Abstract

A group $G$ is called smooth if its subgroup lattice $L(G)$ has a smooth chain, and we call $G$ totally smooth if all maximal chains in $L(G)$ are smooth. We call $G$ a minimal non-totally smooth group if $G$ is not totally smooth with totally smooth proper subgroups. In this paper we determine all finite minimal non-totally smooth groups.

Mathematics Subject Classification: 20D30, 20E15

1. Introduction.
All groups considered in this article will be finite. We use conventional notions and notation as in Doerk, K. and Hawkes, T [1]. In addition, the maximal length of the subgroup lattice $L(G)$ of a group $G$ will be denoted by $n$, and the set of distinct primes dividing the order of $G$ will be denoted by
A maximal chain 1 = $G_0 < G_1 < G_2 < ... < G_n = G$ of subgroups of a group $G$ is called smooth if the interval $[G_{i+j}/G_i] = [G_j/1]$ for all $i, j \in \mathbb{N}$ such that $i + j \leq n$. The group $G$ is called smooth if it has a smooth chain. Finite smooth groups have been studied by Schmidt (see [5]). Hence, if all maximal chains of subgroups of $G$ are smooth, $G$ is called totally smooth.

Totally smooth groups have been studied by Elkholy (see [2]). We call $G$ is a minimal non-totally smooth group if $G$ is not totally smooth with its proper subgroups are totally smooth.

Recall that a P-group is a group lattice-isomorphic to an elementary abelian group (see [4], p. 49).

In this article we determine all finite minimal non-totally smooth groups. So we prove the following:

**The main Theorem.**

Suppose that $G$ is a minimal non-totally smooth group with maximal length $n \geq 3$. Then one of the following holds:

(a) $n = 3$ and $|G| = p^3$.

(b) $n = 3$ and $|G| = p^2q$, where $p$ and $q$ are distinct primes.

(c) $n = 3$ and $G$ is not abelian of order $p_1p_2p_3$, where $p_1$, $p_2$, and $p_3$ are distinct primes.

(d) $G = PQ$, where $P$ is an elementary abelian normal Sylow $p$-subgroup and $Q$ is cyclic of order $q$ which operates irreducibly on $P$.

(e) $G = PQ$, where $P$ is an elementary abelian normal Sylow $p$-subgroup of order $p^q$, $Q$ is cyclic of order $q^2$ which operates irreducibly on $P$ and $q | p - 1$.

Since every lattice of length at most 2 is totally smooth and we determine the structure of minimal non-totally smooth groups, it follows that the maximal length $n \geq 3$.

2. **Preliminaries**

In this part, we state the following lemmas which will be used to prove the main Theorem.

**Lemma 1.** A finite group $G$ is totally smooth if and only if one of the following holds:
(a) $G$ is cyclic of prime power order.

(b) $G$ is a $P$-group.

(c) $G$ is cyclic of square free order (See [2]; Theorem 1).

**Lemma 2.** Let $p$ and $q$ be different primes dividing $|G|$ such that $G = PQ$ where $P$ is an elementary abelian normal subgroup of $G$ of order $p^n (n \in \mathbb{N})$ and $Q = \langle x \rangle$ is a cyclic $q$-group. Then the following properties are equivalent:

(i) Every subgroup of $Q$ is either irreducible on $P$ or normalizes every subgroup of $P$.

(ii) One of the following holds:

(a) $G = P \times Q$ or $x$ induces a power automorphism in $P$.

(b) $q | p - 1, |P| = p^n$, and $x$ induces an automorphism of order $q^{k+1}$ in $P$ where $k$ is the largest integer such that $q^k | p - 1$.

(c) $n = 2, q \nmid p^r - 1 (1 = r < n), q^m | p^n - 1$, and $x$ induces an automorphism of order $q^m$ in $P$ ($m \in \mathbb{N}$) (see [6]; Lemma 3.1).

### 3. The proof of the main Theorem

The proof of the main Theorem will be included in Theorems A-C.

**Theorem A.** Suppose that $G$ is a minimal non-totally smooth $p$-group with maximal length $n \geq 3$. Then $n = 3$ and $|G| = p^3$.

**Proof.** By Lemma 1, we get each proper subgroup of $G$ is cyclic or elementary abelian. We have three cases:

Case 1. All the maximal subgroups of $G$ are elementary abelian.

Thus $G$ is nonabelian and $G^a \neq 1$. If $M$ is any maximal subgroup of $G$, $|G/M| = p$ which implies that $G^a \leq M$ for each maximal subgroup $M$ of $G$. We argue that $|G^a| = p$.

If not, then $G^a$ contains a subgroup $H$ of order $p$ which is normal in $G$ as $H \triangleleft M$. Since $G/H$ is abelian, $G^a \leq H$, a contradiction. Thus $|G^a| = p$. Hence $G/G^a$ would be of order $p^2$. Therefore $n = 3$ and $|G| = p^3$.

Case 2. All maximal subgroups are cyclic.

It follows that by [3, Sats 8.2, p. 310], $G$ has exactly one subgroup of order $p$. Since $G$ is not totally smooth, we get $n = 3$ and $|G| = p^3$.
Case 3. $G$ has two different types of totally smooth maximal subgroup. So let $M_1$ be an elementary abelian maximal subgroup of $G$ and $M_2$ be a cyclic maximal subgroup of $G$. Then $M_1 \cap M_2$ would be of order $p$ and so $G$ is of order $p^3$. 

Now we can assume that $|G|$ is divisible by $m \geq 2$ different primes $p_1, p_2, \ldots, p_m$. First, we consider the case $m = 2$.

**Theorem B.** Suppose that $G$ is a minimal non-totally smooth group with maximal length $n \geq 3$ such that $|G|$ is divisible by two different primes $p$ and $q$. Then one of the following holds:

(a) $|G| = p^2q$.

(b) $G = PQ$, where $P$ is an elementary abelian normal Sylow $p$-subgroup and $Q$ is cyclic of order $q$ which operates irreducibly on $P$.

(c) $G = PQ$, where $P$ is an elementary abelian normal Sylow $p$-subgroup of order $p^9$, $Q$ is cyclic of order $q^2$ which operates irreducibly on $P$ and $q \mid p - 1$.

**Proof.** Clearly, $G$ is solvable and hence $G$ has a minimal normal subgroup $N$ which is elementary abelian. Let $P$ be a Sylow $p$-subgroup of $G$ and $Q$ be a Sylow $q$-subgroup of $G$. Suppose that $N \leq P$.

Assume that $N = P$. If $|Q| = q$, we are done and (b) holds. So suppose that $|Q| > q$ and let $Q_1$ be a maximal subgroup of $Q$. By hypothesis and Lemma 1, $PQ_1$ is cyclic of order $pq$ or a nonabelian $P$-group with $p > q$ as $P \trianglelefteq PQ_1$. If $|PQ_1| = pq$, $|G|$ would be of order $pq^2$ and (a) holds. Therefore $PQ_1$ is a nonabelian $P$-group. Since $P$ is minimal normal subgroup of $G$, $Q$ operates irreducibly on $P$ and $Q_1 \leq N_G(P)$ and hence $q \mid p - 1$. Then by Lemma 2, $|P| = p^9$ and (c) holds.

Thus $N < P$ and our hypothesis show that $NQ$ is a totally smooth proper subgroup of $G$. By Lemma 1, $NQ$ is cyclic of order $pq$ or a nonabelian $P$-group. Suppose first that $NQ$ is a nonabelian $P$-group. It follows that $|Q| = q$ and hence $P \trianglelefteq G$. We claim that $|N| = p$. If not, there exists a normal subgroup $N_1$ of $NQ$. Since $P$ is totally smooth, we have by Lemma 1 that $P$ is cyclic or elementary abelian which implies that $N_1 \trianglelefteq P$. Therefore, $N_1 \trianglelefteq G$ which contradicts the minimality of $N$. Thus $|N| = p$.

If $P$ is cyclic, there is a maximal normal subgroup of $P$ such that $P_1 \trianglelefteq G$. Thus $P_1Q$ is totally smooth and hence $P_1$ would be of order $p$. Therefore, $|G| = p^2q$ and we are done. So $P$ is elementary abelian. Since $G$ is a minimal non-totally smooth group, $NQ$ is a totally smooth proper subgroup of $G$. By Lemma 1, $NQ$ is cyclic of order $pq$ or a nonabelian $P$-group. Suppose first that $NQ$ is a nonabelian $P$-group. It follows that $|Q| = q$ and hence $P \trianglelefteq G$. We claim that $|N| = p$. If not, there exists a normal subgroup $N_1$ of $NQ$. Since $P$ is totally smooth, we have by Lemma 1 that $P$ is cyclic or elementary abelian which implies that $N_1 \trianglelefteq P$. Therefore, $N_1 \trianglelefteq G$ which contradicts the minimality of $N$. Thus $|N| = p$.

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smooth group, \( Q \) centralizes some subgroup \( H \) of \( P \). Hence \( HQ \) is cyclic of order \( pq \) by hypothesis. Thus \(|P| = p^2\) and so \(|G| = p^2q\).

To complete the proof, let \( NQ \) be a cyclic group of order \( pq \).

If \( Q \triangleleft G \), then by Lemma 1 \( P_1Q \) is maximal totally smooth subgroup of \( G \) where \( P_1 \) is a maximal subgroup of \( P \). Thus \(|P_1| = p\) and \(|G| = p^2q\). Hence \( Q \) is not normal in \( G \).

If \( P \) would not be normal in \( G \), then \( P = \text{N}_G(P) \). By Burnside’s theorem \( Q \triangleleft G \), a contradiction. Thus \( P \triangleleft G \). Clearly, if \( P \) is cyclic, then \( n = 3 \) and \(|G| = p^2q\). Therefore \( P \) is elementary abelian. If \(|P| > p^2\), there exists a proper subgroup \( L \) of \( P \) containing \( N \) such that \( L \triangleleft G \). Clearly, \( LQ \) is a totally smooth subgroup of \( G \). Since \( Q \) centralizes \( N \), \([LQ/1]\) is not smooth. Then \( N \) would be a maximal subgroup of \( P \). Once again \(|G| = p^2q\). This completes our proof.

Now we consider the case that \( G \) is solvable and \(|G|\) is divisible by at least three different primes.

**Theorem C.** Suppose that \( G \) is a minimal non totally smooth group with maximal length \( n \geq 3 \) and \(|\pi(G)| \geq 3 \). Then \( n = 3 \) and \(|G| = p_1p_2p_3\).

**Proof.** If \( n = 3 \) and since \( G \) is a minimal non-totally smooth group, we get \( G \) would be not cyclic of order \( p_1p_2p_3 \) and we are done.

Consider \( n \geq 4 \) and \( G \) of order \( p_1, p_2, ..., p_r \) with \( r \geq 3 \). From the solvability of \( G \), \( G \) has a sylow basis and hence \( P_iP_j \) is a totally smooth subgroup of \( G \); \( i, j = 1, 2, ..., r \). By Lemma 1, \( P_iP_j \) is cyclic of order \( p_ip_j \) or a nonabelian \( P \)-group. If \( P_iP_j \) is cyclic for all \( i, j = 1, 2, ..., r \), then every sylow subgroup of \( G \) is of prime order. Since \( n \geq 4 \), there exits \( k \in 1, 2, ..., r \) with \( i \neq k \neq j \) such that \( P_iP_jP_k \) is totally smooth subgroup of \( G \) which cyclic. Then \([P_i, P_j] = 1\) for each \( i, j = 1, 2, ..., r \) which implies that \( G \) is cyclic; a contradiction since \( G \) is not totally smooth. Thus \( P_iP_j \) is a nonabelian \( P \)-group for some \( i, j = 1, 2, ..., r \).

Suppose for a contradiction, that \(|P_i| \geq p_i^2\) for some \( i = 1, 2, ..., r \). It follows that \( P_i \) has a normal subgroup \( L \) of \( G \). Since \( LP_jP_k < G \) and \( LP_j \) is a nonabelian \( P \)-group so \([LP_jP_k/1]\) is not smooth which contradicts our hypothesis. Thus \(|P_i| = p_i\) for each \( i = 1, 2, ..., r \). Hence \( G \) would be of order \( p_1p_2p_3 \) and \( n = 3 \), a contradiction since \( n \geq 4 \). This final contradiction complete our proof.
References


Received: May 11, 2013