Pure Injective Modules Relative to Torsion Theories

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Abstract

Let τ be a hereditary torsion theory on the category $\text{Mod-}R$ of right R -modules. A right R -module M is called pure τ -injective if it is injective with respect to every pure exact sequence having a τ -torsion cokernel. Every module has a pure τ -injective envelope. A module M is called purely quasi τ -injective if it is fully invariant in its pure τ -injective envelope. A module M is called quasi pure τ -injective if maps from τ -dense pure submodules of M into M are extendable to endomorphisms of M . The class of pure τ -injective modules is properly contained in the class of purely quasi τ -injective modules which is in turn properly contained in the class of quasi pure τ -injectives. A torsion theoretic version of each of the concepts of regular and pure semisimple rings is characterized using the above generalizations of pure injectivity.

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1 Introduction

We will denote by R an associative ring with a nonzero identity and by $\tau = (\mathcal{T}, \mathcal{F})$ a hereditary torsion theory on the category $\text{Mod-}R$ of right R -modules. Unless otherwise stated, all modules considered will be right unital R -modules. A submodule N of a module M is said to be τ -dense in M (denoted $N \leq^{\tau d} M$) if M/N is τ -torsion. The largest τ -torsion submodule of a module M is denoted $t(M)$ and has the property that $M/t(M)$ is τ -torsion free. The intersection of any finite number of τ -dense submodules is again τ -dense. If N and K are submodules of a module M such that $N \leq^{\tau d} M$ then $N \cap K \leq^{\tau d} K$.

Any submodule that contains a τ -dense submodule is itself τ -dense. An R -module M is called τ -injective if it is injective with respect to every short exact sequence having a τ -torsion cokernel. This is equivalent to saying that every map from a τ -dense right ideal of R to M can be extended to a map $R \rightarrow M$ (generalized Baer criterion). Any module M has a τ -injective envelope, i.e. a τ -injective module that contains M both as τ -dense and an essential submodule. A module M is called τ -quasi injective if every map from a τ -dense submodule of M into M can be extended to an endomorphism of M . This is equivalent to saying that M is fully invariant in its τ -injective envelope. For preliminaries about torsion theories, we refer to [3].

A short exact sequence $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ of right R -modules is called pure exact if it remains exact when tensored with any (finitely presented) left R -module [5]. A module M is called N -pure injective if it is injective with respect to every pure exact sequence $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$. The module M is called pure injective if it is N -pure injective for every R -module N if and only if it is a direct summand of every module containing it as a pure submodule. The module M is called quasi pure injective if it is M -pure injective [8]. Given a class of modules $\mathcal{X}$, a map $f : M \rightarrow X$, where $X \in \mathcal{X}$, is called an $\mathcal{X}$ -preenvelope of the module M if for any $X' \in \mathcal{X}$ every map $f' : M \rightarrow X'$ factors through X . If furthermore, when $X = X'$ and $f = f'$, the only such factorizations are automorphisms we say that $f : M \rightarrow X$ is an $\mathcal{X}$ -envelope. If $f : M \rightarrow X$ is an $\mathcal{X}$ -envelope we sometimes say that X is an $\mathcal{X}$ -envelope of M . If every module has an $\mathcal{X}$ -(pre)envelope we say that the class $\mathcal{X}$ is (pre)enveloping. An envelope of a module when it exists is unique up to isomorphism (p.129 of [6]). The class of pure injective modules is enveloping [9]. The pure injective envelope of a module M is denoted $PE(M)$. A module is called purely quasi injective if it is fully invariant in its pure injective envelope [2]. Every purely quasi injective module is quasi pure injective but not conversely [2].

In this paper we generalize the above concepts to a torsion theoretic setting. We say that a module is pure τ -injective if it is injective with respect to every pure exact sequence that has a τ -torsion cokernel. We will prove that the class of pure τ -injective modules is enveloping. Then we study modules that are fully invariant in their pure τ -injective envelopes. The last section is about characterizing certain types of rings in terms of pure τ -injective and purely quasi τ -injective modules.

2 Relative Pure τ -Injectivity

Definition 2.1 *Let M and N be right R -modules. The module M is called:*

1. *N -pure τ -injective if M is injective with respect to all pure exact se-*

quences $0 \rightarrow N' \rightarrow N \rightarrow N/N' \rightarrow 0$ such that N/N' is τ -torsion.

2. *quasi pure τ -injective if M is M -pure τ -injective. pure τ -injective if it is N -pure τ -injective for every R -module N .*

Of course the class of N -pure τ -injective modules is closed under direct products, direct summands and finite direct sums. Any N -pure injective module is N -pure τ -injective. If τ is the torsion theory in which every R -module is τ -torsion then a module is N -pure τ -injective if and only if it is N -pure injective.

For the next Theorem, consider the following condition on a module N and an exact sequence $0 \rightarrow M \xrightarrow{\alpha} P$:

(*) There is a map $\beta : N \rightarrow P$ such that $P = \alpha(M) + \beta(N)$ and $\beta^{-1}(\alpha(M))$ is a pure and τ -dense submodule of N .

Theorem 2.2 *For R -modules M and N , the following are equivalent:*

1. *M is N -pure τ -injective.*
2. *Every exact sequence $0 \rightarrow M \xrightarrow{\alpha} P$ that satisfies condition (*) splits.*
3. *Every pure exact sequence $0 \rightarrow M \xrightarrow{\alpha} P$ that satisfies condition (*) splits.*
4. *Every exact sequence $0 \rightarrow M \xrightarrow{\alpha} P$ that satisfies condition (*) and has a τ -torsion cokernel splits.*
5. *Every pure exact sequence $0 \rightarrow M \xrightarrow{\alpha} P$ that satisfies condition (*) and has a τ -torsion cokernel splits.*

Proof. (1) $\Rightarrow$ (2) Put $K = \beta^{-1}(\alpha(M))$. By hypothesis the map $\alpha^{-1}\beta : K \rightarrow M$ extends to $\gamma : N \rightarrow M$. By assumption $P = \alpha(M) + \beta(N)$, i.e. for each $x \in P$ there exist m and n in M and N respectively, such that $x = \alpha(m) + \beta(n)$. Define $\delta : P \rightarrow M$ by $\delta(x) = m + \gamma(n)$. To prove δ is well-defined let $x = \alpha(m) + \beta(n) = \alpha(m') + \beta(n')$, hence $\alpha(m - m') = \beta(n' - n) \in \alpha(M) \cap \beta(N)$ so that $m - m' = \alpha^{-1}\beta(n' - n) = \gamma(n' - n)$. Thus $m + \gamma(n) = m' + \gamma(n')$. It is clear that $\delta\alpha = 1_M$. (2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are trivial. (5) $\Rightarrow$ (1) Let K be a pure τ -dense submodule of N and $f : K \rightarrow M$ be a given map and form the pushout diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{i} & N \\ & & \downarrow f & & \downarrow \beta \\ 0 & \longrightarrow & M & \xrightarrow{\alpha} & P \end{array}$$

where $P = (M \oplus N)/W$, with $W = (k, f(k)), k \in K$. If the second row splits then we get a map $\alpha' : P \rightarrow M$ such that $\alpha'\alpha = 1_M$, hence $\alpha'\alpha i = f$ gives an extension of f . So we need to show that condition (*) together with the conditions in (5) hold. Now as the diagram is a pushout and the upper sequence is pure exact with a τ -torsion cokernel, so is the lower sequence. But it is easy to see that $P = \alpha(M) + \beta(N)$ and $\beta^{-1}(\alpha(M)) = K$ which is pure and τ -dense in N . Hence the result. $\square$

Corollary 2.3 *A module is pure τ -injective if and only if it is a direct summand of every module containing it as a pure τ -dense submodule.*

The following Proposition shows that the class of pure τ -injective modules is preenveloping.

Proposition 2.4 *Every module can be embedded as a pure τ -dense submodule of a pure τ -injective module. Such an embedding is a pure τ -injective preenvelope.*

Proof. Consider $t(PE(M)/M)$, the τ -torsion part of $PE(M)/M$ and let P be the complete inverse image of $t(PE(M)/M)$ under the natural map $PE(M) \rightarrow PE(M)/M$. It is clear that P is a submodule of $PE(M)$ containing M as a pure τ -dense submodule and that $PE(M)/P$ is τ -torsion free. Now consider the row exact commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \xrightarrow{i} & N & \longrightarrow & N/L \longrightarrow 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h \\ 0 & \longrightarrow & P & \longrightarrow & PE(M) & \longrightarrow & PE(M)/P \longrightarrow 0 \end{array}$$

where L is τ -dense and pure in N , f is a given map, g is the map whose existence is ensured by pure injectivity of $PE(M)$ and h is the map defined by $x + L \mapsto g(x) + P$. Since $N/L \in \mathcal{T}$ and $PE(M)/P \in \mathcal{F}$ we must have $h = 0$. Hence $g(N) \subseteq P$ and so P is pure τ -injective. It remains to show that P is a pure τ -injective preenvelope of M . But any map $M \rightarrow P'$ into a pure τ -injective module P' can be extended to a map $P \rightarrow P'$ since M is pure and τ -dense in P . $\square$

Next we show that every module admits a pure τ -injective envelope, but before this we need a lemma due to Enochs and Jenda (p.139 of [6]):

Lemma 2.5 *Let $\mathcal{X}$ be a class of R -modules that is closed under summands, and M an R -module. Suppose that M has an $\mathcal{X}$ -preenvelope and if $((X_i), (\phi_{ji}))$ is any well-ordered inductive system of $\mathcal{X}$ -preenvelopes of M , then for some $\mathcal{X}$ -preenvelope $M \rightarrow X$ there is a factorization $M \rightarrow \varinjlim X_i \rightarrow X$. Then M has an $\mathcal{X}$ -envelope.*

Theorem 2.6 *Every module has a pure τ -injective envelope.*

Proof. Given an inductive system $(P_i, (\phi_{ji}))$ of pure τ -injective envelopes of a module M , the following sequences

$$0 \rightarrow M \rightarrow P_i \rightarrow P_i/M \rightarrow 0$$

are all pure exact with τ -torsion cokernels. The limit of this system of sequences is again pure exact with a τ -torsion cokernel

$$0 \rightarrow M \rightarrow \varinjlim P_i \rightarrow \varinjlim P_i/M \rightarrow 0$$

Hence any map $M \rightarrow P$ into a pure τ -injective preenvelope of M is extendable to a map $\varinjlim P_i \rightarrow P$. So M has a pure τ -injective envelope by Lemma 2.5. $\square$

Let us denote the pure τ -injective envelope of the module M by $P_\tau(M)$. We can now study modules that are fully invariant in their pure τ -injective envelopes, compare with [2].

Definition 2.7 *Let M and N be right R -modules. The module M is called:*

1. *Purely N - τ -injective if for any map $f : P_\tau(N) \rightarrow P_\tau(M)$, we have $f(N) \subseteq M$.*
2. *Purely quasi τ -injective if M is purely M - τ -injective, i.e. if M is fully invariant in its pure τ -injective envelope.*

It is easy to see that a module M is purely N - τ -injective if and only if for any map $f : N \rightarrow P_\tau(M)$, we have $f(N) \subseteq M$. Any purely N -injective is purely N - τ -injective. If τ is the improper torsion theory then a module M is purely N - τ -injective (respectively, purely quasi τ -injective) if and only if M is purely N -injective (respectively, purely quasi injective).

Every pure τ -injective (and hence every pure injective) module is purely N - τ -injective for every R -module N . The converse is not true in general as one may take a von Neumann regular ring R that is not noetherian so that there is a quasi injective R -module which is not injective [4]. Certainly this serves as a counter example where τ is the improper torsion theory. Another example for a ring which is not regular is example 2 of [2].

Proposition 2.8 *For any two modules M and N , if M is purely N - τ -injective then M is N -pure τ -injective.*

Proof. Let $f : K \rightarrow M$ be a homomorphism from a pure τ -dense submodule of N into M . Therefore f can be extended to a map $g : N \rightarrow P_\tau(M)$ and hence by assumption, $g(N) \subseteq M$. Hence g is the desired extension of f . $\square$

Corollary 2.9 *Any purely quasi τ -injective module is quasi pure τ -injective.*

The converse of corollary 2.9 (and Proposition 2.8) is not true in general as in the following example which is given with a different proof in example 3 of [2]:

Example 2.10 It is easy to check that $\mathbb{Z}$ is quasi pure injective but not pure injective hence quasi pure τ -injective but not pure τ -injective, where τ is the improper torsion theory on $\text{Mod-}\mathbb{Z}$. But $\mathbb{Z}$ is not fully invariant in $PE(\mathbb{Z})$. Indeed if we take any element a in $PE(\mathbb{Z})$ that is not in $\mathbb{Z}$ and define a map $\mathbb{Z} \rightarrow PE(\mathbb{Z})$ by $1 \mapsto a$ which extends to an endomorphism of $PE(\mathbb{Z})$ then the image of $\mathbb{Z}$ under this extension, is not contained in $\mathbb{Z}$.

Proposition 2.11 *The direct sum $M_1 \oplus M_2$ of two modules M_1 and M_2 is purely quasi τ -injective if and only if M_i is purely M_j - τ -injective for each $i, j = 1, 2$.*

Proof. Suppose that $M_1 \oplus M_2$ is purely quasi τ -injective. For arbitrary i and j , let $f : M_i \rightarrow P_\tau(M_j)$ be a map. Now f can be easily extended to a map $M_1 \oplus M_2 \rightarrow P_\tau(M_1) \oplus P_\tau(M_2)$ which we again call it f . By assumption $f(M_1 \oplus M_2) \subseteq M_1 \oplus M_2$. Hence $f(M_i) \subseteq M_j$. Conversely let $f : M_1 \oplus M_2 \rightarrow P_\tau(M_1 \oplus M_2)$ be a map. Restrict f to M_i and follow it by the projection $M_1 \oplus M_2 \rightarrow M_j$ to get a map $f_{ij} : M_i \rightarrow M_j$. Since $f_{ij}(M_i) \subseteq M_j$, we must have $f(M_1 \oplus M_2) \subseteq M_1 \oplus M_2$. $\square$

3 Characterization of Rings

Regular rings R are characterized by any one of the following:

1. Every ideal of R is pure. [7]
2. Every exact sequence of R -modules is pure exact. [7]
3. Every pure injective R -module is injective. [10]
4. Every purely quasi injective R -module is quasi injective. [2]
5. Every quasi pure injective R -module is quasi injective. [1]

Here we give a torsion theoretic version of the above characterizations.

Lemma 3.1 *For an exact sequence $0 \rightarrow K \rightarrow L \rightarrow N \rightarrow 0$ such that N is τ -torsion, the following statements are equivalent:*

1. *The sequence is pure exact.*
2. *Every pure τ -injective R -module is injective with respect to the given sequence.*

Proof. (1) $\Rightarrow$ (2) Clear. (2) $\Rightarrow$ (1) In particular, every pure injective R -module is injective with respect to the sequence, so by 34.7 of [10] the result follows. $\square$

Theorem 3.2 *For a ring R the following statements are equivalent:*

1. *Any exact sequence $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$ such that R/I is τ -torsion, is pure exact.*
2. *Any exact sequence $0 \rightarrow K \rightarrow L \rightarrow N \rightarrow 0$ such that N is τ -torsion, is pure exact.*
3. *Every pure τ -injective R -module is τ -injective.*
4. *Every pure τ -injective R -module is τ -quasi injective.*
5. *Every purely quasi τ -injective R -module is τ -quasi injective.*
6. *Every quasi pure τ -injective R -module is τ -quasi injective.*

Proof. (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (4) and (2) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (4) are trivial. (3) $\Rightarrow$ (2) is clear by Lemma 3.1. (4) $\Rightarrow$ (3) Let M be a pure τ -injective R -module and put $P = P_\tau(R)$ so that $M \oplus P$ is pure τ -injective, and hence τ -quasi injective by assumption. If we could show that M is τ -injective with respect to R then it would be τ -injective by the generalized Baer criterion. So let $I \leq^{\tau d} R$ hence $I \leq^{\tau d} P$ and then $M \oplus I \leq^{\tau d} M \oplus P$. Now any map $f : I \rightarrow M$ can be easily extended to a map $M \oplus I \rightarrow M \oplus P$ given by $(m, i) \mapsto (f(i), 0)$. Since $M \oplus P$ is τ -quasi injective this map is extendable to an endomorphism of $M \oplus P$. Restrict this endomorphism to R and follow it by the projection map $M \oplus P \rightarrow M$ to get the desired extension of f . $\square$

Next, we give torsion theoretic version of pure semisimple rings. Recall that a ring R is called pure semisimple if every pure exact sequence of R -modules splits. Pure semisimple rings are characterized by the property that every module is pure injective (see p.529 of [10]). This is equivalent to saying that every R -module is purely quasi injective [2], which is equivalent to the statement that every R -module is quasi pure injective [1]. Now by a τ -pure semisimple ring we mean a ring R over which every pure exact sequence of R -modules with τ -torsion cokernel splits.

Theorem 3.3 *For a ring R the following are equivalent:*

1. *R is τ -pure semisimple.*
2. *Every R -module is pure τ -injective.*
3. *Every R -module is purely quasi τ -injective.*

4. Every R -module is quasi pure τ -injective.

Proof. (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. (1) $\Rightarrow$ (2) Let M be an R -module. Now $0 \rightarrow M \rightarrow P_\tau(M) \rightarrow P_\tau(M)/M \rightarrow 0$ is a pure exact sequence with $P_\tau(M)/M \in \mathcal{T}$, hence the sequence splits, which means that M is pure τ -injective. (4) $\Rightarrow$ (2) For any R -module M with $P = P_\tau(M)$, we have $M \oplus P$ is quasi pure τ -injective by assumption. Consider the map $M \oplus M \rightarrow M \oplus P$ given by $(m', m) \mapsto (m, 0)$. Since $M \oplus M$ is both pure and τ -dense in $M \oplus P$, the above map extends to an endomorphism of $M \oplus P$. Restrict this endomorphism on P and follow it by the projection $M \oplus P \rightarrow M$ to get a map $P \rightarrow M$ that extends the identity map of M . Hence M is a direct summand of P . $\square$

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