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On Primary Decomposition and Polynomial of a Matrix

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Abstract

The goal of this paper is to study some unknown questions on the primary decomposition of matrices over a field K and to give the analogous of some well known results of spectral, algebraic and geometric multiplicity order of an eigenvalue to any P-component of the characteristic polynomial C_A of a matrix A over a field K. More precisely we compute the dimension of the kernel of a polynomial of a square matrix A over any arbitrary commutative field K in terms of its invariant factors. As an application we determine the value of the P-algebraic and P-geometric multiplicity order of any P-component of the characteristic polynomial C_A of a matrix A.

Keywords: Primary decomposition, invariant factors, algebraic multiplicity, geometric multiplicity

1 Introduction

Let K be a field. Let $A \in \mathcal{M}_n(K)$ and P be an irreducible polynomial of $K[X]$. We will say that A is P- primary matrix if the characteristic polynomial C_A of A is a power of P. The Primary decomposition Theorem states that if $A \in \mathcal{M}_n(K)$ is a non zero matrix and $m_A(X) = \prod_{i=1}^{s} P_i^{\alpha_i}$ is the prime decomposition of its minimal polynomial $m_A(X)$ then the matrix A is similar to a block diagonal of P-primary matrices $diag(A_1, A_2, ..., A_s)$. The dimension of sequence vector spaces $Ker\, P^s(A)$- is unknown.

In the first part of this paper, we use some deep results on module theory over a PID to compute the dimension of the kernel of a polynomial of a square matrix A over a commutative field K in terms of its invariant factors.

In the second part, we give the analogous of some well known results of spectral, algebraic and geometric multiplicity order of an eigenvalue, to any P-component of the characteristic polynomial C_A of a matrix A over any arbitrary commutative field K. Some new results on the P-algebraic and P-geometric multiplicity order are also established.

2 Preliminary Notes

Let K be a field. Let M be a finite dimension vector space over K and f a K-endomorphism of M. The vector space M is endowed by a structure of $K[X]$-module via the endomorphism f by $X.m = f(m)$ for any $m \in M$. We will denote by M_f the $K[X]$-module on M induced by f. As the ring $K[X]$ is a PID, then by applying the structure theorem of finitely generated torsion modules over a PID, the very useful following theorem is deduced (see [[6], §2, p. 556],[[8],§ 14], [[1], p. 235] and [3]):

Theorem 2.1 (Rational canonical form) *Let M be a finite-dimensional vector space over a field K and f be a K-endomorphism of M. Let M_f be the $K[X]$-module induced by f then there exists a unique sequence of polynomials $q_1, \cdots, q_r$ such that:*

$$M_f \simeq \frac{K[X]}{(q_1)} \oplus \frac{K[X]}{(q_2)} \oplus \cdots \oplus \frac{K[X]}{(q_r)}$$

and

- $q_i \mid q_{i+1}$
- $q_r = m_f(X)$ *the minimal polynomial of f and* $\prod_{i=1}^{r} q_i = c_f(X)$ *the characteristic polynomial of f.*

The ascending sequence of polynomials $q_1, \cdots, q_r$ are unique and called the invariant factors of f.

If $q_1, \cdots, q_r$ are the invariant factors of f then we will write $IF(f) = (q_1, \cdots, q_r)$.

Let $A \in \mathcal{M}_n(K)$ be a no zero matrix, and for any linear transformation that has matrix A relative to some basis, we denote M_A the $K[X]$-module induced by A. Then by theorem2.1:

$$M_A \simeq \frac{K[X]}{(q_1)} \oplus \frac{K[X]}{(q_2)} \oplus \cdots \oplus \frac{K[X]}{(q_r)}$$

such that $q_i \mid q_{i+1}$, $q_r = m_A(X)$ the minimal polynomial of A and $\prod_{i=1}^r q_i = c_A(X)$ the characteristic polynomial of A. The sequence of polynomials $q_1, \cdots, q_r$ are called the invariant factors of A. The invariant factors of A are unique up similarity. Indeed if $q_1, \cdots, q_r$ are the invariant factors of A then A is similar to a block diagonal matrix $diag(A_1, A_2, ..., A_m)$ where $A_i = Comp(q_i)$ is the companion matrix of q_i.

Let K be a field. Let $A \in \mathcal{M}_n(K)$ and P be an irreducible polynomial of $K[X]$. We will say that A is P-primary matrix if the characteristic polynomial C_A of A is a power of P.

Proposition 2.2 (Primary decomposition Theorem) *Let $A \in \mathcal{M}_n(K)$ be a non zero matrix. Let $m_A(X) = \prod_{i=1}^s P_i^{\alpha_i}$ be the prime decomposition of $m_A(X)$. Let $E_i = Ker\, P_i^{\alpha_i}(A)$. Then the subspaces E_i are invariant under A and A is similar to a block diagonal of P-primary matrices $diag(A_1, A_2, ..., A_s)$.*

Proof. See [[7], Theorem 1.5.1,p29]. ∎

Throughout this paper, E is a finite-dimensional vector space over a field K. If $f \in End_K(E)$, m_f and C_f stand respectively for the minimal and the characteristic polynomial of f.

3 Main Results

This is the main result of this paper.

Theorem 3.1 *Let K be a field. Let $A \in \mathcal{M}_n(K)$ be a non zero matrix and $IF(A) = (q_1, \cdots, q_r)$ its invariant factors. Then*

$$dim_K Ker P(A) = \sum_{i=1}^r deg\,(gcd(P, q_i))$$

for any $P \in K[X]$. In particular $dim_K Ker A$ is the number of i such that $q_i(0) = 0$.

To prove this Theorem we need the following lemmas

Lemma 3.2 *Let u be an endomorphism of a finite dimensional vector space E over K. Assume that $E = \oplus_{i=1}^{n} E_i$ such that E_i are $u-$invariant subspaces of E. Then $u = \oplus_{i=1}^{n} u_i$ with $u_i = res_{E_i} u$ the restriction of u to E_i and*

- $u(x) = \sum_{i=1}^{n} u_i(x_i)$ *for all* $x = \sum_{i=1}^{n} x_i$.
- $P(u) = \oplus_{i=1}^{n} P(u_i)$ *for all* $P \in K[X]$
- $Ker P(u) = \oplus_{i=1}^{n} Ker P(u_i)$

Proof. Easy to prove (see [[8], Proposition 1. 3. 2] and [5]). ∎

Lemma 3.3 *Let R be a PID and let a, b be nonzero elements of R. If $d = (a, b) = gcd\{a, b\}$, then*

$$\{\overline{c} \in R/bR \mid a\overline{c} = \overline{0}\} \simeq R/dR.$$

Proof. Indeed let $M_a := \{\overline{c} \in R/bR \mid a\overline{c} = \overline{0}\}$ clearly M_a is a submodule of the R-module R/bR. Let $b' = \frac{b}{d}$. Then

$$\begin{array}{cccc} \phi : & R & \longrightarrow & M_a \\ & x & \longmapsto & \overline{b'x} \end{array}$$

ϕ is an R-homomorphism. Notice that $a\overline{b'x} = \overline{b\frac{a}{d}x} = \overline{0}$. So $\overline{b'x} \in M_a$.

Furthermore if $\overline{ax} = \overline{0}$ then $ax \in bR$ so $x \in b'R$. Hence ϕ is an onto homomorphism. $Ker\phi = \{x \in R \mid b'x \in bR\} = dR$. Hence $M_a \simeq R/dR$. ∎

Lemma 3.4 *Let $A \in \mathcal{M}_n(K)$ and let M_A be the $K[X]$-module induced by A. If $M_A \simeq K[X]/(q)$. Let $P \in K[X]$,then*

$$Ker(P(A)) \simeq Ker\widetilde{P(X)}$$

where $\widetilde{P(X)} : K[X]/(q) \to K[X]/(q), \overline{T} \mapsto P(X).\overline{T}$

Proof. Let φ denotes the $K[X]$-isomorphism between M_A and $K[X]/(q)$ We have $m \in Ker P(A)$ if and only if $P(A)(m) = 0$ if and only if$\varphi(P(X).m) = \overline{0}$if and only if $\varphi(P(X).m) = \overline{0}$ if and only if $P(X).\varphi(m) = \overline{0}$ if and only if $\widetilde{P(X)}(\varphi(m)) = 0$ if and only if $\varphi(m) \in Ker\widetilde{P(X)}$, where $\widetilde{P(X)} : K[X]/(q) \to K[X]/(q), \overline{T} \mapsto P(X).\overline{T}$ hence $Ker(P(A)) \simeq Ker\widetilde{P(X)}$. ∎

Lemma 3.5 *Let $A \in \mathcal{M}_n(K)$ and let M_A be the $K[X]$-module induced by A. If $M_A \simeq K[X]/(q)$ then for all $P \in K[X]$*

$$Ker(P(A)) \simeq \begin{cases} (0) & if \quad gcd(P, q) = 1 \\ K[X]/(D) & if \quad gcd(P, q) = D \end{cases}$$

Proof. By lemma 3.4 and lemma 3.3 we have $Ker\widetilde{P(X)} \simeq K[X]/(D)$ where $D = gcd(P, q)$. ∎

Now let's give the proof of the theorem 3.1

Proof. Let E be a K-vector space of finite dimension. Let $f \in End_K(E)$ and $\mathcal{B}$ a basis of E such that $mat_{\mathcal{B}}(f) = A$. The space E can be viewed as a K[X]-module $(K[X] \times E \longrightarrow E, (P, x) \longmapsto P.x = P(f)(x))$. Then $E = M_f \simeq \bigoplus_{i=1}^{r} K[X]/(q_i)$ as K[X]-modules, where $q_1, q_2, ..., q_r$ are the invariant factors of A . Hence $E = \oplus_{i=1}^{r} E_i$ where E_i's are f-invariant subspaces and $E_i \simeq K[X]/(q_i)$ as K[X]-modules Hence by lemma 3.2 $f = \oplus_{i=1}^{r} f_i$ and $P(f) = \oplus_{i=1}^{r} P(f_i)$ where $f_i = res_{E_i} f$. So it turns to study the case where f admits one invariant factor (A is companion). By lemma 3.5 $KerP(f_i) \simeq K[X]/(D_i)$ where $gcd(P, q_i) = D_i$. We have by lemma 3.2 $KerP(f) = \oplus_{i=1}^{r} KerP(f_i) \simeq \oplus_{i=1}^{r} K[X]/(D_i)$. Hence $dim_K KerP(f) = \sum_{i=1}^{r} dim_K(K[X]/(D_i)) = \sum_{i=1}^{r} deg\,(D_i) = \sum_{i=1}^{r} deg\,(gcd(P, q_i)$. ∎

4 Generalized algebraic and geometric multiplicity order

Let K be a field. Let Q be a polynomial of $K[X]$ and P be an irreducible polynomial of $K[X]$ which occur in the prime decomposition of Q. We will say that the power polynomial P^s is the P-component of Q if $Q = P^s Q_1$ where Q_1 is a polynomial of $K[X]$ coprime with P. The integer s is said the P-valuation of Q and will be denoted by $\upsilon_P(Q)$.

In order to give the analogous of some well known results of spectral, algebraic and geometric multiplicity order of an eigenvalue. We introduce the P-algebraic and P-geometric multiplicity order relative to any P-component of the characteristic polynomial C_A of the matrix A.

Definition 4.1 *Let $A \in \mathcal{M}_n(K)$. Let C_A be the characteristic polynomial of the matrix A. If P is an irreducible monic factor of C_A then*

- *The P-algebraic multiplicity order of the matrix A (or the algebraic multiplicity order of A at the factor P) is $dim_K KerP(A)^{\upsilon_P(C_A)}$.*

- *The P-geometric multiplicity order of the matrix A (or the geometric multiplicity order of A at the factor P) is $dim_K KerP(A)$.*

Throughout this work we will follow the notations used by the authors of [1]:

1) $\nu_{alg}(P)$ denote the P-algebraic multiplicity order of the matrix A.

2) $\nu_{geom}(P)$ denote the P-geometric multiplicity order of the matrix A.

Proposition 4.2 *Let $f \in End_K(E)$ and $IF(f) = (q_1, \cdots, q_r)$ its invariant factors. Let $P \in K[X]$ be an irreducible monic factor of C_f. If $s_i = \upsilon_P(q_i)$. Then for any positive integer l*

$$dim_K KerP^l(f) = \begin{cases} r \times l \times degP & if \quad l < s_1 \\ (\sum_{i=1}^{k} s_i + (r-k)l)degP & if \quad l \geq s_1 \end{cases}$$

where k is the number of i such that $s_i \leq l$.

Proof. Indeed, by theorem 3.1 $dim_K KerP^l(f) = \sum_{i=1}^{r} deg\,(gcd(P^l,\, q_i)) = \sum_{i=1}^{r} inf(l, s_i)degP$ so we deduce the result. ∎

Corollary 4.3 *Let $f \in End_K(E)$ and $P \in K[X]$ be an irreducible monic factor of C_f. Let $s = \upsilon_P(m_f)$. Then*

$$dim_K KerP^l(f) = \upsilon_P(C_f)\, degP$$

for any positive integer $l \geq s$.

Proof. Indeed, if $t = \upsilon_P(C_f)$ and $IF(f) = (q_1, \cdots, q_r)$ are the invariant factors of f and $l \geq s = s_r$ then $l \geq s_i$ for all $i = 1, \cdots, r$ so by proposition 4.2 r=k hence $dim_K KerP^l(f) = (\sum_{i=1}^{r} s_i)degP = tdegP$ since $\sum_{i=1}^{r} s_i = t$. ∎

Corollary 4.4 *Let $A \in \mathcal{M}_n(K)$. Let C_A be the characteristic polynomial of A. If P is an irreducible monic factor of C_A then P-algebraic multiplicity order of the matrix A is $\upsilon_P(C_A)degP$.*

Proof. Indeed, let f be the endomorphism canonically associated to A. By the corollary 4.3 and since $\upsilon_P(C_f) \geq \upsilon_P(m_f)$ we have $dim_K KerP^t(f) = tdegP$ where $t = \upsilon_P(C_f)$. ∎

Let $f \in End_K(E)$ and $N_k = Kerf^k$. As E is a finite dimension vector space over K, the sequence N_k is stationary. It is well known that if $N_k = N_{k+1}$

then $N_s = N_k$ for any number $s \geq k$. Hence if k is the small number such that $N_k = N_{k+1}$ then the sequence N_m is a strictly increasing sequence in the interval $[0, \ k]$.

Corollary 4.5 *Let $f \in End_K(E)$. Let $P \in K[X]$ be an irreducible monic factor of C_f. Let $s = \upsilon_P(m_f)$. Let $N_k = KerP^k(f)$. Then the sequence N_k is a strictly increasing sequence in the interval $[0, \ s]$ and $N_l = N_s$ for any positive integer $l \geq s$.*

Proof. Indeed, $KerP^s(f) \subseteq KerP^l(f)$ and by corollary 4.3 if $l \geq s = \upsilon_P(m_f)$ then $dim_K KerP^l(f) = dim_K KerP^s(f)$ and hence $KerP^s(f) = KerP^l(f)$ for any positive integer $l \geq s$. ∎

Corollary 4.6 *Let $f \in End_K(E)$ and $IF(f) = (q_1, \cdots, q_r)$ its invariant factors. Let $P \in K[X]$ be an irreducible monic factor of m_f. If $s_i = \upsilon_P(q_i)$ then*

$$\nu_{geom}(P) = \begin{cases} rdegP & if \quad s_1 > 1 \\ (\sum_{i=1}^{k} s_i + (r-k))degP & if \quad s_1 \leq 1 \end{cases}$$

where k is the number of indices i such that $s_i \leq 1$. In particular if $\upsilon_P(m_f) = 1$ then $\nu_{geom}(P) = \upsilon_P(C_f)\, degP$.

Proof. Indeed, if $\upsilon_P(m_f) = 1$ then by corollary 4.3, we have $\nu_{geom}(P) = dim_K KerP(f) = tdegP$. ∎

If the characteristic polynomial C_f of f splits completely (as in the case where K is an algebraically closed field) we refind the classical known results in the following corollary

Corollary 4.7 *Let $f \in End_K(E)$ factors. Let $P \in K[X]$ be an irreducible factor of C_f. Let $s = \upsilon_P(m_f)$. Then $dim_K Ker(f - \lambda I)$ is the number of i such that $q_i(\lambda) = 0$. If further $s = 1$ then the geometric multiplicity order of λ is $\upsilon_P(C_f)$.*

Proof. If $P = X - \lambda$ then by theorem 3.1 we have $dim_K Ker(f - \lambda I) = \sum_{i=1}^{r} deg\,(gcd(X - \lambda, q_i)) = number\, of\, i\, such\, that\, q_i(\lambda) = 0$.
If $s = 1$ we apply the corollary 4.6. ∎

Proposition 4.8 *Let $f \in End_K(E)$. Let $P \in K[X]$ be an irreducible monic factor of C_f. Then $\nu_{alg}(P) = \nu_{geom}(P)$ if and only if $\upsilon_P(m_f) = 1$.*

Proof. Indeed, if $t = \upsilon_P(C_f)$ and $\upsilon_P(m_f) = 1$ then by corollary 4.6 $\nu_{geom}(P) = tdegP = \nu_{alg}(P)$. Conversely if $\nu_{alg}(P) = \nu_{geom}(P)$ then $(\sum_{i=1}^{k} s_i + (r-k))degP = tdegP$ and hence $\sum_{i=1}^{k} s_i + (r-k) = t$. If $k < r$ then $\sum_{i=k+1}^{r} s_i = r - k$ and $1 < s_i$ for any $k < i$. But the sum $\sum_{i=k+1}^{r} s_i = r - k$ contradicts $1 < s_i$ for any $k < i$. Therefore $k = r$ and $s_r \leq 1$. As P is a component of the characteristic polynomial C_f of f we conclude that $\upsilon_P(m_f) = s_r = 1$. ∎

Proposition 4.9 *Let $f \in End_K(E)$. Let $P \in K[X]$ be an irreducible monic factor of C_f. Then $\nu_{geom}(P) = degP$ if and only if $\upsilon_P(m_f) = \upsilon_P(C_f)$.*

Proof. Indeed $\nu_{geom}(P) = l\, degP$ where $l = \sum_{i=1}^{k} s_i + (r-k)$ and k is the number of indices i such that $s_i \leq 1$. If $\nu_{geom}(P) = degP$ then $l = 1$ hence if $k = r$ then $\sum_{i=1}^{r} s_i = 1$ then $s_r = 1$ and $s_i = 0, \ \forall i \leq r-1$ since the sequence s_i is non negative and increasing. So $\upsilon_P(m_f) = 1 = \upsilon_P(C_f)$.
If $k < r$ then $l = \sum_{i=1}^{k} s_i + (r-k) = 1$ implies that $k = r-1$ and $s_i = 0 \ \forall i \leq r-1$. Hence $\upsilon_P(C_f) = \sum_{i=1}^{r} s_i = s_r = \upsilon_P(m_f)$.
Conversely if $\upsilon_P(m_f) = \upsilon_P(C_f)$ then $\sum_{i=1}^{r-1} s_i = 0$ so $s_i = 0 \ \forall i \leq r-1$. If $k < r$ then $k = r-1$ so $\nu_{geom}(P) = (\sum_{i=1}^{r-1} s_i + (r-(r-1)))degP = degP$. If $k = r$ then $s_r \leq 1$ and since P is a component of the characteristic polynomial C_f of f we conclude that $s_r = 1$ and by consequence $l = 1$ and $\nu_{geom}(P) = degP$. ∎

References

[1] W. A. Adkins and S. H. Weintraub, *Algebra: an approach via module theory*, Graduate Texts in Mathematics, 136, Springer-Verlag, New York, 1992.

[2] J. M. Arnaudiès, J. Bertin, *Groupes, Algèbre et Géomètrie*, Tome1, Ellipses, Paris, 1994.

[3] M. E. Charkani and S. Bouarga *On Sylvester operator and Centralizer of matrices*, Submitted paper in "Annales Mathématiques du Quebec" (August 2013).

[4] W. H. Greub, *Linear Algebra,* Third Edition, Springer-Verlag, New York, 1967.

[5] P. Lancaster, M. Tismenetsky,*The Theory of Matrices,* 2nd Edition, Academic Press, New York, 1985.

[6] S. Lang, *Algebra*, Graduate Texts in Mathematics springer, revised third edition, 2002.

[7] K. O'Meara, J. Clark, C. Vinsonhaler, *Advanced Topics in Linear Algebra: Weaving Matrix Problems through the Weyr Form* , Oxford, New York: Oxford University Press, 2011.

[8] V. Prasolov, *Problems and Theorems in Linear Algebra*, American Mathematical Society; 1st edition Translations of Mathematical Monographs, Vol. 134,1994.

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