The Algebras $M_{n,n}(E)$ and $M_n(E) \otimes E$

in Positive Characteristic

Sérgio M. Alves

Departamento de Ciências Exatas e Tecnológicas
Universidade Estadual de Santa Cruz
Ilhéus, BA, Brasil

Abstract

In this paper we discuss the PI-equivalence of the algebras $M_n(E) \otimes E$
and $M_{n,n}(E)$.

Mathematics Subject Classification: 16R10, 16R40, 15A75

Keywords: PI Theory, Verbally prime algebra, Grassmann algebras

1 Introduction

In characteristic zero the algebras $M_n(E) \otimes E$ and $M_{n,n}(E)$ appear in the list of non-trivial verbally-prime algebras. Verbally-prime algebras play a proeminent role in Kemer’s results about structure of the varieties of associative algebras in characteristic zero (see [8] for a detailed account of it). Recently, some new results have been obtained about the behavior of polynomial identities satisfied by the verbally-prime algebras, giving a new proof that the so called Kemer’s Tensor Product Theorem is no longer valid in positive characteristic (see [1], [2], [3], [4], [5], [6] and [7] for a detailed account of this research).

2 Preliminary Notes

Verbally prime algebras form one of the most important classes of algebras in the PI theory. The structure theory of $T$-ideals developed by Kemer classified
the verbally prime algebras over a field of characteristic zero (see, for example, [10] for a detailed account of it).

We recall some of the main definitions and notations that will be used in what follows. Unless stated otherwise, we consider associative unitary algebras. Throughout this text, $K$ is a fixed infinite field with char $K ≠ 2$. The algebras, vector spaces and tensor products will be considered over $K$. We denote by $K\langle X \rangle$ the free associative algebra of infinite rank. The elements of $K\langle X \rangle$ are called polynomials. We refer to the book of Drensky [8] for background information on PI algebras.

We recall that an algebra $A$ is verbally prime if its $T$-ideal is prime in the class of all $T$-ideals in the free associative algebra.

Let $A$ be an algebra and $f \in K\langle X \rangle$ a polynomial without constant term. Then $f(x_1, \ldots, x_m)$ is a central polynomial for $A$ whenever $f(a_1, \ldots, a_m)$ is central for every $a_1, \ldots, a_m \in A$. The set $C(A)$ of all central polynomials for $A$ is a vector subspace of $K\langle X \rangle$, which is closed under algebra endomorphisms of $K\langle X \rangle$. A vector subspace of $K\langle X \rangle$ with this property is called $T$-space. $C(A)$ is the $T$-space of the central polynomials for $A$.

Let $E$ be the Grassmann (or exterior) algebra of a vector space $V$ with basis $\{e_1, e_2, \ldots\}$. Then 1 and the monomials $e_{i_1} e_{i_2} \cdots e_{i_k}$ $(i_1 < i_2 < \ldots < i_k, \ k \geq 1)$ form a basis of $E$, and the multiplication in $E$ is induced by $e_i e_j = -e_j e_i$, $e_i^2 = 0$, for all $i, j = 1, 2, \ldots$. We denote by $E_0$, respectively $E_1$, the span of all monomials with $k$ even, respectively odd. Then $E_0$ is the center of $E$, the elements of $E_1$ anticommute, and $E = E_0 \oplus E_1$. It follows by Kemer’s theory that the only nontrivial verbally prime $T$-ideals in $K\langle X \rangle$, char $K = 0$, are: $T(M_n(K))$ for $M_n(K)$ being the matrix algebra of order $n \geq 1$; $T(M_n(E))$, $n \geq 1$ and $T(M_{a,b}(E))$, $a,b \geq 1$. Here $M_n(E)$ is the algebra of $n \times n$ matrices over $E$. The algebra $M_{a,b}(E)$ is the subalgebra of $M_{a+b}(E)$ that consists of the block matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in M_a(E_0)$, $D \in M_b(E_0)$, $B \in M_{a \times b}(E_1)$ and $C \in M_{b \times a}(E_1)$. We denote by $\Delta_0$ the set of $(i,j) \in \mathbb{N} \times \mathbb{N}$ such that $1 \leq i, j \leq a$ or $a+1 \leq i, j \leq a+b = n$, and by $\Delta_1$ the set of $(i,j)$ such that $1 \leq i \leq a$, $a+1 \leq j \leq a+b = n$ or $1 \leq j \leq a$, $a+1 \leq i \leq a+b$. Then $M_{a,b}(E)$ consists of the matrices in $M_n(E)$ such that the $(i,j)$th entry belongs to $E_\beta$ when $(i,j) \in \Delta_\beta$. Here $E'$ denotes the Grassmann algebra without unity.

If two algebras $A$ and $B$ satisfy the same polynomial identities we say that $A$ is PI equivalent to $B$ and we denote by $A \sim B$. An important consequence of the Kemer’s structure theory is the Tensor Product Theorem (TPT).

**Theorem 2.1 (Kemer)** Let char $K = 0$. Then we have $M_{1,1}(E) \sim E \otimes E$; $M_{a,b}(E) \otimes E \sim M_{a+b}(E)$ and $M_{a,b}(E) \otimes M_{c,d}(E) \sim M_{ac+bd,ad+bc}(E)$.

Theorem 2.1 admits proofs that are independent of Kemer’s structure theory. For a detailed account of this research and the study of the behaviour
of the corresponding $T$-ideals over infinite fields of positive characteristic we refer to [5, Introduction].

In this paper we proved with the algebras $M_n(E) \otimes E$ and $M_{n,n}(E)$ are PI equivalents in characteristic zero and are not PI equivalents in positive characteristic $p > 2$.

## 3 The algebras $M_{n,n}(E)$ and $M_n(E) \otimes E$

In this section we investigate the relationship between the $T$-ideals of the algebras $M_{n,n}(E)$ and $M_n(E)$.

The proof of the following Lemma is immediate.

**Lemma 3.1** $M_n(E \otimes E) \cong M_n(E) \otimes E$.

In [3, Section 2] it was proved the following fact.

**Lemma 3.2** $M_n(M_{1,1}(E)) \cong M_{n,n}(E)$.

**Theorem 3.3** Let $\text{char } K = 0$. Then we have $M_{n,n}(E) \sim M_n(E) \otimes E$.

*Proof.* Since $E \otimes E \sim M_{1,1}(E)$, taking the tensor product with $M_n(K)$ on the both sides, we have $M_n(E \otimes E) \sim M_n(M_{1,1}(E))$. Now the result follows by Lemmas 3.1 and 3.2. \hfill \square

Until the end of this section, we assume that $\text{char } K = p > 2$. As in [6] we denote by $K \oplus M_{1,1}(E')$ the unitary algebra obtained from $M_{1,1}(E')$ by formal adjoint of the unity $I_2$, the identity matrix of order 2. We recall that it was shown in [6, Corollary 11] that $E \otimes E \sim K \oplus M_{1,1}(E')$. It is easy to see that $M_n(K \oplus M_{1,1}(E')) \cong M_n(K) \oplus M_n(M_{1,1}(E'))$.

**Lemma 3.4** $M_n(E) \otimes E \sim M_n(K \oplus M_{1,1}(E'))$.

*Proof.* Since $E \otimes E \sim K \oplus M_{1,1}(E')$, taking the tensor product with $M_n(K)$ on the both sides, we have $M_n(E \otimes E) \sim M_n(K \oplus M_{1,1}(E'))$. Now the result follows by Lemma 3.1. \hfill \square

**Lemma 3.5** $M_n(K \oplus M_{1,1}(E')) \cong M_n(K) \oplus M_{n,n}(E')$.

*Proof.* Since $M_n(K \oplus M_{1,1}(E')) \cong M_n(K) \oplus M_n(M_{1,1}(E'))$, the result follows by applying Lemma 3.2 for $E'$. \hfill \square

**Corollary 3.6** $M_n(E) \otimes E \sim M_n(K \oplus M_{n,n}(E'))$. 

Now we will study the central polynomials for $M_{n,n}(E)$ and $M_n(E) \otimes E$. It was shown in [6, Theorem 13] that $T(M_{1,1}(E)) \subseteq T(E \otimes E)$. Using this inclusion and Lemma 3.2, we have $T(M_{n,n}(E)) \subseteq T(M_n(E) \otimes E) = T(M_n(K) \oplus M_{n,n}(E'))$. Therefore $C(M_{n,n}(E)) \subseteq C(M_n(E) \otimes E) = C(M_n(K) \oplus M_{n,n}(E'))$.

Now we consider $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in M_n(K) \right\} \subset M_{2n}(K)$ and 

$$B = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in M_n(K) \right\} \subset M_{n,n}(E).$$

**Lemma 3.7** Let $A$ be as defined above and let $f \in C(M_n(K)) \setminus C(M_{2n}(K))$ be a multilinear polynomial. Then $f^t \in C(A)$, for all $t \in \mathbb{N}$.

**Proof.** Given $A_1, \ldots, A_m \in A$, there exist $B_1, \ldots, B_m \in M_n(K)$ such that $A_i = \begin{pmatrix} B_i & 0 \\ 0 & B_i \end{pmatrix}$, for all $i = 1,2,\ldots,m$. Then

$$f(A_1, \ldots, A_m) = \begin{pmatrix} f(B_1, \ldots, B_m) & 0 \\ 0 & f(B_1, \ldots, B_m) \end{pmatrix} = \begin{pmatrix} \alpha I_n & 0 \\ 0 & \alpha I_n \end{pmatrix}$$

for some $\alpha \in K$, where $I_n$ denotes the identity matrix of order $n$. Thus, given $A_1, \ldots, A_m \in A$, we have $f(A_1, \ldots, A_m) = \alpha I_{2n}$ for some $\alpha \in K$, and it follows that $f^t(A_1, \ldots, A_m) = \alpha^t I_{2n}$ is central, for all $t \in \mathbb{N}$. \hfill \square

The next remark is a direct consequence of [11, Theorem 2.1] and will be useful in the sequel.

**Remark 1** Recall that $\text{char } K = p > 2$.

1. Since $x^p$ is an identity for $E'$ it follows that, given $D \in M_{n,n}(E')$ there exists $s = rp$ such that $D^s = 0$.

2. $M_{n,n}(E')$ is an ideal of $M_{n,n}(E)$.

Given $X_1, \ldots, X_m \in M_n(K) \oplus M_{n,n}(E') = A \oplus M_{n,n}(E')$, we write $X_i = Y_i + Z_i$, where $Y_i \in A$ and $Z_i \in M_{n,n}(E')$, for $i = 1, \ldots, m$. Let $f(x_1, \ldots, x_m)$ be as in the hypothesis of Lemma 3.7. By Remark 1 we have $f(X_1, \ldots, X_m) = f(Y_1, \ldots, Y_m) + D = \alpha I_{2n} + D$, for some $\alpha \in K$ and $D \in M_{n,n}(E')$. Moreover, if $s = rp$ then $f^s(X_1, \ldots, X_m) = \alpha^s I_{2n}$. Therefore the polynomial $f^s(x_1, \ldots, x_m)$ is central for $M_n(K) \oplus M_{n,n}(E')$, and consequently it is central for $M_n(E) \otimes E$.

On the other hand let $A_1, \ldots, A_m \in B$ be such that $A_i = \begin{pmatrix} B_i & 0 \\ 0 & C_i \end{pmatrix}$, for all $i = 1,2,\ldots,m$. We note that

$$f(A_1, \ldots, A_m) = \begin{pmatrix} f(B_1, \ldots, B_m) & 0 \\ 0 & f(C_1, \ldots, C_m) \end{pmatrix} = \begin{pmatrix} \alpha I_n & 0 \\ 0 & \beta I_n \end{pmatrix}$$
The algebras $M_{n,n}(E)$ and $M_n(E) \otimes E$ in positive characteristic

is not central for $M_{n,n}(E)$ by taking distinct $\alpha, \beta \in K$, and this implies that $f^*(x_1,\ldots,x_m)$ is not central for $M_{n,n}(E)$.

The above arguments are the proof of the following theorem.

**Theorem 3.8** Let $\text{char } K = p > 2$. Then we have the strict inclusion

$$C(M_{n,n}(E)) \subset C(M_n(E) \otimes E).$$

**Corollary 3.9** Let $\text{char } K = p > 2$. Then we have the strict inclusion

$$T(M_{n,n}(E)) \subset T(M_n(E) \otimes E).$$

**Proof.** Let $f(x_1,\ldots,x_m)$ be a central polynomial for $M_n(E) \otimes E$ which is not central for $M_{n,n}(E)$. Then the polynomial $g(x_1,\ldots,x_m) = [f(x_1,\ldots,x_m), x_{m+1}]$ is in $T(M_n(E) \otimes E) \setminus T(M_{n,n}(E))$, and we are done. \(\square\)

**Corollary 3.10** Let $\text{char } K = p > 2$. Then the algebras $M_{n,n}(E)$ and $M_n(E) \otimes E$ are not PI equivalents.

Following the above ideas we can construct central polynomials for $M_n(E)$ from multilinear central polynomials for $M_n(K)$. For this let $f(x_1,\ldots,x_m)$ be a multilinear central polynomial for $M_n(K)$. See Halpin [9] for an example of such polynomial.

Let $A_1,\ldots,A_m \in M_n(E)$. We can choose $B_1,\ldots,B_m \in M_n(K)$ and $C_1,\ldots,C_m \in M_n(E')$ such that $A_i = B_i + C_i$, for $i = 1,2,\ldots,m$. Since $M_n(E')$ is an ideal of $M_n(E)$ and $f$ is multilinear, the same reasoning of the proof of Theorem 3.8 gives the following.

**Theorem 3.11** Let $f(x_1,\ldots,x_m)$ be a multilinear central polynomial for $M_n(K)$. Then there exists a positive integer $s = s(p,n)$ which depends on $\text{char } K = p > 2$ and on $n$, such that $f^*(x_1,\ldots,x_m)$ is a central polynomial for $M_n(E)$.

**ACKNOWLEDGEMENTS.** Sérgio M. Alves is partially supported by PROPP/UESC Nr 00220.1300.978.

**References**


Received: December 31, 2013