A Generalization of Reduced Modules

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Abstract

This paper introduces two new types of modules. The first, is called a primally reduced module, which is a generalization of reduced modules and the second, is called a radically reduced module. Some properties of these types of modules are proved and in addition, some relations concerning these modules are determined.

Keywords: primally reduced, radically reduced, locally prime, locally reduced, locally multiplication

1 Introduction

Let $R$ be a commutative ring with identity and $M$ be an $R$–module. $M$ is called a multiplication module if for each submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IM$ [2], equivalently, if for any submodule $N$ of $M$, we have $N = (N : M)M$ [12], where $(N : M) = \{r \in R : rM \subseteq N\}$ is an ideal of $R$. A proper submodule $N$ of $M$ is called a prime submodule if for $r \in R, m \in M$, the condition $rm \in N$ implies that $m \in N$ or $rM \subseteq N$ (equivalently, $r \in (N : M)$) [11] and it is called a weakly prime submodule if $0 \neq rm \in N$, where $r \in R, m \in M$, then $m \in N$ or $rM \subseteq N$ [2], or equivalently, if for $r, s \in R$ and $x \in M, rsx \in N$ implies $rx \in N$ or $sx \in N$ [12] and in this case $(N : M) = \{r \in R : rM \subseteq N\}$ is a prime ideal of $R$.
$M$ is called a reduced module if the intersection of all prime submodules of $M$ is zero [11], that is $\bigcap \text{Spec}(M) = 0$, where $\text{Spec}(M) = \{ P : P $ is a prime submodule of $M \}$ and it is called locally reduced if $M_P$ is reduced for each maximal ideal $P$ of $R$ [4]. For a submodule $N$ of $M$, $(0:N)$ is defined as $(0 : N) = \{ r \in R : rN = 0 \}$. A non-empty subset $S$ of $R$ is called a multiplicative closed set in $R$ if $0 \notin S$ and $a, b \in S$ implies that $ab \in S$ [7]. If $S$ is a multiplicative set in $R$, then one can easily make $M_S$ as an $R_S$-module under the module operations $\frac{r}{s} + \frac{u}{t} = \frac{tr + su}{st}$ and $\frac{r}{s} \cdot \frac{x}{y} = \frac{rx}{sy}$, for $\frac{r}{s} \in R_S$ and $\frac{x}{y}, \frac{u}{t} \in M_S$ [8], so that when we say $M_S$ is a module we mean $M_S$ is an $R_S$-module. An element $r \in R$ is called prime to $N$ if $rm \in N$, for $m \in M$, implies that $m \in N$ [1], equivalently, $r \in R$ is not prime to $N$ if $rm \in N$ for some $m \in M - N$. If we denote the set of all elements of $R$ that are not prime to $N$ by $S_M(N)$, then we have $S_M(N) = \{ r \in R : rm \in N \}$, for some $m \in M - N$, specially, if $N = 0$, then $S_M(0) = \{ r \in R : rx = 0 \}$, for some $0 \neq x \in M \}$ and $N$ is called a primal submodule of $M$ if $S_M(N)$ forms an ideal of $R$ [1]. A proper ideal $P$ of $R$ is called a weakly prime ideal if $0 \neq ab \in P$, for $a, b \in R$, then $a \in P$ or $b \in P$[2]. A proper submodule $N$ of $M$ is called an $S_M(N)$-locally prime submodule of $M$, if $N_P$ is a prime submodule of $M_P$ for each maximal ideal $P$ of $R$ with $S_M(N) \subseteq P$ and it is called an $S_M(N)$-weakly prime submodule of $M$, if $N_P$ is a weakly prime submodule of $M_P$ for each maximal ideal $P$ of $R$ with $S_M(N) \subseteq P$ [5]. The annihilator of $M$ is denoted by $\text{Ann}(M)$ or $(0 : M)$ and defined as $\text{Ann}(M) = \{ r \in R : rM = 0 \}$ and if $m \in M$, then the left annihilator of $m$ is defined as $(0 : m) = \{ r \in R : rm = 0 \}$ [9]. Finally, a proper submodule $N$ of $M$ is called a maximal submodule if it is not properly contained in any proper submodule of $M$ [13] and the Jacobson radical of $M$, denoted by $\text{Rad}_j(M)$, is defined to be the intersection of all the maximal submodules of $M$ [3].

Throughout this paper, all rings are commutative with identity and all modules are left $R$-modules.

## 2 The Results

First, we introduce the following definition.

**Definition 2.1.** Let $M$ be an $R$-module. The primal spectrum of $M$ is denoted by $p\text{Spec}(M)$, and is defined as $p\text{Spec}(M) = \{ N : N $ is a primal submodule of $M \}$ and we say that $M$ is a primally reduced $R$-module if $\bigcap p\text{Spec}(M) = 0$.

**Proposition 2.2.** Every reduced $R$-module $M$ is primally reduced.
Proof. We have \( \bigcap \text{Spec}(M) = 0 \). Since every prime submodule of \( M \) is primal [1], so \( \text{Spec}(M) \subseteq \text{pSpec}(M) \) and thus \( \bigcap \text{pSpec}(M) \subseteq \text{Spec}(M) = 0 \), that is \( \bigcap \text{pSpec}(M) = 0 \). Hence \( M \) is primally reduced.

Here we give an example of a primally reduced module which is not reduced.

**Example 2.3.** Let us consider \( Z_{12} \) as a \( Z_{12} \)-module, that is take \( M = Z_{12} \) and \( R = Z_{12} \). The proper submodules of \( Z_{12} \) are, \( \{0\} \), \( \{0, 2, 4, 6, 8, 10\} \), \( \{0, 3, 6, 9\} \), \( \{0, 4, 8\} \) and \( \{0, 6\} \). Among these proper submodules we have only \( \{0, 2, 4, 6, 8, 10\} \) and \( \{0, 3, 6, 9\} \) are prime and thus \( \text{Spec}(Z_{12}) = \{N, L\} \), so we have \( \bigcap \text{Spec}(Z_{12}) = N \cap L = \{0, 6\} \neq \{0\} \). Hence \( Z_{12} \) is not reduced module. On the other hand, as \( N \) and \( L \) are prime submodules of \( Z_{12} \), they are primal [1], so \( N, L \in \text{pSpec}(Z_{12}) \) and since for the submodule \( K = \{0, 4, 8\} \), we have \( S_M(K) = \{0, 2, 4, 6, 8, 10\} \), which is an ideal of \( Z_{12} \), so \( K \) is a primally reduced submodule of \( Z_{12} \) and thus \( K \in \text{pSpec}(Z_{12}) \), so we have \( \{N, L, K\} \subseteq \text{pSpec}(Z_{12}) \) and that gives \( \bigcap \text{pSpec}(Z_{12}) \subseteq N \cap L \cap K = 0 \). That means \( \bigcap \text{pSpec}(Z_{12}) = 0 \). Hence \( Z_{12} \) is primally reduced but not reduced.

In view of Proposition 2.2 and Example 2.3 we can say that primally reduced modules are generalizations of reduced modules.

Now, we prove the following result which will be used in the next results.

**Lemma 2.4.** Let \( M \) be an \( R \)-module and \( P \) a maximal ideal of \( R \). If \( \phi \neq A \subseteq R \) and \( \phi \neq N \subseteq M \) with \( S_R(A) \subseteq P \) and \( S_M(N) \subseteq P \). Then:

1. If \( N_P \) is a submodule of \( M_P \), then \( N \) is a submodule of \( M \).
2. If \( A_P \) is an ideal of \( R_P \), then \( A \) is an ideal of \( R \).

Proof. (1) Let \( \xi, \eta \in N \), then \( \frac{\xi}{1}, \frac{\eta}{1} \in N_P \) and then \( \frac{\xi - \eta}{1} = \frac{x - y}{1} \in N_P \), so \( q(x - y) \in N \), for some \( q \notin P \). If \( x - y \notin N \), then \( q \in S_M(N) \subseteq P \), that is a contradiction. Hence \( x - y \in N \). If \( r \in R \) is any element then \( \frac{r \xi}{1} = \frac{rx}{1} \in N_P \), so \( prx \in N \), for some \( p \notin P \). If \( rx \notin N \), then we get \( p \in S_M(N) \subseteq P \), that is a contradiction and thus \( rx \in N \). Hence \( N \) is a submodule of \( M \).

By using the same technique we can prove (2).

**Proposition 2.5.** Let \( M \) be an \( R \)-module and \( N \) a primal submodule of \( M \). If \( P \) is a maximal ideal of \( R \) such that \( S_M(N) \subseteq P \), then \( N_P \) is a primal submodule of \( M_P \).

Proof. As \( N \) is primal, it is a proper submodule of \( M \), so by [5, Proposition 2.17], \( N_P \) is a proper submodule of \( M_P \) and by [6, Proposition 2.20], we have \( S_{M_P}(N_P) = (S_M(N))_P \) and as \( N \) is primal, \( S_M(N) \) is an ideal of \( R \) (note that \( S_M(N) \) is a proper ideal since \( 1 \notin S_M(N) \)), then we have \( (S_M(N))_P \) is an ideal of \( R_P \), that is, \( S_{M_P}(N_P) \) is an ideal of \( R_P \). Hence \( N_P \) is a primal submodule of \( M_P \).

**Proposition 2.6.** Let \( M \) be an \( R \)-module and \( P \) is a maximal ideal of \( R \).
If $\overline{N}$ is a primal submodule of $M_P$, then there exists a primal submodule $N$ of $M$ such that $S_M(N) \subseteq P$ and $\overline{N} = N_P$.

Proof. By [5, Proposition 2.16], we have $\overline{N} = N_P$, for the submodule $N = \{x \in M : \frac{r}{p} \in \overline{N}\}$ of $M$. By [6, Lemma 2.27], we have $S_M(N) \subseteq P$ and by [6, Proposition 2.20], $S_{M_P}(N_P) = (S_M(N))_P$. As $N_P$ is primal we have $S_{M_P}(N_P)$ is an ideal of $R_P$ and thus $(S_M(N))_P$ is an ideal of $R_P$, so that by Lemma 2.4, we have $S_M(N)$ is an ideal of $R$. Hence $N$ is a primal submodule of $M$.

Combining Proposition 2.5 and Proposition 2.6 we get the following theorem.

**Theorem 2.7.** Let $M$ be an $R-$module and $P$ a maximal ideal of $R$, then there is a one to one correspondence between the set of all primal submodules $N$ of $M$ for which $S_M(N) \subseteq P$ and the primal submodules of $M_P$.

Proof. Let $Q = \{N : N$ is a primal submodule of $M$ for which $S_M(N) \subseteq P\}$ and $\overline{Q} = \{\overline{N} : \overline{N}$ is a primal submodule of $M_P\}$. Define $f : Q \rightarrow \overline{Q}$ as: if $N \in Q$, then $N$ is a primal submodule of $M$ such that $S_M(N) \subseteq P$, so by Proposition 2.5, we have $N_P$ is a primal submodule of $M_P$ and thus $N_P \in \overline{Q}$, so we set $f(N) = N_P$, then one can easily prove that $f$ defines a one to one correspondence between $Q$ and $\overline{Q}$.

**Proposition 2.8.** Let $M$ be an $R-$module and $P$ is a maximal ideal of $R$, then we have $(\bigcap pSpec(M))_P \subseteq \bigcap pSpec(M_P)$.

Proof. Let $\frac{r}{p} \in (\bigcap pSpec(M))_P$, for $x \in M$ and $p \notin P$. Then $qx \in \bigcap pSpec(M)$, for some $q \notin P$. Let $\overline{N} \in pSpec(M_P)$, that is $\overline{N}$ is a primal submodule of $M_P$. Then by Proposition 2.6, we have $\overline{N} = N_P$, for the primal submodule $N = \{m \in M : \frac{m}{p} \in \overline{N}\}$ of $M$ with $S_M(N) \subseteq P$, so that $N \in pSpec(M)$ and thus $qx \in N$, from which we get $\frac{r}{p} = \frac{q}{q} \frac{r}{p} = \frac{qr}{qp} \in N_P = \overline{N}$ and so $\frac{r}{p} \in \bigcap pSpec(M_P)$ and thus we have $(\bigcap pSpec(M))_P \subseteq \bigcap pSpec(M_P)$.

**Corollary 2.9.** Let $M$ be an $R-$module. If $M_P$ is primally reduced for every maximal ideal $P$ of $R$, then $M$ is primally reduced.

Proof. Let $P$ be any maximal ideal of $R$, then $M_P$ is primally reduced, that is $\bigcap pSpec(M_P) = 0$. Then by Proposition 2.8, we get $(\bigcap pSpec(M))_P = 0$. So by [5, Corollary 2.3], we get $\bigcap pSpec(M) = 0$ and thus $M$ is primally reduced.

Next, we introduce the following definition.

**Definition 2.10.** Let $M$ be an $R-$module. The Jacobson spectrum of $M$, denoted by $jpSpec(M)$, is defined as the set $jpSpec(M) = \{N \in pSpec(M) : S_M(N) \subseteq Rad_j(R)\}$, where $Rad_j(R)$ is the Jacobson radical of $R$ and we say $M$ is radically reduced if $\bigcap jpSpec(M) = 0$. 
Proposition 2.11. If $M$ is an $R$–module and $P$ is a maximal ideal of $R$, then $\bigcap p\text{Spec}(M_P) \subseteq (\bigcap jp\text{Spec}(M))_P$.

Proof. Let $x \in \bigcap p\text{Spec}(M_P)$, where $x \in M, p \notin P$. Now, let $N \in jp\text{Spec}(M)$, then $N \subseteq p\text{Spec}(M)$ and $S_M(N) \subseteq \text{Rad}_j(R) \subseteq P$. By Proposition 2.5, $N_P$ is a primal submodule of $M_P$, that is $N_P \in p\text{Spec}(M_P)$, so that $x \in N_P$. Then by [6, Lemma 2.1], we get $x \in N$, so that $x \in \bigcap jp\text{Spec}(M)$ and thus $x \in (\bigcap jp\text{Spec}(M))_P$. Hence $\bigcap p\text{Spec}(M_P) \subseteq (\bigcap jp\text{Spec}(M))_P$.

Corollary 2.12. A radically reduced $R$–module $M$ is primally reduced.

Proof. We have $\bigcap jp\text{Spec}(M) = 0$. Let $P$ be any maximal ideal of $R$. By Proposition 2.11, we get $\bigcap p\text{Spec}(M_P) \subseteq (\bigcap jp\text{Spec}(M))_P = 0$, that means $\bigcap p\text{Spec}(M_P) = 0$, so that $M_P$ is a primally reduced module for each maximal ideal $P$ of $R$ and thus by Corollary 2.9, we have $M$ is primally reduced.

Corollary 2.13. Let $M$ be an $R$–module and $P$ is a maximal ideal of $R$. If $M$ is a multiplication $R$–module, then $M_P$ is a multiplication $R_P$–module.

Proof. Let $N$ be a submodule of $M_P$, then by [6, Lemma 2.27], $N = N_P$ and $S_M(N) \subseteq P$, for the submodule $N = \{x \in M : \frac{x}{1} \in N\}$ of $M$. But then as $M$ is a multiplication $R$–module, there exists an ideal $I$ of $R$ such that $N = IM$, from which we get $N_P = (IM)_P = IM_P$ (by [5, Corollary 2.3]). Hence $M_P$ is a multiplication $R_P$–module.

Theorem 2.14. Let $M$ be a multiplication $R$–module and $N$ a primal submodule of $M$, then the following statements are equivalent.

1. $N$ is $S_M(N)$–locally prime.
2. $(N : M)$ is a prime ideal of $R$.
3. $N = PM$, for some prime ideal $P$ of $R$ with $\text{Ann}(M) \subseteq P$.

Proof. (1)$\Rightarrow$(2). Let $N$ be $S_M(N)$–locally prime. As $N$ is primal, $S_M(N)$ is a proper ideal of $R$, so that $S_M(N) \subseteq P$, for some maximal ideal $P$ of $R$. By Corollary 2.13, $M_P$ is a multiplication $R_P$–module, so by [11, Lemma 2.1], we have $(N_P : M_P)$ is a prime ideal of $R_P$ and since $S_M(N) \subseteq P$, by [5, Theorem 2.21], we get $(N : M)_P$ is a prime ideal of $R_P$ and by [5, Proposition 2.24], we get $(N : M)$ is a prime ideal of $R$.

(2)$\Rightarrow$(3). The proof follows from [11, Lemma 2.1].

(3)$\Rightarrow$(1). By [11, Lemma 2.1], $N$ is prime and by [5, Proposition 2.10], $N$ is $S_M(N)$–locally prime.

Theorem 2.15. Let $M$ be an $R$–module and $N$ a proper submodule of $M$ such that $S_M(0) \subseteq (N : M)$. Then $N$ is weakly prime if and only if $S_M(N) \subseteq (N : M)$.

Proof. Suppose that $N$ is weakly prime. Let $r \in S_M(N)$, then $rx \in N$, for some $x \notin N$. If $rx = 0$, then $rx \in \{0\}$ and as $x \notin N$, we have $x \neq 0$, that
is \( x \notin \{0\} \), so that \( r \in S_M(0) \) and thus \( r \in (N : M) \) and if \( 0 \neq rx \in N \) and since \( x \notin N \) and \( N \) is weakly prime we get \( rM \subseteq N \), that is, \( r \in (N : M) \), so that \( S_M(N) \subseteq (N : M) \). Conversely, suppose that \( S_M(N) \subseteq (N : M) \). Let \( 0 \neq rx \in N \), where \( r \in R \) and \( x \in M \). If \( x \notin N \), then \( r \in S_M(N) \subseteq (N : M) \), so that \( rM \subseteq N \). Hence \( N \) is a weakly prime submodule of \( M \).

**Theorem 2.16.** Let \( R \) be a local ring with \( P \) as its unique maximal ideal and \( M \) be an \( R \)–module. The following statements are equivalent.

1. \( M \) is radically reduced.
2. \( M \) is primally reduced.
3. \( M \) is reduced.

Proof. Since \( j\text{p} \text{Spec}(M) \subseteq p\text{Spec}(M) \subseteq \text{Spec}(M) \), so \( \bigcap \text{Spec}(M) \subseteq \bigcap j\text{p} \text{Spec}(M) \). Then clearly, we have \( (1) \Rightarrow (2) \Rightarrow (3) \).

To show \( (3) \Rightarrow (1) \). Let \( M \) be reduced, so that \( \bigcap \text{Spec}(M) = 0 \). To show \( \bigcap j\text{p} \text{Spec}(M) = 0 \). Let \( x \in \bigcap j\text{p} \text{Spec}(M) \). Let \( N \in \text{Spec}(M) \), that means \( N \) is a prime submodule of \( M \) and then by [1], we have \( N \) is a primal submodule of \( M \), so that \( N \in p\text{Spec}(M) \) and \( S_M(N) \) is a (proper) ideal of \( R \) and thus \( S_M(N) \subseteq P(= \text{Rad}_j(R)) \), which means that \( N \in j\text{p} \text{Spec}(M) \). Hence we get that \( x \in N \) and that gives \( x \in \bigcap \text{Spec}(M) = 0 \), thus we get \( x = 0 \), that means \( \bigcap j\text{p} \text{Spec}(M) = 0 \). Hence \( M \) is radically reduced.

**Proposition 2.17.** Let \( M \) be a multiplication and a locally reduced \( R \)–module and \( N \) a submodule of \( M \). If \( M \) is a primally reduced \( R \)–module, then \( N \bigcap \text{Ann}(N)M = 0 \).

Proof. We have \( \bigcap \text{pSpec}(M) = 0 \). If \( P \) is any maximal ideal of \( R \), then by [6, Proposition 2.3] we have \( M_P \) is a multiplication \( R \)–module and as \( M \) is locally reduced, we have \( M_P \) is reduced. Now, by [11, Lemma 2.3], we have \( N_P \bigcap \text{Ann}(N_P)M_P = 0 \). By [6, Proposition 2.4], we have \( (\text{Ann}(N))_P \subseteq \text{Ann}(N_P) \), so by using [5, Corollary 2.3], we have \( [N \bigcap \text{Ann}(N)M]_P = N_P \bigcap \text{Ann}(N)_P M_P \subseteq N_P \bigcap \text{Ann}(N_P)M_P = 0 \). So we get \( N \bigcap \text{Ann}(N)M = 0 \).

**References**


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Received: December, 2013