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Notes on s-Semiembedded Subgroups of Finite Groups[1]

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Abstract

In this notes, some examples were obtained to show that some theorems in the paper entitled "On s-Semiembedded Subgroups of Finite Groups" [Algebra Colloquium 20:1(2013)65-74] are not true in general. Then the revised version of the related theorems are given.

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1 Introduction

All groups considered in this paper will be finite. We use conventional notions and notation, as in Huppert [4]. G always denotes a finite group, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$, G_p is a Sylow p-subgroup of G for some $p \in \pi(G)$.

Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ *a formation* provided that (i) if $G \in \mathcal{F}$ and $H \lhd G$, then $G/H \in \mathcal{F}$, and (ii) if G/M and G/N are in $\mathcal{F}$, then

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$G/(M \cap N)$ is in $\mathcal{F}$ for all normal subgroups M, N of G. A formation $\mathcal{F}$ is said to be *saturated* if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In this paper, $\mathcal{U}$ will denote the class of all supersolvable groups. Clearly, $\mathcal{U}$ is a saturated formation (ref. [4, p.713, Satz 8.6]).

Two subgroups H and K of G are said to be *permutable* if $HK = KH$. A subgroup H of G is said to be *s-permutable* (or *s-quasinormal, π-quasinormal*) [5] in G if H permutes with every Sylow subgroup of G; H is said *c-normal* [8] in G if G has a normal subgroup T such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of H in G. Recently, Guo etc in [3] introduces the following concept, which covers both s-permutability and c-normality: Let H be a subgroup of G. H is called s-embedded in G if there is a normal subgroup T of G such that HT is s-permutable in G and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G.

As a generalization of s-permutability, a subgroup H of G is said *s-semipermutable* in G ([1]) if H permutes with every Sylow p-subgroup G_p of G with $(|H|, p) = 1$. We know that an s-permutable subgroup of G is subnormal in G, but s-semipermutable subgroups of G are not necessarily subnormal in G, the Sylow 2-subgroup of S_3 is a counterexample. In [6], the author give a new concept which covers properly both s-semipermutability and s-embeddedness.

Definition ([6]) *Let H be a subgroup of G. We say H is s-semiembedded in G if there is a normal subgroup T of G such that HT is s-semipermutable and $H \cap T \leq H_{ssG}$, where H_{ssG} is the subgroup of H generated by all those subgroups of H which are s-semipermutable in G.*

The author first gives some properties of s-semipermutable subgroup and s-semiembedded subgroup of finite groups:

[6, Lemma 2.1] *Suppose that H is an s-semipermutable subgroup of G and N is a normal subgroup of G. Then*

(a) If $H \leq K \leq G$, then H is s-semipermutable in K;
(b) HN is s-semipermutable in G;
(c) HN/N is s-semipermutable in G/N;
(d) $H \cap N$ is s-semipermutable in G;
(e) If $H \leq O_p(G)$, then H is s-permutable in G.

[6, Lemma 2.3] *Let U be a s-semiembedded subgroup of G and N a normal subgroup of G. Then* *(a) If $U \leq H \leq G$, then U is s-semiembedded in H.*

(b) Suppose that $N \leq U$. Then U/N is s-semiembedded in G/N. Conversely, if U/N is s-semiembedded in G/N, then U also is s-semiembedded in G.

(c) Suppose that U is a π-group and N is a $\pi'-$subgroup. Then UN/N is s-semiembedded in G/N.

(d) If $U \leq O_p(G)$ for some prime p, then U is s-embedded in G.

Based on the lemmas above, the author get the following results:

[6, Theorem 1.6] *Let G be a group. Then G is solvable if and only if every maximal subgroup M of G is s-semiembedded in G.*

[6, Theorem 1.7] *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that, for any non-cyclic Sylow subgroup P of E, there holds either every maximal subgroup of P or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) is s-semiembedded in G. Then $G \in \mathcal{F}$.*

[6, Theorem 1.8] *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose, for any non-cyclic Sylow subgroup P of $F^*(E)$, there holds either every maximal subgroup of P or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) is s-semiembedded in G. Then $G \in \mathcal{F}$.*

The following results related to p-nilpotency of groups are main steps in the proofs of [6, Theorem 1.6, 1.7 and 1.8].

[6, Theorem 1.9] *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are s-semipermutable in G, then G is p-nilpotent.*

[6, Theorem 1.10] *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. Then G is p-nilpotent if and only if all maximal subgroups of P are s-semiembedded in G.*

[6, Theorem 1.11] *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. Then G is p-nilpotent if and only if there holds either every maximal subgroup of P or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$)*

is s-semiembedded in G.

Unfortunately, [6, Lemma 2.1(c) and (d)] and [6, Lemma 2.3(a), (b) and (c)] are not true in general, neither are [6, Theorem 1.6, 1.7, 1.8, 1.10 and 1.11]. But [6, Theorem 1.9] is true.

2 Counterexamples

Example 2.1 *Let $G = A_4 \times C_3$, where A_4 is the alternative group on 4 letters and C_3 is the cyclic group of order 3. Let x, y are two elements of order 2 in A_4 and let $H = \langle x \rangle \times C_3$. Put $N_1 = 1 \times C_3$ and $N_2 = \langle x, y \rangle$. Then N_1, N_2 are two normal subgroups of G.*

Since $\pi(H) = \pi(G)$, H is s-semipermutable in G. Then $G/N_1 \cong A_4$ and $H/N_1 \cong \langle x \rangle$. But $\langle x \rangle$ does not permute with any Sylow 3-subgroup of A_4 since A_4 has no any subgroup of order 6. Hence H/N_1 is not s-semipermutable in G/N_1. Thus [6, Lemma 2.1(c)] is false in general.

We know that $H \cap N_2 = \langle x \rangle$. If $\langle x \rangle$ is s-semipermutable in G, then $\langle x \rangle$ is s-semipermutable in A_4. So $\langle x \rangle$ permutes with any Sylow 3-subgroup of A_4, a contradiction. Thus [6, Lemma 2.1(d)] is false in general.

Since C_3 is normal in G and $\langle x \rangle \times C_3 = H$ is s-semipermutable in G and $\langle x \rangle \cap C_3 = 1$, $\langle x \rangle$ is s-semiembedded in G. But $\langle x \rangle$ is not s-semiembedded in A_4. In fact, if $\langle x \rangle$ is s-semiembedded in A_4. Then there exists a normal subgroup T of A_4 such that $\langle x \rangle T$ is s-semipermutable in A_4 and $\langle x \rangle \cap T \leq \langle x \rangle_{ssG}$. If $T = 1$, then $\langle x \rangle T = \langle x \rangle$ is s-semipermutable in A_4, a contradiction. If $T \neq 1$, then $T = \langle x, y \rangle = N_2$ or A_4. So $\langle x \rangle \cap T = \langle x \rangle$ is s-semipermutable in A_4, a contradiction. Thus [6, Lemma 2.3(a)] is false in general.

Since H is s-semipermutable in G, H is s-semiembedded in G. But $H/N_1 \cong \langle x \rangle$ and $N_1 \leq H$, $G/N_1 \cong A_4$ and $\langle x \rangle$ is not s-semiembedded in A_4. Hence [6, Lemma 2.3(b)] is false in general.

Since $\langle x \rangle$ is s-semiembedded in G and $(|\langle x \rangle|, |N_1|) = (2, 3) = 1$, $\langle x \rangle N_1/N_1 \cong \langle x \rangle$, $G/N_1 \cong A_4$ and $\langle x \rangle$ is not s-semiembedded in A_4. Thus [6, Lemma 2.3(c)] is false in general. □

Example 2.2 *Let $G = A_4 \times C_3$. Let $p = 2$ and P is a Sylow 2-subgroup of G. Let x be any element of order 2 in A_4. Then $\langle x \rangle$ is s-semiembedded in G. Noticing $\langle x \rangle$ is maximal in P, but G is not 2-nilpotent. Hence [6, Theorem 1.10] is false in general.*

Furthermore, $\langle x \rangle$ is a subgroup of P of order 2 and P is abelian, hence [6, Theorem 1.11] is false in general. □

Example 2.3 *Let $\mathcal{F} = \mathcal{U}$ be the class of all supersolvable groups. Put $G = E = A_4 \times C_3$. Let x, y are two elements of order 2 in A_4. Then $G/E \in \mathcal{U}$.*

$F^*(E) = \langle x, y\rangle \times C_3$. The non-cyclic Sylow subgroup of $F^*(E)$ is $\langle x, y\rangle$, the Sylow 2-subgroup of $F^*(E)$. It is easy to see that $\langle x\rangle$ is maximal in $\langle x, y\rangle$ and $\langle x\rangle$ is s-semiembedded in G. But G is not supersolvable. Hence [6, Theorem 1.8] is false in general.

Furthermore, Let z is an element of order 3 in A_4. Then $\langle x, y\rangle$ is the Sylow 2-subgroup of G and $\langle z\rangle \times C_3$ is a Sylow 3-subgroup of G. Noticing every subgroup of $\langle x, y\rangle$ of order 2 is s-semiembedded in G and every subgroup of $\langle z\rangle \times C_3$ of order 3 is s-semipermutable in G, $\langle x, y\rangle$ is abelian. But G is not supersolvable. Hence [6, Theorem 1.7] is false in general. □

Example 2.4 *Let $G = L_2(7)$, the simple group of order 168. Suppose that M is a maximal subgroup of G. Then $[G : M] = 7$ or 8 (ref. [2, page 3]). Hence M is s-semipermutable in G. Therefore, [6, Theorem 1.6] is false in general.* □

3 New Results

Now we introduce the following new concept which is stronger than s-semiembeddedment.

Definition 3.1 *Let H be a subgroup of G. H is called weakly s-embedded in G if there is a normal subgroup T of G such that HT is s-permutable and $H \cap T \leq H_{ssG}$, where H_{ssG} is the subgroup of H generated by all those subgroups of H which are s-semipermutable in G.*

Following [7], a a subgroup H of G is called to be a *weakly s-semipermutable subgroup* of G if there exits a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{ssG}$.

By the definitions, obviously, we have s-permutable subgroup ⇒ s-semipermutable subgroup ⇒ weakly s-semipermutable subgroup ⇒ weakly s-embedded subgroup, s-embedded subgroup ⇒ weakly s-embedded subgroup. But the converse does not hold in general.

Example 3.2 (1). Suppose that $G = A_5$, the alternative group of degree 5. Then A_4 is weakly s-embedded in G, but not s-embedded in G.

(2). Let E be the following elementary abelian group of order 25:

$$E = \langle a, b | a^5 = b^5 = 1, ab = ba\rangle$$

and let α be an automorphism of E of order 4 satisfying

$$a^\alpha = a^2, b^\alpha = b^4.$$

Denote that $G = [E]\langle\alpha\rangle$. Let $A = \langle ab\rangle$. Then A is weakly s-embedded in G but not weakly s-semipermutable in G.

In fact, pick $T = \langle a\rangle$. Then $AT = E$ is normal in G and $A \cap T = 1$. Hence A is weakly s-embedded in G. Now we prove that A is not weakly s-semipermutable in G.

Suppose that A is weakly s-semipermutable in G. Then there is a subnormal subgroup T of G such that $G = AT$ and $A \cap T \leq A_{ssG}$. If $T = G$, then $A = A_{ssG}$ is s-semipermutable in G. Then $A\langle\alpha\rangle$ is a group. By the well-known Sylow theorem, we have A is normal in $A\langle\alpha\rangle$. But this is a contradiction as $(ab)^\alpha = a^2b^4 \notin A$. So we have $A_{ssG} = 1$ and T is of order 20 and T is maximal in G. So T is normal in G. Since $E \not\leq T$, we may assume $a \notin T$ without loss of generality. Now we have

$$[a, \alpha] \in \langle a\rangle \cap T = 1,$$

a contradiction. Hence A is not weakly s-semipermutable in G.

Example 2.1 indicates that [6, Lemma 2.1 (c) and (d)] are not true in general. But they are hold under extra conditions.

Lemma 3.3 *Suppose that H is an s-semipermutable subgroup of G and N is a normal subgroup of G and X is a subgroup of G. If H is a p-group for some prime p, then*

(a) HN/N is s-semipermutable in G/N;

(b) $H \cap N$ is s-semipermutable in G.

Proof. (a) is [9, Property 2]. (b) is [7, Lemma 2.2(4)]. □

Using the same arguments in the proof of [6, Lemma 2.3], we have

Lemma 3.4 *Let U be a weakly s-embedded subgroup of G and N a normal subgroup of G. Then*

(a) If $U \leq H \leq G$, then U is weakly s-embedded in H.

(b) Suppose that $N \leq U$. Then U/N is weakly s-embedded in G/N. Conversely, if U/N is weakly s-embedded in G/N, then U also is weakly s-embedded in G.

(c) Suppose that U is a π-group and N is a $\pi'-$subgroup. Then UN/N is weakly s-embedded in G/N.

(d) If $U \leq O_p(G)$ for some prime p, then U is s-embedded in G.

Using the same proofs of corresponding results and considering that [6, Theorem 1.9] is true, [6, Theorem 1.7, 1.8 and 1.11] can be revised as follows, respectively.

Theorem 3.5 *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. Then G is p-nilpotent if and only if there holds either every maximal subgroup of P or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) is weakly s-embedded in G.*

Theorem 3.6 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that, for any non-cyclic Sylow subgroup P of E, there holds either every maximal subgroup of P or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) is weakly s-embedded in G. Then $G \in \mathcal{F}$.*

Theorem 3.7 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose, for any non-cyclic Sylow subgroup P of $F^*(E)$, there holds either every maximal subgroup of P or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) is weakly s-embedded in G. Then $G \in \mathcal{F}$.*

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