M-Purity and Torsion Purity in Modules

Ashok Kumar Pandey and Manoj Pathak

Department of Mathematics
Ewing Christian College Allahabad
Allahabad (India) 211 002
ashokpandeyecc@gmail.com

Copyright © 2013 Ashok Kumar Pandey and Manoj Pathak. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

The aim of this paper is to relativize the concept of M-purity and σ-purity defined and studied by Azumaya [2] with respect to an arbitrary hereditary torsion theory given by a left exact torsion radical σ and also relate this concepts with the notions of σ-purity as given by B. B. Bhattacharya and D. P. Choudhury [3] and Ashok Kr. Pandey [1].

Mathematics Subject Classification: 16D99

Keywords: Left R-modules, (FG, σ)-purity, σ-pure projective modules, Finitely σ-compact modules, torsion purity, σ-pure injective modules

1 Introduction

The notion of purity plays a fundamental role in the theory of abelian groups as well as in module categories. In the first section of this paper we examine the purities by torsion modules, finitely generated torsion modules and cyclic torsion modules. Work in this direction was initiated by Walker, Stenstrom, Azumaya [2], B. B. Bhattacharya and D. P. Choudhury [3] and Ashok Kr. Pandey [1]. In this there is an attempt to relativize the usual Cohn [4] purity with respect to a torsion theory We also develope the theory of (M, σ)-purity and (μ, σ)-purity relative to a torsion theory with radical σ which is
weaker than the usual purity and given a sufficient condition for these two coicide (when \( M \) is a left \( R \)-module and \( \mu \) is an \( i \times j \) matrix determined by \( M \)). In the second section of this present paper we relativize the concept of weak \((M, \sigma)\)-purities corresponding to direct products of matrices of left modules which are row finite or those of right modules are column finite.

2 \((M, \sigma)\)-purity

In this paper \( \sigma \) will denote a given left exact torsion radical and a torsion module means a module \( M \) for which \( \sigma(M) = M \). Suppose that \( M, B \) and \( C \) are left \( R \)-modules.

**Definition 2.1.** An epimorphism \( p : B \twoheadrightarrow C \) is said to be \((M, \sigma)\)-pure if for each homomorphism \( \varphi : M \rightarrow C \) with image \((\varphi)\) a torsion module that is \( \varphi[M] \subseteq \sigma[B] \), there exists a homomorphism \( \phi : M \rightarrow B \) such that \( p\phi = \varphi \)

\[
\begin{array}{ccc}
M & \xrightarrow{\varphi} & B \\
& | & \\
& \downarrow & \\
& & C \\
\end{array}
\]

We may extend the lower sequence by taking the kernel of \( P \) denoted by \( A \) and refer the short exact sequence

\[
0 \rightarrow A \rightarrow B \xrightarrow{p} C \rightarrow 0
\]

as \((M, \sigma)\)-pure.

If the epimorphism \( p : B \rightarrow C \) factorizes as

\[
B \xrightarrow{g} N \xrightarrow{h} C
\]

that is \( p = (h0g) \) with \( g \)-epic, then we can easily see that whenever \( g \) and \( h \) are \((M, \sigma)\)-pure then \( p \) is \((M, \sigma)\)-pure.

Conversely in the above situation, if \( p \) is \((M, \sigma)\)-pure then \( h \) is also \((M, \sigma)\)-pure. If \( B \) is torsion then \( p : B \rightarrow C \) splits (that is kernel \( p \) is direct summand of \( A \)) if and only if \( p \) is \((B, \sigma)\)-pure and this is equivalent to the condition that \( p \) is \((M, \sigma)\)-pure for every left \( R \)-module \( M \).

Given a row finite \( I \times J \) matrix \( \mu = (r_{ij}) \), by a system of linear equations given by \( \mu \) in a left module \( Y \), we mean a system \( \sum (r_{ij}x_j) = y_i \) where \( y_i \in Y \) for each \( i \in I \) and \( x_j \) \((j \in J)\) are unknowns.

**Definition 2.2.** We say that a submodule \( A \) is \((\mu, \sigma)\)-pure in a module \( B \), if any system of linear equation \( \sum_j r_{ij}x_j = a_i \) given by the row finite matrix
μ in A, whenever solvable in B in the form $x_j = b_i$ for which there are left ideals $D_i \in \mathcal{D}$ where $\mathcal{D}$ is the Gabriel filter [5] of dense left ideals corresponding to the left exact torsion radical $\sigma$, such that $D_i b_j \subseteq A$, the system is also solvable in $A$ that is there are $a'_j \in A$ with $\sum r_{ij} a'_j = a_i$ for every $i \in I$.

This exactly means that given vectors $(b_i) \in \prod_j B$ and $(a_i) \in \prod_i A$ and $\mu(b_j) = (a_i)$ with $D_j b_i \subseteq A$ for some $d_j \in \mathcal{D}$, there exists $(a'_j) \in \prod_j A$ such that $\mu(a'_j) = (a_i)$ where the vector $\mu(a'_j)$ is obtained by matrix product of the row finite matrix $\mu$ and column vector $(a'_j)$. We may rephrase the above condition of $A$ being $(\mu, \sigma)$ -pure in $B$ or that $B$ is a $(\mu, \sigma)$ -pure extension of $A$ as follows.

We view $\mu$ as mapping $\prod_j B$ to $\prod_i B$ by left matrix multiplication. Then we have:

**Theorem 2.3.** A submodule $A$ is $(\mu, \sigma)$ -pure in $B$ if and only if $\mu[\prod_j (B_j)] \cap \prod_i A \subseteq \mu[\prod_j (A)]$ whenever $B'_j$ are submodules of $B$ containing $A$ such that $A$ is dense in $B_j$.

*Proof.* Any element of the left hand side is of the form $(a_i)_j = \mu(b_j)_j = \sum r_{ij} b_j$ and $A$ dense in $B_j$ means $B_j/A$ is torsion and hence for each element $(b_j + A) \in B_j/A$, there exists $D_j \in \mathcal{D}$ such that $D_j (b_j + A) = 0$ that is $D_j b_j \subseteq A$.

The following result links $(\mu, \sigma)$ -purity with $(M, \sigma)$ -purity.

**Proposition 2.4.** Let $\mu = (r_{ij})$ be a row finite $I \times J$ matrix where $I$ and $J$ are arbitrary sets. Then a submodule $A$ is $(\mu, \sigma)$ -pure in a module $B$ if and only if the sequence $0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$ is $(M, \sigma)$ -pure where

$$
\begin{align*}
\oplus_i R &\xrightarrow{\mu'} \oplus_j R \xrightarrow{\phi} M \longrightarrow 0
\end{align*}
$$

is exact with $\mu'$ given by the matrix $\mu$.

*Proof.* The condition of exactness of the above sequence is equivalent to the condition that $L = \ker(\phi)$ has a system of generators $y_i = \sum r_{ij} x_j$ where $x_j \mid j \in J$ is a system of generators of $\oplus_j R$. Suppose that $A$ is $(\mu, \sigma)$ -pure in $B$.

To show that $A$ is $(M, \sigma)$ -pure in $B$, we take any homomorphism $f : M \rightarrow B/A$ such that image $f$ is torsion that is $\text{Im}(f) \subseteq \sigma(B/A)$. We complete the diagram using the projectivity of $\oplus_j R$ and taking $L = \text{Im}(\mu')$.

$$
\begin{align*}
0 &\longrightarrow L \xrightarrow{j} \oplus_j R \xrightarrow{\phi} M \longrightarrow 0 \\
&\downarrow h \quad \downarrow g \quad \downarrow f \\
0 &\longrightarrow A \xrightarrow{i} B \xrightarrow{p} B/A \longrightarrow 0
\end{align*}
$$
In fact \( h : L \rightarrow \mathcal{A} \) exists because \( \text{po}(\text{go}j) = f \circ (\phi \circ \alpha) = f.0 = 0 \) and \( \mathcal{A} = \ker(p) \). Here \( j : L \rightarrow \oplus J \mathcal{R} \) is the inclusion of \( \text{Im}(\mu') \) in \( \oplus J \mathcal{R} \). If \((x_j)_J \) and \((x_i)_I \) are system of generators of \( \oplus J \mathcal{R} \) and \( \mathcal{L} \) respectively then \( h(y_i) = \text{i}o\text{h}(y_i) = \text{go}j(y_i) = \text{go}(\sum r_i j x_j) = \sum r_i j g(x_j) \) and \( \text{po}(g(x_j)) = f \circ \phi \circ (x_j) \subseteq \sigma(B/\mathcal{A}) \). So there exists for each \( j \in J, D_j \in \mathcal{D} \) such that \( D_j p(g(x_j)) = 0 \). Therefore \( D_j (g(x_j) + A) = 0 \) that is \( D_j (g(x_j) + A) \subseteq A \). As \( \mathcal{A} \) is \((\mu, \sigma)\)-pure in \( \mathcal{B} \). There exists \( a_j' \in \mathcal{A} \) such that \( \sum r_i' j a_j' = h(y_i) \). Now mapping \( x_j \) to \( a_j' \) we get a homomorphism \( \alpha : \oplus J \mathcal{R} \rightarrow \mathcal{A} \) such that \( (\alpha o j)(y_i) = \sum r_i j \alpha(x_j) = \sum r_i j \alpha_j' = h(y_i) \), for each \( i \in I \). As \( y_i \) generate \( \mathcal{L} \). Therefore \( (\alpha o j) = h \). This is equivalent to the existence of \( q : \mathcal{M} \rightarrow \mathcal{B} \) with \((\text{po}q = f) \). Therefore the lower sequence is \((\mathcal{M}, \sigma)\)-pure. Conversely, if the lower sequence is \((\mathcal{M}, \sigma)\)-pure where \( \mathcal{M} \) is given by a sequence

\[
\oplus_i \mathcal{R} \xrightarrow{\mu'} \oplus J \mathcal{R} \xrightarrow{\phi} \mathcal{M} \rightarrow 0
\]

which is exact and given the system of relations \( \sum r_i j b_j = a_i \), where \( \mu = (r_i j, b_j) \in \mathcal{B} \) and \( a_i \in \mathcal{A} \) and \( D_j B_j \subseteq \mathcal{A} \) for \( D_j \in \mathcal{D} \) for each \( j \in J \). We define a map \( g : \oplus J \mathcal{R} \rightarrow \mathcal{B} \) by \( g(x_j) = b_j \) for every \( j \in J \) where \( x_j \)’s generate \( \oplus J \mathcal{R} \). Therefore \( g(\sum r_i j x_j) = \sum r_i j g(x_j) = \sum r_i j b_j = a_i \in \mathcal{A} \), hence \( g(y_i) \in \mathcal{A} \) for every \( i \in I \) where \( y_i = \sum r_i j x_j \). Now \( \text{Im}(\mu') = \text{submodule generated by } \sum r_i j x_j = y_i \)’s = \( \ker(\phi) \). This shows that the upper sequence of the above is exact where \( L = \langle y_i \rangle = \text{submodule of } \oplus J \mathcal{R} \text{ generated by } y_i \)’s = \( \ker(\phi) \) and also \( \mathcal{M} = \text{Coker}(j) \) where \( j : L \rightarrow \mathcal{L} \) is the inclusion map.

Define \( h : L \rightarrow A \) by \( g|_L \). Now \( \text{po}(\text{go}j(y_i)) = \text{po}(\text{i}o\text{h}(y_i)) = 0 \). Therefore there exists \( f : \mathcal{M} \rightarrow B/A \) such that \((f o \phi) = (\text{po}g) \). As \( \phi \) is surjective, \( \text{Im}(f) = \text{Im}(f o \phi) = \text{Im}(\text{po}g) = \langle g(x_i) + A \rangle = \langle b_j + A \rangle \subseteq \sigma(B/A) \) as \( D_j (b_j + A) = 0 \) for each \( j \in J \). Therefore \( \text{Im}(f) \) is torsion and hence by \((\mathcal{M}, \sigma)\)-purity of the lower sequence there exists \( g : \mathcal{M} \rightarrow \mathcal{B} \) such that \((\text{po}q = f) \). This is equivalent to the existence of \( \alpha : \oplus J \mathcal{R} \rightarrow \mathcal{A} \) such that \((\alpha o j) = h \). Put \( \alpha(x_j) = \alpha_j' \in \mathcal{A} \). Then \( \sum_j r_i j a_j' = \alpha(\sum r_i j x_j) = (\alpha o j)(y_i) = h(y_i) = a_i \) and hence the given system is solvable in \( \mathcal{A} \). Therefore \( \mathcal{A} \) is \((\mu, \sigma)\)-pure in \( \mathcal{B} \).

\[\square\]

The next proposition connects \((\mathcal{M}, \sigma)\) and \((\mu, \sigma)\)-purity with \( \mu \)-purity and \( \mu \)-purity of \textbf{Azumaya}[3]

**Proposition 2.5.** A submodule \( \mathcal{A} \) of a module \( \mathcal{B} \) is \((\mu, \sigma)\)-pure if and only if \( \mathcal{A} \) is \( \mu \)-pure in the closure \( \overline{\mathcal{A}} \) of \( \mathcal{A} \). (Here \( \mu \) is a given row finite matrix and \( \mathcal{A} \) is given by \( \overline{\mathcal{A}} / \mathcal{A} = \sigma(B/A) \).

**Proof.** Given any system of equations \( \sum r_i j g(x_j) = a_i \) in \( \mathcal{A} \) which is solvable in \( \overline{\mathcal{A}} \) with \( x_j = \overline{x_j} \in \overline{\mathcal{A}} \) we see that \( \overline{x_j} + \mathcal{A} \in \overline{\mathcal{A}} / \mathcal{A} = \sigma(B/A) \). So there exists
$D_i \in \mathcal{D}$ such that $D_j(\overline{a}_j + A) = 0$ that is $D_j\overline{a}_j \subseteq A$. Since $A$ is $(\mu, \sigma)$ -pure in $B$, we see that there exists $a'_j \in A$ such that $\sum r_{ij}a'_j = a_i$ and hence the system is solvable in $A$. Therefore the above system of equations given by the matrix $\mu = (r_{ij})$ is solvable in $A$ whenever it is solvable in $\overline{A}$ that is $A$ is $\mu$ -pure in $\overline{A}$.

Conversely if $A$ is $\mu$ -pure in $\overline{A}$, and we have relations $\sum r_{ij}b_j = a_i$ with $b_j \in B, a_i \in A$ and $D_jb_j \subseteq A$ for some $D_i \in \mathcal{D}$, we have $D_j(b_j + A) = 0 \implies (b_j + A) \in \sigma(B/A) = (\overline{A}/A)$ so $b_j \in \overline{A}$. Thus by $\mu$ -purity of $A$ in $\overline{A}$, there exists $a'_j \in A$ such that $\sum r_{ij}a'_j = a_i$ and hence $A$ is $(\mu, \sigma)$ -pure in $B$. 

**Proposition 2.6.** If a module $M$ is given by a defining matrix $\mu$, then a sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

of left $R$-modules is $M$ -pure if and only if it is $\mu$ -pure.

**Corollary 2.7.** If $A$ is a submodule of a module $B$ and $\overline{A}$ is the closure of $A$ in $B$, then the followings are equivalent for a module $M$ given by a row finite defining matrix $\mu$:

(i) $A$ is $(M, \sigma)$ -pure in $B$

(ii) $A$ is $\mu$ -pure in $\overline{A}$

(iii) $A$ is $(M, \sigma)$ -pure in $B$

(iv) $A$ is $M$ -pure in $\overline{A}$

3 \hspace{1cm} Weak $(M, \sigma)$- purities

We now consider conditions weaker than $(M, \sigma)$ and $(\mu, \sigma)$ -purities.

**Definition 3.1.** A sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

is said to be weakly $(N, \sigma)$ -pure if the sequence

$$0 \longrightarrow N \otimes A \longrightarrow N \otimes \overline{A}$$

is exact, when $N$ is a given right $R$-module. Here $\overline{A}$ is the closure of $A$ in $B$, that is $\overline{A}/A = \sigma(B/A)$. 


**Definition 3.2.** Given a column finite matrix \( \nu = (s_{ij}) \), an exact sequence

\[
0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0
\]

is said to be **weakly** \((\nu, \sigma)\)-pure if given a system of equations \( \sum s_{ij}x_j = a_i \) with \((x_j) \in \oplus_J B, (a_i) \in \oplus_I A\) such that for each \( j \in J \), there is \( D_j \in \mathcal{D} \) with \( D_jx_j \subseteq A \), there exists \((a'_j) \in \oplus_J A\) with \( \sum s_{ij}a'_j = a_i \). Note that we are restricting the vectors \((x_j), (a_i)\) and \((a'_j)\) to the corresponding direct sums of copies of \( B, A\) and \( A'\) taken over \( J, I\) and \( J\) respectively, where as in case of \((\mu, \sigma)\)-purity they could belong to the corresponding direct products. In case of \( I \) and \( J \) are finite then of course the two notions coincide.

Just as defining matrices of left modules are row finite, those of right modules are column finite. But in this case the defining matrix will be an \( I \times J \) matrix if there is an exact sequence of right modules

\[
\oplus_J R \xrightarrow{\nu'} \oplus_I R \longrightarrow N \longrightarrow 0
\]

where \( \nu'(e_j) = \sum e_is_{ij} \) and \( \nu = (s_{ij}) \) and to keep the sum finite, there should be at most finitely many non-zero \((s_{ij}')\)'s for each \( j \) that is \( \nu \) should be column finite.

**Proposition 3.3.** For an exact sequence

\[
0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0
\]

of left \( R \)-modules and a column finite matrix \( \nu \), the following statements are equivalent:

(i) The sequence is weakly \((\nu, \sigma)\)-pure.

(ii) The sequence is weakly \((N, \sigma)\)-pure for a right \( R \) module \( N \) given by a column finite matrix \( \nu \).

(iii) \( A \) is weakly \( N \)-pure in \( \overline{A} \) (where \( \overline{A}/A = \sigma(B/A) \)).

**Proof.** The notion of weak \( N \)-purity referred to in the statement (iii) above is the one defined in [Azumaya][2] and (ii) \(\iff\) (iii) follows from the definition of weak \((N, \sigma)\)-purity.

By proposition 2, [Azumaya][2], \( A \) is weakly \( N \)-pure in \( \overline{A} \) if and only if \( A \) is weakly \( \nu \)-pure in \( \overline{A} \). Now the last condition means that given \((x_j) \in \oplus_J \overline{A}, (a_i) \in_I A\), with \( \sum r_{ij}x_j = a_i \), there exists \((a'_j) \in \oplus_J A\) with \( \sum r_{ij}a'_j = a_i \).

But \( x_j \in A \) means that \((x_j + A) \in \sigma(B/A)\) that is \( D_jx_j \subseteq A \) and hence \( A \) is weakly \( \nu \)-pure in \( \overline{A} \) if and only if \( A \) is weakly \((\nu, \sigma)\)-pure in \( B \). This proves the equivalence of (i) and (ii).

\( \square \)
Theorem 3.4. The following are equivalent for an exact sequence

\[ 0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0 \]

(i) The sequence is \((M, \sigma)\)-pure for all finitely presented modules \(M\).

(ii) The sequence is \((\mu, \sigma)\)-pure for all finite matrices \(\mu\).

(iii) \(A\) is pure in \(\overline{A} \).

References


Received: April 15, 2013