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An Envelope for Left Alternative Algebras

Josef Rukavicka

Department of Mathematics
Faculty of Electrical Engineering
Czech Technical University in Prague
josef.rukavicka@seznam.cz, rukavij@fel.cvut.cz

Abstract

Let A be the free non-associative algebra and let T be the T-ideal generated by the identity $(x, y, z)+(y, x, z)$. Given an ideal $J \supseteq T$, then $C_J = A/J$ is a left alternative algebra. We construct an ideal Γ_J and we define a universal enveloping algebra of C_J as A/Γ_J. We introduce a linear map $\omega : C_J \to A/\Gamma_J$, such that $\omega((a, b, c)) = (a, b, c)-(b, a, c)$. As a conjecture we state that ω is injective. The injection of C_J into A/Γ_J is similar to the injection of a Lie algebra into an associative algebra by $[a, b] = ab - ba$; moreover we show how to construct a spanning set of A/Γ_J from a basis of C_J and we define a universal property analogously like in the case of Lie algebras.

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1 Introduction

There is a well known construction of a Lie algebra, [2]: given an associative algebra A_{asc}, a Lie algebra may be obtained from A_{asc} by defining a new bilinear multiplication $[x, y] = xy - yx$ on the underlying vector space of A_{asc}. The algebra obtained in this way is usually denoted as A^-_{asc}.
Actually the famous PBW (Poincaré Birkhoff Witt) theorem implies that any

Lie algebra $\Im$ is a subalgebra of A^-_{asc} for some unital associative algebra A_{asc}, [2]. The corresponding associative algebra A_{asc} is called universal enveloping algebra for the Lie algebra $\Im$. The universal enveloping algebras possess a universal property. The PBW theorem has been extended to Malcev algebras, [3] and then to Bol algebras, [4].

We apply a similar approach like in the case of Lie algebras to construct universal enveloping algebras for left alternative algebras. A left alternative algebra is embedded into its envelope by means of the linear map ω which sends an associator (a, b, c) to $(a, b, c) - (b, a, c)$ (note the similarity with $[x, y] = xy - yx$).

The paper is organized as follows. In the second section a special basis B of the free non-associative algebra A is constructed; B contains "maximal" number of associators in the sense that given an associator $(a, b, c) \in A$ than either $(a, b, c) \in B$ or (a, b, c) is equal to a linear combination of associators from B; it means $(a, b, c) = \sum_{ijk} r_{ijk}(a_i, b_j, c_k)$ where $(a_i, b_j, c_k) \in B$ and r_{ijk} are scalars. This feature turns out to be essential for several proofs in the paper.

The third section defines the T-ideal T generated by a defining identity $(x, y, z) + (y, x, z)$ and a linear map $\pi : A \to A$ with the kernel denoted Π, such that $\pi((a, b, c)) = (\pi(a), \pi(b), \pi(c)) - (\pi(b), \pi(a), \pi(c))$ where $(a, b, c) \in B$. Bases of vector spaces $\pi(A)$ and Π are constructed. Next it is proved that $\Pi \subset T$ and $\pi(T) \not\subset T$.

In the fourth section we consider a left alternative algebra $C_J = A/J$ where $J \supseteq T$ is an ideal in A. We introduce an ideal Γ_J generated by the subspace $\pi(J)$. A linear map $\omega : C_J \to A/\Gamma_J$ based on the linear map π is presented. As a conjecture we state that $\pi(A) \cap \Gamma_J = \pi(J)$, what implies immediately that ω is injective. We define a universal enveloping algebra of C_J as A/Γ_J. It is showed how to build up a set Υ from a basis of C_J in such a way that $\{a + \Gamma_J \mid a \in \Upsilon\}$ spans the vector space A/Γ_J. Although the construction of Υ is "analogous" to the construction of a basis of the universal enveloping algebra from a basis of a Lie algebra, [2], unfortunately the elements $\{a + \Gamma_J \mid a \in \Upsilon\}$ are not linearly independent. Thus it remains as an open question how build up a basis of A/Γ_J from a basis of C_J.

Finally we prove that A/Γ_J possesses a universal property as follows: Given two left alternative algebras C_J and C_I with injections ω and ω' into their universal enveloping algebras, respectively, and a homomorphism $\sigma : C_J \to C_I$. Then there is a unique homomorphism $\mu : A/\Gamma_J \to A/\Gamma_I$ such that

$\mu \circ \omega = \omega' \circ \sigma$.

2 Associator basis of the free non-associative algebra

In the paper all algebras and vector spaces are considered over a field K with characteristic 0.

Let A be the free non-associative algebra on a set of generators $X = \{x_1, x_2, \dots x_k\}$. The elements of A will be called polynomials. It is well known that $A = \bigoplus_{i>0} A_i$ where A_i is a homogeneous component of A spanned by all words of the length i.

Let $(a, b, c) = (ab)c - a(bc)$ denote the associator in A where $a, b, c \in A$.

Let $B = \bigcup_{i>0} B_i$ denote a set of polynomials defined as follows where $B_i = B \cap A_i$.

- $B_1 = X$
- $(a, b, c) \in B$ where $a, b, c \in B$
- $ax \in B$ where $a \in B$ and $x \in X$

Proposition 2.1. *The set B spans the vector space A.*

Proof. The proposition obviously holds for A_1 and A_2. We suppose it holds for A_i where $i < n$ and we prove that it holds for A_n where $n > 2$. Let $w \in A_n$ be a word or an associator that is not in B. Three cases may occur:

1. $w = (w_1, w_2, w_3)$, then replace w_1, w_2, w_3 by linear combination of elements of B and multiply out. The result follows from the fact that $(a, b, c) \in B$ if $a, b, c \in B$.

2. $w = w_1 x$ where $x \in X$, then replace w_1 by linear combination of elements of B and multiply out. The result follows from the fact that $ax \in B$ if $a \in B$ and $x \in X$.

3. $w = w_1 w_2$ where $w_2 \notin X$, then $w_2 = w_3 w_4$.
   Thus $w = w_1 w_2 = w_1(w_3 w_4) = (w_1 w_3) w_4 - (w_1, w_3, w_4)$. The associator (w_1, w_3, w_4) turns into the case 1 and $(w_1 w_3) w_4$ turns into the case 2 or 3 with the difference that we decreased the length of the last multiplicand. By iterating the process we eventually achieve the case 2, where the last multiplicand is from X.

□

Let $S = \bigcup_{i>0} S_i$ denote the set of all non-associative words on X where S_i contains the words of the length i. It is known that the set S is a basis of A.

Proposition 2.2. *The set B forms a basis of the vector space A.*

Proof. Consider a map $\tau : B \to S$ defined by

- $\tau(x) = x$ where $x \in X$
- $\tau(ab) = \tau(a)\tau(b)$ where $ab \in B$
- $\tau((a,b,c)) = \tau(a)(\tau(b)\tau(c))$ where $(a,b,c) \in B$.

Since the polynomials of the form $a(bc)$ cannot appear in B, it is easy to see the map τ is an injection. The fact that B_i spans A_i implies that τ is in fact a bijection. This proves that B is a basis of A. □

Corollary 2.3. *Given $(a,b,c) \in A$ such that $(a,b,c) \notin B$. Then*

$$(a,b,c) = \sum_{ijk} \alpha_{ijk}(a_i, b_j, c_k)$$

where $(a_i, b_j, c_k) \in B$ and α_{ijk} is a non-zero scalar.
In this sense B contains "maximal" number of associators.

3 A linear map $\pi : A \to A$

Definition 3.1. *Let $\pi : A \to A$ be a linear map defined on the basis B as follows. The kernel of π will be denoted $\Pi = \bigoplus_i \Pi_i$ where $\Pi_i = \Pi \cap A_i$.*

- $\pi(x) = x$ *where* $x \in X$
- $\pi((a,b,c)) = (\pi(a), \pi(b), \pi(c)) - (\pi(b), \pi(a), \pi(c))$ *where* $(a,b,c) \in B$
- $\pi(ax) = \pi(a)\pi(x) = \pi(a)x$ *where* $ax \in B$, $x \in X$.

Remark 3.2. *Note that due to Corollary 2.3 it holds $\pi((a,b,c)) = (\pi(a), \pi(b), \pi(c)) - (\pi(b), \pi(a), \pi(c))$ for any $a, b, c \in A$.*

For $a \in B$, $\pi(a) \neq 0$, we define sets $\dot{B}_a, \ddot{B}_a \subset B$ as follows:
$\dot{B}_a = \{b \mid b \in B, \pi(a) = \pi(b)\}$
$\ddot{B}_a = \{b \mid b \in B, \pi(a) = -\pi(b)\}$

It is easy to verify that $\pi(a) = \sum_{b \in \dot{B}_a} b - \sum_{b \in \ddot{B}_a} b$ where $\pi(a) \neq 0$.

Example 3.3. *Let* $X = \{x, y, z, t, u, v\}$ *and let* $a = ((x, y, z)t, u, v) \in B$, *then* $\pi(a) = (\pi((x, y, z)t), \pi(u), \pi(v)) - (\pi(u), \pi((x, y, z)t), \pi(v)) = ((x, y, z)t, u, v) - ((y, x, z)t, u, v) - (u, (x, y, z)t, v) + (u, (y, x, z)t, v)$.
And it holds that $\dot{B}_a = \{((x, y, z)t, u, v), (u, (y, x, z)t, v)\}$ *and*
$\ddot{B}_a = \{((y, x, z)t, u, v), (u, (x, y, z)t, v)\}$.

Let B be ordered. And let $L = \bigcup_i L_i$ where $L_i \subseteq B_i$ be a set defined as follows:

- $L_1 = X$
- $(a, b, c) \in L$ where $a, b, c \in L$ and $a > b$ (with regard to the order of B)
- $ax \in L$ where $a \in L$ and $x \in X$

Definition 3.4. *We define a multiple as in [1], p. 304.*
Given polynomials $g, f \in A$. *We call* g *a multiple of* f *if there is a sequence* $p = (f_0, f_1, \ldots, f_r)$, $r \geq 0$, *such that:*

- $f_0 = f, f_r = g,\ f_i \in A$
- $f_i = h_i f_{i-1}$ *or* $f_{i-1} h_i$ *with* $h_i \in A$.

Proposition 3.5. *A set* $\pi(L) = \{\pi(a) \mid a \in L\}$ *forms a basis of the vector space* $\pi(A)$.

Proof. A set $\pi(B) = \{\pi(a) \mid a \in B\}$ clearly spans $\pi(A)$. The fact that the set $\pi(L)$ spans $\pi(A)$ follows from that $(a, b, c) + (b, a, c) \in \Pi$: given $p \in B$ having a multiple of the form (a, b, c), if we replace (a, b, c) by (b, a, c) then $\pi(p)$ only changes a sign, hence it is enough to include in L only associators where $a < b$. To prove the linear independence of elements of $\pi(L)$, just realize that for any $a, b \in L, a \neq b$ it holds $(\dot{B}_a \cup \ddot{B}_a) \cap (\dot{B}_b \cup \ddot{B}_b) = \emptyset$. □

Corollary 3.6. *The set* $\{a + \Pi \mid a \in L\}$ *is a basis of* A/Π.

Let $T = \bigoplus_i T_i$ denote a T-ideal in A generated by the defining identity $(x, y, z) + (y, x, z)$ where $T_i = T \cap A_i$.

Definition 3.7. *We define:*
$\dot{\Delta} = \{a + \alpha b \mid a \in B \setminus L, b \in L, \alpha \in \{-1, 1\}, \pi(a) \neq 0, \pi(a + \alpha b) = 0\}$
$\ddot{\Delta} = \{a \mid a \in B, \pi(a) = 0\}$

$$\Delta = \dot{\Delta} \cup \ddot{\Delta}$$

Remark 3.8. *To see that the definition of* $\dot{\Delta}$ *makes sense, note that any* $a \in B \setminus L$, $\pi(a) \neq 0$ *is a multiple of at least one associator* $(c_1, c_2, c_3) \in B$, *thus* $b \in L$ *exists and is uniquely determined as well as the scalar* α *(less formally said* b *arises from* a *when we order all associators* (c_1, c_2, c_3) *in* a, *so that* $c_1 > c_2$*).*

Proposition 3.9. *The set* Δ *forms a basis of* Π.

Proof. Given any $a+\alpha b\in\dot{\Delta}$ or $a\in\ddot{\Delta}$ where $a\in B\setminus L, b\in L, \alpha\in\{-1,1\}$, then a appears in no other polynomial from $\Delta\setminus\{a+\alpha b, a\}$. It follows that elements of Δ are linearly independent.
The facts that $\pi(\Delta)=0$, $|\Delta|+|L|=|B|$, and $span(\Delta\cup L)=A$ imply that Δ forms a basis of Π. □

Corollary 3.10. $\Pi\subset T$

Proof. It is easy to see that for the basis Δ of Π it holds $\Delta\subset T$. □

Example 3.11. *To see that* Π *is a proper subset of* T*:*
$x(x,x,x)\in T\setminus\Pi$, *since* $\pi(x(x,x,x))=$
$\pi(-(x,x,x)x+(x,x,xx)-(x,xx,x)+(xx,x,x))=$
$-2(x,xx,x)+2(xx,x,x)$.
On the other hand note that every polynomial of the form $(a,b,c)+(b,a,c)$ *where* $a,b,c\in A$ *lies in* Π *due to Corollary 2.3.*

Remark 3.12. *It can be proved that* $\pi(T)\not\subset T$ *and* $\pi(T)\cap T\neq 0$. *For example let* $f=y(y,(y,x,x),x)-y((x,y,x),y,x)$, *then* $f\in T$ *and* $\pi(f)\notin T$. *On the other hand let* $f=t(x,y,z)+t(y,x,z)$, *then* $\pi(f)\neq 0$ *and* $f,\pi(f)\in T$. *We omit the laborious proof.*

4 Envelope of left alternative algebras

Let J be an ideal in A such that $T\subseteq J$ and let $C_J=A/J$, then C_J is a left alternative algebra. Let Γ_J be the ideal generated by the subspace $\pi(J)$. Let $\omega: C_J\to A/\Gamma_J$ be a linear map defined as $\omega(b)=\pi(b)+\Gamma_J\in A/\Gamma_J$ where $b+J\in C_J=A/J$. The linear map ω is well defined, since $\pi(J)\subset\Gamma_J$.

Conjecture 4.1. $\pi(A)\cap\Gamma_J=\pi(J)$

Remark 4.2. *The above conjecture has been confirmed by a computer for* J *being the free left alternative algebra on* 2 *and* 4 *generators for* J_i *where* $i\leq 6$ *and* $i\leq 4$, *respectively.*

Corollary 4.3. *The linear map* $\omega: C_J\to A/\Gamma_J$ *is injective.*

Proof. Note that a linear map $\hat{\omega}: C_J\in A/\pi(J)$, $\hat{\omega}(b)=\pi(b)+\pi(J)\in A/\pi(J)$, $b+J\in C_J=A/J$ is clearly injective since $\Pi\subset T\subseteq J$. The injection of ω then follows from the previous conjecture. □

Definition 4.4. *We call* A/Γ_J *a universal enveloping algebra of a left alternative algebra* C_J.

Let $\tilde{C}_J \subset L$ be a set such that the set $\{a + J \mid a \in \tilde{C}_J\}$ forms a basis of $C_J = A/J$. Next we require that for any $(a, b, c), ax \in \tilde{C}_J$ it follows that $a, b, c, x \in \tilde{C}_J$, $x \in X$. It is easy to see such $\tilde{C}_J$ exists, since $L \subset B$ and L itself satisfies this condition.

Let $\dot{\Upsilon} = \{\pi(a) \mid a \in \tilde{C}_J\}$. Recall that B is ordered, and that $\tilde{C}_J \subset L \subset B$. We define a set $\ddot{\Upsilon}$:

- $(\pi(a), \pi(b), \pi(c)) \in \ddot{\Upsilon}$ where $a, b, c \in \tilde{C}_J$, $a \leq b$
- $(a, b, c) \in \ddot{\Upsilon}$ where $a, b, c \in \ddot{\Upsilon}$
- $ax \in \ddot{\Upsilon}$ where $a \in \ddot{\Upsilon}$, $x \in X$

Let $\Upsilon = \dot{\Upsilon} \cup \ddot{\Upsilon}$ and let Ψ be a set defined as follows:

- $\pi(a)x \in \Psi$ where $a \in \tilde{C}_J$, $x \in X$
- $(\pi(a), \pi(b), \pi(c)) \in \Psi$ where $a, b, c \in \tilde{C}_J$, $a > b$

Proposition 4.5. *The set* $\{a + \Gamma_J \mid a \in \Upsilon\}$ *spans the vector space* A/Γ_J.

Proof. Obviously $\dot{\Upsilon}$ generates the algebra A/Γ_J since $\tilde{C}_1 \subset \dot{\Upsilon}$. Hence the it is clear that $span(\{a + \Gamma_J \mid a \in \Upsilon \cup \Psi\}) = A/\Gamma_J$ (note that the set $\Upsilon \cup \Psi$ is constructed in an analogous way like the basis B on generators X, with difference that the generators are from $\dot{\Upsilon}$; less formally we can say that the set $\Upsilon \cup \Psi$ is an associator basis on generators $\dot{\Upsilon}$, we don't consider multiples of elements from Ψ since A/Γ_J is an algebra). Hence it is enough to prove that $\{a + \Gamma_J \mid a \in \Psi\} \subset span(\{a + \Gamma_J \mid a \in \Upsilon\})$.

- $\pi(a)x \in \Psi$, $a \in \tilde{C}_J$, $x \in X$
  Either $ax \in \tilde{C}_J$, then $\pi(ax) = \pi(a)x$ or $p = ax - \sum_i \alpha_i b_i \in T$, $b_i \in \tilde{C}_J$, $\alpha_i \in K$, then $\pi(p) = \pi(a)x - \sum_i \alpha_i \pi(b_i) \in \pi(T) \subset \Gamma_J$ where $\pi(b_i) \in \dot{\Upsilon}$; in consequence $\pi(a)x + \Gamma_J \in span(\{a + \Gamma_J \mid a \in \Upsilon\})$.
- $(\pi(a), \pi(b), \pi(c)) \in \Psi$, $a, b, c \in \tilde{C}_J$, $a > b$
  Either $(a, b, c) \in \tilde{C}_J$, then $\pi((a, b, c)) = (\pi(a), \pi(b), \pi(c)) - (\pi(b), \pi(a), \pi(c))$ and $(\pi(b), \pi(a), \pi(c)) \in \ddot{\Upsilon}$, $\pi((a, b, c)) \in \dot{\Upsilon}$. Or $p = (a, b, c) - \sum_i \alpha_i b_i \in T$, $b_i \in \tilde{C}_J$, $\alpha_i \in K$, then $\pi(p) = (\pi(a), \pi(b), \pi(c)) - (\pi(b), \pi(a), \pi(c)) - \sum_i \alpha_i \pi(b_i) \in \pi(T) \subset \Gamma_J$ where $\pi(b_i) \in \dot{\Upsilon}$, $(\pi(b), \pi(a), \pi(c)) \in \ddot{\Upsilon}$; in consequence $(\pi(a), \pi(b), \pi(c)) + \Gamma_J \in span(\{a + \Gamma_J \mid a \in \Upsilon\})$.

$\square$

Finally we show the universal enveloping algebra possesses a universal property:

Proposition 4.6. *Given two left alternative algebras C_J and C_I with injections ω and ω' into their universal enveloping algebras, respectively, and a homomorphism $\sigma : C_J \rightarrow C_I$. Then there is a unique homomorphism $\mu : A/\Gamma_J \rightarrow A/\Gamma_I$ such that $\mu \circ \omega = \omega' \circ \sigma$.*

Proof. It follows easily from the observation that the set $\Lambda = \{\pi(a) + \Gamma_J \mid a + J \in C_J\}$ generates A/Γ_J, hence any homomorphism defined on elements of Λ extends to a unique homomorphism of universal enveloping algebras. □

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