Central Units in Integral Group Rings II

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Abstract

Recent work on central units of integral group rings is surveyed. In particular we present two methods of constructing central units, induction and lifting, and demonstrate how these constructions can often be used to find generators for large subgroups in the full group of central units of an integral group ring.

Keywords: Integral group ring; Central unit; Bass cyclic unit

The purpose of this brief note is to survey recent work on central units in integral group rings, updating an earlier paper (part I) which appeared in 1999 [8]. In addition to the general references on units in group rings mentioned before [3,14], a more recent book by Polcino Milies and Sehgal [10] is also highly recommended. As before, our notation will follow that in [14].

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1 Trivial Central Units

The first open problem stated in [8] was to characterize groups $G$ for which all central units of $\mathbb{Z}G$ are trivial. For finite groups the result was known [12] but in general the problem was still open. A complete characterization has since been obtained by Dokuchaev, Polcino Milies and Sehgal.

**Theorem 1.1** [1]

All central units of $\mathbb{Z}G$ are trivial if and only if every finite normal subgroup $A$ of $G$ satisfies the condition that for every $a \in A$ and every natural number $j$ relatively prime to $|a|$ the element $a^j$ is conjugate (in $G$) to $a$ or $a^{-1}$.

A crucial step in obtaining the above characterization is the following nice result obtained earlier by Polcino Milies and Sehgal.

**Proposition 1.2** [11]

Any central unit $u$ of $\mathbb{Z}G$ can be written as $u = gw$, $g \in G$, $w \in \mathbb{Z}T$, where $T$ is the torsion subgroup of $\emptyset(G)$, the FC-subgroup of $G$, and $gw = wg$.

The decomposition of $u$ in the above proposition is clearly not unique. In [7] an example was given of a central unit in a group ring $\mathbb{Z}G$ with the property that no such decomposition $u = gw$ can be found if we add the restriction that $g$ should be central in $G$. It is still an open question as to which groups have the property that a decomposition into a product of central units can always be obtained.

2 Construction of Central Units

It was noted in [8] that while several important families of units (Bass cyclic, bicyclic, alternating etc.) are known it seems to be less straightforward to construct simple examples of central units in integral groups rings and Open Problem 4 suggested finding some new interesting families of central units. Since that time two methods of constructing central units have been shown to be useful. The first of these (Induced Central Units) is very simple and is certainly not new - a version of it was helpful in [6]. The second (Lifted Central Units) is also quite simple but does not seem to have been used until recently.

2.1 Induced Central Units

Assume $N \triangleleft G$ and $G/N$ is finite, and let $\{g_1, g_2, \ldots, g_n\}$ be a set of coset representatives of $N$ in $G$. If $u$ is a central unit in $\mathbb{Z}N$ then $\prod_i u^{g_i}$ is a central
unit in $\mathbb{Z}G$.

It is very easy to verify the above statement. The problem with induced units is that may be difficult to know whether a central unit obtained in this way is nontrivial. Nevertheless this kind of construction can be seen in [6] and is also closely related to a construction entitled “central units of the second kind” which was useful in [2].

2.2 Lifted Central Units

Assume $G$ is finite, $N \triangleleft G$ and $\overline{v}$ is a central unit in $\mathbb{Z}(G/N)$. If $s$ is any integer such that $\overline{v^s} - 1$ belongs to $|N|R(G/N)$ then

$$u = 1 + (v^s - 1)^\frac{N}{|N|}$$

is a central unit in $\mathbb{Z}G$. In addition $u - v^s$ belongs to $\Delta(G,N)$, so what we are observing here is that a suitable power of a central unit in $\mathbb{Z}(G/N)$ can be lifted to a central unit in $\mathbb{Z}G$. Since nontrivial central units are of infinite order (see [10], Corollary 7.3.3), this procedure will always construct nontrivial central units in $\mathbb{Z}G$ from nontrivial central units of $\mathbb{Z}(G/N)$.

It is not difficult to verify the above statements. In fact, if $\overline{w}$ is the inverse of $\overline{v}$ in $\mathbb{Z}(G/N)$ then $1 + (w^s - 1)^\frac{N}{|N|}$ is the inverse of $u$ in $\mathbb{Z}G$. This kind of construction is closely related to “central units of the first kind” which were helpful in [2].

If $G$ is a finite group, $L$ is any subnormal subgroup of $G$ and $K \triangleleft L$, then the two constructions just described can be used to turn central units of $\mathbb{Z}(L/K)$ into central units of $\mathbb{Z}G$ in several different ways. A few applications of these constructions have already been mentioned and we will see others in the next section. Perhaps other applications, or other similar but more sophisticated constructions, can be found.

3 Finite Index

Ideally we would like to obtain a nice generating set for $\mathbb{Z}(\mathcal{U}(\mathbb{Z}G))$, the group of central units of $\mathbb{Z}G$. Since this is an intractable problem even for abelian groups, most attention has been paid to trying to obtain such a generating set for a subgroup of finite index in $\mathbb{Z}(\mathcal{U}(\mathbb{Z}G))$. As noted in [8], for finite $G$ a finite set of generators was described by Ritter and Sehgal [13], but these generators are not easy to compute in practice. Open Problem 3 asked for a “simple” set of generators for a subgroup of finite index in $\mathbb{Z}(\mathcal{U}(\mathbb{Z}G))$ for interesting classes of groups, noting that such a set had been found when $G$ is finitely generated nilpotent [6] and when $G = A_n$ for some $n$ [4].
The following shows how induced and lifted central units can be used in obtaining finite index results.

**Proposition 3.1**

Let \( N \subseteq M \subseteq G \) where \( G \) is finite, \( N \vartriangleleft G \) and \( M \vartriangleleft G \). Assume we have a finite set of generators for a subgroup of finite index of \( \mathcal{Z}(\mathcal{U}(\mathbb{Z}(G/N))) \) and also for a subgroup of finite index in \( \mathcal{Z}(\mathcal{U}(\mathbb{Z}M)) \). Assume as well that any central unit of \( \mathbb{Z}G \) which lies in \( 1 + \Delta(G, N) \) actually lies in \( \mathbb{Z}M \). Then, using these generators, we can construct a finite set of generators for a subgroup of finite index in \( \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \).

**Proof**

Let \( z \) be a central unit in \( \mathbb{Z}G \). There exists an integer \( r \) such that \( z^r = \prod_{i=1}^{k} a_i^{\alpha_i} \cdot \prod_{j=1}^{k} b_j^{\beta_j} \), for some integers \( \alpha_i, \beta_j \), where \( a_1, a_2, \ldots, a_k \) are among the finite set of generators for a subgroup of finite index in \( \mathcal{Z}(\mathcal{U}(\mathbb{Z}(G/N))) \). For each \( i, 1 \leq i \leq k \), we lift some power of \( a_i \) to a central unit \( c_i \) of \( \mathbb{Z}G \) by the method described in 2.2. It follows that there exist integers \( w, \beta_1, \beta_2, \ldots, \beta_k \) such that \( z^w = c_1^{\beta_1} c_2^{\beta_2} \cdots c_k^{\beta_k} v \) for some central unit \( v \) of \( \mathbb{Z}G \) which belongs to \( 1 + \Delta(G, N) \).

By the hypothesis \( v \) belongs to \( 1 + \Delta(M) \). It follows that there exists an integer \( x \) such that \( v^x = e_1^{\gamma_1} e_2^{\gamma_2} \cdots e_\ell^{\gamma_\ell} \) where \( \gamma_1, \ldots, \gamma_\ell \) belong to \( \mathbb{Z} \) and \( e_1, e_2, \ldots, e_\ell \) are among the finite set of generators for a subgroup of finite index in \( \mathcal{Z}(\mathcal{U}(\mathbb{Z}M)) \).

Since all conjugates \( e_i^g, g \in G \), of all \( e_i \) commute with each other, and \( v \) and all \( c_i \) are central in \( \mathbb{Z}G \), we use 2.1 to obtain \( z^y = c_1^{\theta_1} c_2^{\theta_2} \cdots c_k^{\theta_k} f_1^{\varepsilon_1} f_2^{\varepsilon_2} \cdots f_\ell^{\varepsilon_\ell} \) where \( f_i = \prod_g e_i^g \) for each \( i \) and \( y, \theta_1, \theta_2, \ldots, \theta_k, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell \) are integers.

Finally we note that the above process allows us to put a bound on \( y \). The proof is complete.

The following lemma, whose proof appears in [5], leads to a sufficient condition for the crucial final property of Proposition 3.1 to be satisfied.

**Lemma 3.2**

Let \( G \) be a finite group with normal subgroup \( N \). Assume that \( x \in G \) has the property that \( C_N(x) = \{1\} \). Then \( x \) can not appear in the support of any central unit of \( \mathbb{Z}G \) which also belongs to \( 1 + \Delta(G, N) \).

Ferraz and Simón [2] have recently constructed a nice set of generators for a subgroup of finite index in \( \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \) in the case where \( G \) is not a \( p \)-group and is meta-(cyclic of prime order). Their work was especially interesting because the generators were actually independent. As a first application of Proposition 3.1 we obtain a result for a more general class of groups (unfortunately, our
result does not include independence).

**Theorem 3.3**

Assume $K < G$ is such that $G/K$ and $K$ are both finite cyclic and $(|K|, |G/K|) = 1$. Then we can construct a finite set of generators for a subgroup of finite index in $\mathcal{Z}(U(ZG))$.

**Proof**

We may assume $G$ is not abelian since otherwise Bass cyclic units would suffice (see [10, Theorem 8.3.15]).

Proceed by induction on $|G|$, noting that $|K| \neq 1$ and $|G/K| \neq 1$. We may assume $G/K = <x>$ where $m = |x| = |G/K|$ in $G$.

Let $K = S_1 \times \cdots \times S_t$ be the decomposition of $K$ as a product of Sylow subgroups. Note that $x$ acts on each $S_i$ by conjugation. Since $G$ is not abelian, $x$ must not commute with the generator of some $S_k = <y>$. Assume $|y| = p^n$ and let $\ell$ be the smallest positive integer such that $x^\ell$ commutes with $y$.

We claim that the conditions of Proposition 3.1 apply to $G$ with $N = <y>$ and $M = <K, x^\ell>$. Clearly $N \triangleleft G$, $M \triangleleft G$ and $N \subseteq M$. Since $G/N$ and $M$ are both metacyclic with relatively prime factors and $M \neq G$, the induction hypothesis tells us that the result holds for both these groups. To complete the proof we must show that any central unit of $ZG$ which belongs to $1 + \Delta(G, S_k)$ is actually in $Z <K, x^\ell>$. Because of Lemma 3.2 it is enough to show that if $1 \leq i \leq \ell - 1$ and $1 \leq j \leq p^n - 1$ then $x^iy^j \neq y^jx^i$.

We know that $x^{-i}yx = y^s$ where $s \neq 1$ and $s^\ell \equiv 1 \pmod{p^n}$. Since $p$ does not divide $\ell$, it is easy to see that $s^i \not\equiv 1 \pmod{p^n}$ for any $1 \leq i \leq \ell - 1$.

Assume finally that $x^{-i}y^jx^i = y^j$ for $i$ as above and $1 \leq j \leq p^n - 1$. Then $y^js^i = y^j$, so $j(s^i - 1) \equiv 0 \pmod{p^n}$ and hence $j \equiv 0 \pmod{p^n}$. The proof is complete.

Although Theorem 3.3 does not immediately produce an independent set of generators it is sometimes reasonably straightforward in particular examples to obtain such a set. We will briefly describe two such examples.

For $G = <g>$ cyclic of order $n$ and $a$ relatively prime to $n$, let $B(a, n)$ denote the Bass cyclic unit

$$(1 + g + \cdots + g^{a-1})^{\phi(n)} + \frac{1-a}{n} \phi(n)\hat{g}.$$ 

It was noted earlier that whenever $G$ is finite abelian the Bass cyclic units generate a subgroup of finite index in $U(ZG)$.
Examples 3.4

1. Let $G = \langle x, y | x^7 = y^9 = 1, x^y = x^2 \rangle$. The conditions of Theorem 3.3 are satisfied with $N = \langle x \rangle$, so the construction in Proposition 3.1 can be applied with $N = \langle x \rangle$ and $M = \langle x, y^3 \rangle$. In this case the torsion-free rank of $U(\mathbb{Z}(G/N))$ is 2 (see the formula on p.253 of [10]) and suitable powers of the independent Bass cyclic units $B(2, 9)$ and $B(4, 9)$ of $\mathbb{Z}(G/N)$ are lifted to independent central units of $\mathbb{Z}G$. Since $[B(2, 9)]^{57} = 1$ and $[B(4, 9)]^{19} = 1$ in $\mathbb{Z}_7 < y >$, the lifted units are $v_1 = 1 + ([B(2, 9)]^{57} - 1) \frac{7}{7}$ and $v_2 = 1 + ([B(4, 9)]^{19} - 1) \frac{7}{7}$.

The overall torsion-free rank of $\mathbb{Z}(U(\mathbb{Z}G))$ is 3 (this can be deduced from Section 2 of [2]). Since $|\frac{M}{N}| = 3, \mathbb{Z}(\frac{M}{N})$ has only trivial units so every Bass cyclic unit of $\mathbb{Z}M$, and hence every central unit of $\mathbb{Z}G$ induced from a Bass cyclic unit of $\mathbb{Z}M$, becomes trivial in $\mathbb{Z}(G/N)$. It follows that any nontrivial central unit of $\mathbb{Z}G$ which is induced from a Bass cyclic unit of $\mathbb{Z}M$ will form, together with $v_1$ and $v_2$, an independent generating set for a subgroup of finite index in $\mathbb{Z}(U(\mathbb{Z}G))$. A straightforward calculation shows that the induced unit $B^*(2, 21) = [B(2, 21)]B(2, 21)B^*(2, 21)$ is nontrivial, so $\{v_1, v_2, B^*(2, 21)\}$ is the required independent set of generators.

2. Now let $G = \langle x, y | x^{13} = y^9 = 1, x^y = x^3 \rangle$. Again the construction described in Proposition 3.1 is applied with $N = \langle x \rangle$ and $M = \langle x, y^3 \rangle$. As far as lifted units go, the situation is exactly the same as in the previous example except that the powers needed for lifting are different. Here the independent lifted central units are

$$v_1 = 1 + ([B(2, 9)]^{183} - 1) \frac{13}{13}$$

and

$$v_2 = 1 + ([B(4, 9)]^{61} - 1) \frac{13}{13}.$$ 

The torsion-free rank of $\mathbb{Z}(U(\mathbb{Z}G))$ is now equal to 6 (as before this can be determined from [2]), so we need 4 independent central units induced from Bass cyclic units of $\mathbb{Z}M$. After some calculation it is seen that there are only 4 distinct nontrivial induced units - in fact $B^*(3, 13) = B^*(4, 13) = B^*(16, 39) = B^*(17, 39) = 1, B^*(2, 13) = B^*(5, 13) = B^*(6, 13), B^*(2, 39) = B^*(5, 39) = B^*(7, 39), B^*(8, 39) = B^*(11, 39) = B^*(19, 39), B^*(4, 39) = B^*(10, 39) = B^*(14, 39)$. (Here $B(a, 13)$ is a Bass cyclic unit in $\mathbb{Z} < x > \subseteq \mathbb{Z}M$). Hence

$$\{v_1, v_2, B^*(2, 13), B^*(2, 39), B^*(8, 39), B^*(4, 39)\}$$

is the required independent generating set.

3. We remark that finding an independent generating set is much more difficult when $\mathbb{Z}(M/N)$ has nontrivial units.
Theorem 3.3 has been included here as a particular case because it is relatively easy to prove. In [5] a finite set of generators for a subgroup of finite index in \( \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \) is given for much more general classes of groups, including strongly monomial groups. A special case which is relatively easy to state is that of finite metabelian groups (see [5] for the definition of a strongly Shoda pair).

**Theorem 3.5 [5]**

Let \( G \) be a finite metabelian group. Then the group generated by all units of the form \( \prod_g (1 - \frac{\hat{K}}{|K|} + b^{n(K)} \frac{\hat{K}}{|K|})^g \), with \((H,K)\) a strong Shoda pair and \( H \) normal in \( G \), \( b \) a Bass cyclic unit in \( \mathbb{Z}G \) and \( n(K) \) the smallest positive integer so that \( 1 - \frac{\hat{K}}{|K|} + b^{n(K)} \frac{\hat{K}}{|K|} \) is in \( \mathbb{Z}H \), is a central subgroup of finite index in \( \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \).

We can also apply Proposition 3.1 to Frobenius groups. Recall that a finite group \( G \) is called a Frobenius group if it has a normal subgroup \( N \) (\( N \neq \{1\}, G \)) with the property that if \( 1 \neq g \in N \) then \( C_G(g) \subseteq N \). Properties of Frobenius groups are discussed in [9]. In particular it is shown that such a group contains a subgroup \( H \) (called a Frobenius complement) such that \( G = HN \) and \( H \cap N = \{1\} \).

**Theorem 3.6**

If \( G \) is a Frobenius group with Frobenius complement \( H \), and \( |H| \) is odd, then we can construct a finite set of generators for a subgroup of finite index in \( \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \).

**Proof**

We will use Proposition 3.1 with \( N \) as standard for a Frobenius group and \( M = N \).

It is known [9] that \( N \) is nilpotent so we can construct a finite set of generators for a subgroup of finite index in \( \mathcal{Z}(\mathcal{U}(\mathbb{Z}N)) \) [6]. Since \( |G/N| \) is odd it also follows from [9] that \( G/N \) is metacyclic with relatively prime factors, so using Theorem 3.3 we can construct a finite set of generators for a subgroup of finite index in \( \mathcal{Z}(\mathcal{U}(\mathbb{Z}(G/N))) \). Finally we observe that the definition of Frobenius group tells us that if \( x \notin N \) then \( x \notin C_G(g) \) for any \( 1 \neq g \in N \), so Lemma 3.2 allows us to conclude that central units of \( \mathbb{Z}G \) which lie in \( 1 + \Delta(G, N) \) must lie in \( \mathbb{Z}N \).

Proposition 3.1 gives the result.
The generators in Theorem 3.6 can also be stated in terms of strong Shoda pairs (as in Theorem 3.5) and a proof of Theorem 3.6 which leads to a concrete listing of such a set of generators can be found in [5].

It is still unclear how to extend Theorem 3.6 to arbitrary Frobenius groups.

REFERENCES


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