Fuzzy Medial Ideals Characterized by its Intuitionistic

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Abstract

We consider the Intuitionistic fuzzification of the concept medial ideals, the image (preimage) of fuzzy medial ideals in BCI-algebra, and investigate some of their properties. We introduce the notion of product of intuitionistic fuzzy medial ideals in BCI-algebras, and investigate some related properties.


Keywords: medial BCI-algebras, fuzzy medial ideals in BCI-algebras, intuitionistic fuzzy medial ideals, intuitionistic fuzzy image (preimage) of medial ideals

1 Introduction

The concept of fuzzy subset and various operations on it were first introduced by Zadeh in [8]. Since then, several researches were conducted on the generalizations of the notion of fuzzy sets. The idea of “intuitionistic fuzzy set” was first published by Atanassov [1, 2] as a generalization of the notion of fuzzy set. In [6] J.Meng and Y.B.Jun studied medial BCI-algebras. In [7] S.M.Mostafa, Y.B.Jun and Amany Elmenshawy introduce the notion of medial ideals in BCI-algebras, they state the fuzzification of medial ideals and investigates its properties. In this paper, we introduce the notion of intuitionistic fuzzy medial ideals in BCI-algebras and fuzzy intuitionistic image (preimage) of medial ideals in BCI-algebras.
We also introduce the product of two intuitionistic fuzzy medial ideals in medial BCI-algebras and investigate some results.

2 Preliminaries

An algebraic system \((X, *, 0)\) of type \((2, 0)\) is called a BCI-algebra if it satisfies the following conditions:

- \((\text{BCI-1})\) \(((x * y) * (x * z)) * (y * x) = 0,\)
- \((\text{BCI-2})\) \((x * (x * y)) * y = 0,\)
- \((\text{BCI-3})\) \(x * x = 0,\)
- \((\text{BCI-4})\) \(x * y = 0\) and \(y * x = 0\) imply \(x = y.\)

For all \(x, y,\) and \(z \in X.\) In a BCI-algebra \(X,\) we can define a partial ordering”\(\leq\)” by \(x \leq y\) if and only if \(x * y = 0.\) In what follows, \(X\) will denote a BCI-algebra unless otherwise specified.

A BCI-algebra \(X\) is called a medial BCI-algebra if it satisfies the following condition: \((x * y) * (z * u) = (x * z) * (y * u)\) for all \(x, y, z\) and \(u \in X.\)

In a medial BCI-algebra \(X,\) the following holds for all \(x, y, z \in X:\)

1. \(x * (y * z) = z * (x * y),\)
2. \(x * (x * y) = y,\)
3. \(0 * (y * x) = x * y.\)

**Definition 2.1.** A non empty subset \(S\) of a medial BCI-algebra \(X\) is said to be medial subalgebra of \(X,\) if \(x * y \in S,\) for all \(x, y \in S.\)

**Definition 2.2 [3].** A non-empty subset \(I\) of a BCI-algebra \(X\) is said to be a BCI-ideal of \(X\) if it satisfies:

1. \((I_1)\) \(0 \in I,\)
2. \((I_2)\) \(x * y \in I\) and \(y \in I\) implies \(x \in I\) for all \(x, y \in X.\)

**Definition 2.3 [7].** A non empty subset \(M\) of a BCI-algebra \(X\) is said to be a medial ideal of \(X\) if it satisfies:

1. \((M_1)\) \(0 \in M,\)
2. \((M_2)\) \(z * (y * x) \in M\) and \(y * z \in M\) imply \(x \in M\) for all \(x, y \in X.\)

**Proposition 2.4 [7].** Any medial ideal of a BCI-algebra must be a BCI-ideal but the converse is not true.

**Lemma 2.5.** Any BCI-ideal of a medial BCI-algebra is a medial ideal.

**Proof.** Clear.
**Example 2.6** [7]. Let \( X = \{0, 1, 2, 3, 4, 5\} \) be a set with a binary operation \( * \) defined by the following table:

\[
\begin{array}{cccccc}
* & 0 & 1 & 2 & 3 & 4 \\
0 & 0 & 0 & 0 & 0 & 4 \\
1 & 0 & 1 & 0 & 1 & 4 \\
2 & 2 & 2 & 0 & 0 & 4 \\
3 & 3 & 2 & 2 & 1 & 4 \\
4 & 4 & 4 & 4 & 4 & 0 \\
5 & 5 & 4 & 5 & 4 & 1 \\
\end{array}
\]

Then \( (X, *, 0) \) is a BCI-algebra and \( A = \{0, 1, 2, 3\} \) is a medial-ideal of \( X \).

### 3. Fuzzy medial ideals

**Definition 3.1** [5]. Let \( X \) be a BCI-algebra. A fuzzy set \( \mu \) in \( X \) is called a fuzzy BCI-ideal of \( X \) if it satisfies:

(FI1) \( \mu(0) \geq \mu(x) \),

(FI2) \( \mu(x) \geq \min\{\mu(x * y), \mu(y)\} \), for all \( x, y \) and \( z \in X \).

**Definition 3.2** [7]. Let \( X \) be a BCI-algebra. A fuzzy set \( \mu \) in \( X \) is called a fuzzy medial ideal of \( X \) if it satisfies:

(FM1) \( \mu(0) \geq \mu(x) \),

(FM2) \( \mu(x) \geq \min\{\mu(z * (y * x)), \mu(y * z)\} \), for all \( x, y \) and \( z \in X \).

**Lemma 3.3.** Any fuzzy medial-ideal of a BCI-algebra is a fuzzy BCI-ideal of \( X \).

**Proof.** Clear.

### 4. Intuitionistic fuzzy medial ideals in BCI-algebras

An Intuitionistic fuzzy set (briefly IFS) \( A \) in a nonempty set \( X \) is an object having the form \( A = \{(x, \mu_A(x), \lambda_A(x)) | x \in X\} \), where the function \( \mu_A : X \rightarrow [0,1] \), \( \lambda_A : X \rightarrow [0,1] \) denote the degree of membership, degree of non membership, respectively and \( 0 \leq \mu_A(x) + \lambda_A(x) \leq 1 \), for all \( x \in X \).

An IFS \( A = \{(x, \mu_A(x), \lambda_A(x)) | x \in X\} \) in \( X \) can be identified to an order pair \((\mu_A, \lambda_A)\) in \( I^X \times I^X \).

We shall use the symbol \( A = (\mu_A, \lambda_A) \) for IFS \( A = \{(x, \mu_A(x), \lambda_A(x)) | x \in X\} \).
**Definition 4.1.** An IFS \( (\mu, \lambda) \) in a BCI-algebra \( X \) is called an intuitionistic fuzzy medial subalgebra of \( X \) if it satisfies the following:

(IFMS1) \( \mu_*(x \ast y) \geq \mu(x), \mu(y) \),

(IFMS2) \( \lambda_*(x \ast y) \leq \max\{\lambda_*(x), \lambda_*(y)\} \), for all \( x, y \in X \).

**Example 4.2.** Let \( X = \{0, 1, 2, 3, 4, 5\} \) as in example 2.6, and \( (\mu, \lambda) \) be an IFS in \( X \) defined by \( \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0.3 < 0.7 = \mu_0 \), and \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0.5 > 0.2 = \lambda_0 \). Then \( (\mu, \lambda) \) is an intuitionistic fuzzy medial subalgebra of \( X \).

**Lemma 4.3.** Every intuitionistic fuzzy medial subalgebra \( (\mu, \lambda) \) of \( X \) satisfies the inequalities \( \mu_*(0) \geq \mu_*(x) \), and \( \lambda_*(0) \leq \lambda_*(x) \) for all \( x \in X \).

**Proof.** Clear.

**Definition 4.4[4].** An IFS \( (\mu, \lambda) \) in \( X \) is called an intuitionistic fuzzy ideal of \( X \) if it satisfies the following inequalities:

(IFI1) \( \mu_*(0) \geq \mu_*(x) \) and \( \lambda_*(0) \leq \lambda_*(x) \)

(IFI2) \( \mu_*(x) \geq \min\{\mu_*(x \ast y), \mu_*(y)\} \),

(IFI3) \( \lambda_*(x) \leq \max\{\lambda_*(x \ast y), \lambda_*(y)\} \), for all \( x, y \in X \).

**Definition 4.5.** An IFS \( (\mu, \lambda) \) in \( X \) is called an intuitionistic fuzzy medial ideal of \( X \) if it satisfies the following inequalities:

(IFM1) \( \mu_*(0) \geq \mu_*(x) \) and \( \lambda_*(0) \leq \lambda_*(x) \)

(IFM2) \( \mu_*(x) \geq \min\{\mu_*(z \ast (y \ast x), \mu_*(y \ast z))\} \),

(IFM3) \( \lambda_*(x) \leq \max\{\lambda_*(z \ast (y \ast x), \lambda_*(y \ast z))\} \), for all \( x, y, z \in X \).

**Example 4.6:** Let \( X = \{0, 1, 2, 3\} \) be a set with a binary operation \( \ast \) define by the following table:

<table>
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<tr>
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<th>0</th>
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Define IFS \( (\mu, \lambda) \) in \( X \) as follows \( \mu_*(0) = \mu_*(1) = 1, \mu_*(2) = \mu_*(3) = t \).

\( \lambda_*(0) = \lambda_*(1) = 0, \lambda_*(2) = \lambda_*(3) = s \). Where \( t, s \in [0, 1], t + s \leq 1 \). By routine calculations we can prove that \( (\mu, \lambda) \) is an intuitionistic fuzzy medial ideal of \( X \).
Lemma 4.7. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy medial ideal of $X$. If $x \leq y$ in $X$, then $\mu_A(x) \geq \mu_A(y)$, $\lambda_A(x) \leq \lambda_A(y)$, for all $x, y \in X$. That is $\mu_A$ is order reserving and $\lambda_A$ is order preserving.

**Proof.** Let $x, y \in X$ be such that $x \leq y$, then $x * y = 0$. From (IFM2), we have,

$$
\mu_A(x) \geq \min\{\mu_A(0 * (y * x)), \mu_A(y * 0)\} = \min\{\mu_A((x * y), \mu_A(y)\}
$$

$$
= \min\{\mu_A(0), \mu_A(y)\} = \mu_A(y).
$$

Similarly, from (IFM3), we have $\lambda_A(x) \leq \max\{\lambda_A(0 * (y * x)), \lambda_A(y * 0)\}$, hence, $\lambda_A(x) \leq \max\{\lambda_A(x * y), \lambda_A(y)\} = \max\{\lambda_A(0), \lambda_A(y)\} = \lambda_A(y)$.

Lemma 4.8. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy medial ideal of $X$. If the inequality $z \leq x$ holds in $X$, then $\mu_A(x) \geq \min\{\mu_A(y), \mu_A(z)\}$, $\lambda_A(x) \leq \max\{\lambda_A(y), \lambda_A(z)\}$, for all $x, y, z \in X$.

**Proof.** Let $x, y, z \in X$ be such that $x * y \leq z$. Thus, put $z = 0$ in (IFM2) and using lemma 4.7, we get

$$
\mu_A(x) \geq \min\{\mu_A(0 * (y * x)), \mu_A(y * 0)\} = \min\{\mu_A(x * y), \mu_A(y)\} \geq \min\{\mu_A(z), \mu_A(y)\}.
$$

Similarly for $\lambda_A(x)$.

**Theorem 4.9.** Every intuitionistic fuzzy medial ideal of $X$ is an intuitionistic fuzzy medial subalgebra of $X$.

**Proof.** Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy medial ideal of $X$. Since $x * y \leq x$, for all $x, y \in X$, then $\mu_A(x * y) \geq \mu_A(x)$, $\lambda_A(x * y) \leq \lambda_A(x)$.

Put $z = 0$ in (IFM2), (IFM3), we have,

$$
\mu_A(x * y) \geq \mu_A(x) \geq \min\{\mu_A(0 * (y * x)), \mu_A(y * 0)\} = \min\{\mu_A(x * y), \mu_A(y)\}
$$

$$
\geq \min\{\mu_A(z), \mu_A(y)\}.
$$

Similarly for $\lambda_A(x)$. Then $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy subalgebra of $X$.

The converse of theorem 4.9 may not be true. For example, the intuitionistic fuzzy subalgebra $A = (\mu_A, \lambda_A)$ in example 4.2 is not an intuitionistic fuzzy medial ideal of $X$ since $\mu_A(1) = 0.3 < 0.7 = \min\{\mu_A(4 * (4*1)), \mu_A(4*4)\}$.

**Theorem 4.10.** Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy medial subalgebra of $X$ such that $x * y \leq z$ for all $x, y, z \in X$. Then $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy medial ideal of $X$.

**Proof.** Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy medial subalgebra of $X$. Recall that $\mu_A(0) \geq \mu_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$, for all $x \in X$. Since, for all $x, y, z \in X$, we have $x * (z * (y * x)) = (y * x) * (z * x) \leq y * z$, it follows from lemma 4.8 that $\mu_A(x) \geq \min\{\mu_A((z * (y * x)), \mu_A(y * z)\}$, $\lambda_A(x) \leq \max\{\lambda_A((z * (y * x)), \lambda_A(y * z)\}$.

Hence $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy medial ideal of $X$. 

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**Fuzzy medial ideals**

Theorem 4.9. Every intuitionistic fuzzy medial ideal of $X$ is an intuitionistic fuzzy medial subalgebra of $X$.
**Definition 4.11:** For any \( t \in [0,1] \) and a fuzzy set \( \mu \) in a nonempty set \( X \), the set \( U(\mu, t) := \{ x \in X \mid \mu(x) \geq t \} \) is called an upper \( t \)-level cut of \( \mu \), and the set \( L(\mu, t) := \{ x \in X \mid \mu(x) \leq t \} \) is called a lower \( t \)-level cut of \( \mu \).

**Theorem 4.12:** An IFS \( \langle \mu, \lambda \rangle \) is an intuitionistic fuzzy medial ideal of \( X \) if and only if for all \( s, t \in [0,1] \), the set \( U(\mu, s, a) \) and \( L(\lambda, s, a) \) are either empty or medial ideals of \( X \).

**Proof.** Straightforward.

## 5. Homomorphisms of medial BCI-algebras

Let \( (X, *, 0) \) and \( (Y, *, 0') \) be BCI-algebras. A mapping \( f : X \to Y \) is said to be a homomorphism if \( f(x * y) = f(x)' * f(y) \) for all \( x, y \in X \). Note that if \( f : X \to Y \) is a homomorphism of BCI-algebras, then \( f(0) = 0' \). Let \( f : X \to Y \) be a homomorphism of medial BCI-algebras. For any IFS \( \langle \mu, \lambda \rangle \) in \( Y \), define an IFS \( \langle \mu', \lambda' \rangle \) in \( X \) by \( \mu'(x) := \mu(f(x)) \), and \( \lambda'(x) := \lambda(f(x)) \) for all \( x \in X \).

**Theorem 5.1.** Let \( f : X \to Y \) be a homomorphism of BCI-algebras. If an IFS \( \langle \mu, \lambda \rangle \) in \( Y \) is an intuitionistic fuzzy medial ideal of \( Y \), then an IFS \( \langle \mu', \lambda' \rangle \) in \( X \) is an intuitionistic fuzzy medial ideal in \( X \).

**Proof.** For all \( x, y, z \in X \), we have \( \mu'(x) \leq \mu(f(0)) = \mu'(0) \), and \( \lambda'(x) \geq \lambda(f(0)) = \lambda'(0) \).

**Theorem 5.2:** Let \( f : X \to Y \) be an epimorphism of BCI-algebras and let \( \langle \mu, \lambda \rangle \) be an IFS in \( X \). If \( \langle \mu', \lambda' \rangle \) is an intuitionistic fuzzy medial ideal of \( X \), then \( \langle \mu, \lambda \rangle \) is an intuitionistic fuzzy medial ideal in \( Y \).

**Proof.** For any \( a \in Y \), there exists \( x \in X \) such that \( f(x) = a \). Then

\[
\mu'(a) = \mu(f(x)) = \mu'(x) \leq \mu'(0) = \mu(f(0)) = \mu(0),
\]

\[
\lambda'(a) = \lambda(f(x)) = \lambda'(x) \geq \lambda'(0) = \lambda(f(0)) = \lambda(0).
\]

Let \( a, b, c \in Y \), there exists \( x, y, z \in X \) such that \( f(x) = a, f(y) = b, f(z) = c \). It follows that
μₐ(a) = μₐ(f(x)) = μₐ(x) ≥ min{μₐ(z * (y * x)), μₐ(y * z)}
= min{μₐ(f(z * (y * x))), μₐ(f(y * z))} = min{μₐ(f(z) * f(y * x)), μₐ(f(y) * f(z))}
= min{μₐ(f(z) * f(y) * f(x)), μₐ(f(y) * f(z))} = min{μₐ(c * (b * a)), μₐ(b * c)}.

Similarly, λₐ(a) ≤ max{λₐ(c * (b * a)), λₐ(b * c)}. This completes the proof.

6. Cartesian product of intuitionistic fuzzy medial ideals

Let μ and λ be two fuzzy sets in the set X. The product λ × μ : X × X → [0,1] is defined by (λ × μ)(x, y) = min{λ(x), μ(y)}, for all x, y ∈ X.

Let A = (X, λₐ, μₐ) and B = (X, λᵦ, μᵦ) be two IFS of X, the Cartesian product A × B = (X × X, μₐ × μᵦ, λₐ × λᵦ) is defined by μₐ × μᵦ(x, y) = min{μₐ(x), μᵦ(y)} and λₐ × λᵦ(x, y) = max{λₐ(x), λᵦ(y)}, where μₐ × μᵦ : X × X → [0,1], for all x, y ∈ X.

Remark 6.1: Let X and Y be medial BCI-algebras, we define* on X × Y by, for every (x, y), (u, v) ∈ X × Y, (x, y) * (u, v) = (x * u, y * v). Clearly (X × Y; *, (0,0)) is a medial BCI-algebra.

Proposition 6.2: Let A = (X, λₐ, μₐ), B = (X, λᵦ, μᵦ) be intuitionistic fuzzy medial ideals of X, then A × B is intuitionistic fuzzy medial ideal of X × X.

Proof. μₐ × μᵦ(0,0) = min{μₐ(0), μᵦ(0)} ≥ min{μₐ(x), μᵦ(y)} = μₐ × μᵦ(x, y), for all x, y ∈ X.

And λₐ × λᵦ(0,0) = max{λₐ(0), λᵦ(0)} ≤ max{λₐ(x), λᵦ(y)} = λₐ × λᵦ(x, y), for all x, y ∈ X. Now let (x₁, x₂), (y₁, y₂), (z₁, z₂) ∈ X × X, then

min{μₐ × μᵦ((z₁, z₂) * ((y₁, y₂) * (x₁, x₂))}, μₐ × μᵦ((y₁, y₂) * (z₁, z₂))

= min{μₐ × μᵦ(μₐ(z₁ × (y₁ * x₁)), μₐ(z₂ × (y₂ * x₂)))}, μₐ × μᵦ(μᵦ(y₁ × z₁), μᵦ(y₂ × z₂))

= min{μₐ(z₁ × (y₁ * x₁)), μᵦ(z₂ × (y₂ * x₂))}, μₐ(y₁ × z₁), μᵦ(y₂ × z₂)

≤ min{μₐ(z₁ × (y₁ * x₁)), μᵦ(z₂ × (y₂ * x₂))}, μᵦ(y₁ × z₁), μᵦ(y₂ × z₂)

≤ min{μₐ(z₁ × (y₁ * x₁)), μᵦ(z₂ × (y₂ * x₂))}, min{μᵦ(y₁ × z₁), μᵦ(y₂ × z₂)}

≤ min{μₐ(z₁ × (y₁ * x₁)), μᵦ(z₂ × (y₂ * x₂))}, μᵦ(y₁ × z₁), μᵦ(y₂ × z₂)

≤ min{μₐ(z₁ × (y₁ * x₁)), μᵦ(z₂ × (y₂ * x₂))}, μᵦ(y₁ × z₁), μᵦ(y₂ × z₂)

Similarly we can prove that,

max{λₐ × λᵦ(μₐ(z₁ × (y₁ * x₁)), μᵦ(z₂ × (y₂ * x₂))}, λᵦ(y₁ × z₁), λᵦ(y₂ × z₂)} ≥

(λₐ × λᵦ(x₁, x₂)). This completes the proof.

Definition 6.3: Let A = (X, λₐ, μₐ) and B = (X, λᵦ, μᵦ) be IFS of a BCI-algebra X. for s, t ∈ [0,1] the set U(μₐ × μᵦ, s) := {(x, y) ∈ X × X | (μₐ × μᵦ)(x, y) ≥ s} is called upper s-level of (μₐ × μᵦ)(x, y) and the set
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Theorem 6.4: The intuitionistic fuzzy sets \( A = (X, \lambda_A, \mu_A) \) and \( B = (X, \lambda_B, \mu_B) \) are intuitionistic fuzzy medial ideals of \( X \) if and only if the non-empty set upper \( s \)-level cut \( U(\mu_A \times \mu_B, s) \) and the non-empty lower \( t \)-level cut \( L(\lambda_A \times \lambda_B, t) \) are medial ideals of \( X \times X \) for all \( s, t \in [0,1] \).

Proof. Let \( A = (X, \lambda_A, \mu_A) \) and \( B = (X, \lambda_B, \mu_B) \) be intuitionistic fuzzy medial ideals of \( X \), therefore for any \( (x,y) \in X \times X \), we have
\[
\mu_A \times \mu_B(0,0) = \min\{\mu_A(0), \mu_B(0)\} \geq \min\{\mu_A(x), \mu_B(y)\} = \mu_A \times \mu_B(x,y).
\]
Let \( (x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X \) and \( s \in [0,1] \), such that
\[
((z_1, z_2) \ast ((y_1, y_2) \ast (x_1, x_2))) \in U(\mu_A \times \mu_B, s), \quad \text{and} \quad (y_1, y_2) \ast (z_1, z_2) \in U(\mu_A \times \mu_B, s).
\]
Now \( (\mu_A \times \mu_B)(x_1, x_2) \)
\[
\geq \min\{\mu_A \ast \mu_B((z_1, z_2) \ast ((y_1, y_2) \ast (x_1, x_2))), (\mu_A \ast \mu_B)((y_1, y_2) \ast (z_1, z_2))\}
\]
\[
= \min\{\mu_A \ast \mu_B((z_1, z_2) \ast (y_1 \ast x_1, y_2 \ast x_2)), (\mu_A \ast \mu_B)((y_1 \ast z_1, y_2 \ast z_2))\}
\]
\[
= \min\{\mu_A \ast \mu_B((z_1 \ast (y_1 \ast x_1), z_2 \ast (y_2 \ast x_2)), (\mu_A \ast \mu_B)((y_1 \ast z_1, y_2 \ast z_2))\} \geq \min\{s, s\} = s,
\]
Therefore \( (x_1, x_2) \in U((\mu_A \times \mu_B)(x,y), s) \) is a medial ideal of \( X \times X \). Similar to above \( L((\lambda_A \times \lambda_B)(x,y), t) \) is a medial ideal of \( X \times X \). This completes the proof.

References


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