]. Consequently, $xN \subset Nx^2$ ......(3).
Combining (2) and (3) $xN = Nx^2$ for all $x$ in $N$. For the ”if” part, first we
show that ’$f$’ is a $P_3$ mate function. For any $x \in N$ we have $x = xf(x)x$
$\in xN = Nx^2$ [by assumption]. Therefore $x = n_1x^2$ for some $n_1$ in $N$. And
$xf(x)x = nx.xf(x)x = f(x)xnx^2$ [Since $E \subset C(N)$] $= f(x)x^2$ $\Rightarrow$
$[xf(x) - f(x)x]x = 0 \Rightarrow x[xf(x) - f(x)x] = 0$ [by Theorem 5.9 (ii)] $\Rightarrow$
x$f(x)[xf(x) - f(x)x] = 0$ [by Theorem 5.9 (ii)] and $f(x)[xf(x) - f(x)x] = f(x).0 = 0$ [since $N = N_0$]. Consequently, $[xf(x) - f(x)x]^2 = 0$
and hence $xf(x) = f(x)x$ [by R(2)] ...... (4). Hence $f$ is a $P_3$ mate function. Now,
$Nx.y = [Nxy] = [Nf(x)x]y$ [by R(3)] $= [f(x)xN]y$ [since $E \subset C(N)$] $= [xf(x)N]y$
[by (4)] $= xNy$ [by R(3)]. Thus $N$ is a $\beta_1$ near-ring.

With a view to establishing a structure theorem we prove the following theorem.

**Theorem 5.11** Let $N$ be a $\beta_1$ near-ring with a mate function $f$. Then $N$
is subdirectly irreducible if and only if $N$ is a near-field.

**Proof** For the ’only if’ part, first we shall show that no non-zero idempotent of $N$
is a zero divisor. Let $J$ be the set of all non-zero idempotents in $N$ which are zero divisors. Let $I = \cap \{(0 : e)/e \in J\}$. Since
\(\beta_1\) near-rings

\(N\) is subdirectly irreducible, \(R(1)\) demands that \(I \neq \{0\}\). Let \(a \in I - \{0\}\). Then \(ae = 0\) for all \(e \in J\). (1). By Theorem 5.9 (ii), \(ea = 0 \Rightarrow ef(a)a = f(a)a \in J\). Therefore \(af(a)a = 0 [\text{by (1)}] \Rightarrow a = 0\) which is a contradiction to \(a \neq 0\). Consequently, no non-zero idempotent of \(N\) is a zero-divisor... (2).

Let \(M\) be any non-zero \(N\)-subgroup of \(N\) and let \(0 \neq x \in M\). Lemma 5.8 demands that \(x \in Nx^2\) for all \(n \in N\). Therefore, \(x = n'x^2\) for some \(n'\) in \(N\). Now, for all \(n \in N\), \(nx = nn'x^2 = n_1x^2 \Rightarrow (n - n_1)x = 0 \Rightarrow (n - n_1)f(x)x = 0 [\text{by Theorem 5.9 (ii)}]\). Now by (2), \(n - n_1x = 0 \Rightarrow n = n_1x \in NM \subset M\). Thus \(N \subset M\). Consequently, \(N\) has no non trivial \(N\)-subgroups ...(3). Let \(n \in N - \{0\}\). Then by (3), \(Nn = N\).

Now, by Theorem 5.9 (ii), \(E \subset C(N)\) and \(C(N) \subset N_d\). Therefore, \(N_d \neq \{0\}\). Thus \(N\) is a near-field. [by R(6)]. The proof of 'if' part is obvious.

We conclude our discussion by proving the following structure theorem.

**Theorem 5.12** Let \(N\) be a \(\beta_1\) near-ring with a mate function. Then \(N\) is isomorphic to a subdirect product of near-fields.

**Proof** By Theorem 5.7, \(N\) is isomorphic to a subdirect product of subdirectly irreducible \(\beta_1\) near-rings \(N_i\)s. Since \(N\) has a mate function it follows that each \(N_i\) also has a mate function. The rest of the proof is taken care of by Theorem 5.11.

**References**


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