On $P$-semihereditary Rings

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Abstract

In this paper, we introduce the notion of “$P$-semihereditary rings” which is a generalization of the notion of semihereditary rings. We establish the transfer of this notion to trivial ring extensions and provide a class of $P$-semihereditary rings which are not a semihereditary rings.

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1 Introduction

All rings considered below are commutative with unit and all modules are unital. Recall that a ring $R$ is called semihereditary if every finitely generated ideal $I$ of $R$ is projective. Recall that a semihereditary integral domain is a Prüfer domain. We introduce a new concept of a ”$P$-semihereditary” ring. A ring $R$ is called $P$-semihereditary if every finitely generated prime ideal $P$ of $R$ is projective. A semihereditary ring is naturally a $P$-semihereditary ring.

Let $A$ be a ring, $E$ be an $A$-module and $R := A \ltimes E$ be the set of pairs $(a, e)$ with pairwise addition and multiplication given by: $(a, e)(b, f) = (ab, af + be)$. $R$ is called the trivial ring extension of $A$ by $E$. Recall that a prime ideal of $R$ has always the form $Q \ltimes E$, where $Q$ is a prime ideal of $A$ [4, Theorem 25.1]. Considerable work, part of it summarized in Glaz’s book [3] and Huckaba’s
book [4], has been concerned with trivial ring extensions. These have proven to be useful in solving many open problems and conjectures for various contexts in (commutative and non-commutative) ring theory. See for instance [3, 4, 6].

Our aim in this paper is to prove that $P$-semihereditary rings are not semihereditary rings, in general. Further, we investigate the possible transfer of the $P$-semihereditary property to various trivial extension constructions and homomorphic image.

2 Main Results

In this section, we study the possible transfer of the $P$-semihereditary property to various trivial extension contexts and homomorphic image. First, we examine the context of trivial ring extensions of a ring $A$ by a divisible flat $A$-module $E$.

**Theorem 2.1** Let $A$ be a ring, $E$ be a divisible flat $A$-module, and $R := A \propto E$ be the trivial ring extension of $A$ by $E$. Then:
1) $R$ is a $P$-semihereditary ring if and only if so is $A$.
2) $R$ is a never semihereditary ring.

We need the following lemma before proving Theorem 2.1. An $R$-module $M$ is called $P$-flat if, for any $(s, x) \in R \times M$ such that $sx := 0$, $x \in (0 : s)M$. If $M$ is flat, then $M$ is naturally $P$-flat. In the domain case $P$-flat is equivalent to torsion-free and when $R$ is an arithmetical ring (i.e., the lattice formed by its ideals is distributive), then any $P$-flat module is flat (by [2, p. 236]). Also, every $P$-flat cyclic module is flat (by [2, Proposition 1(2)]).

**Lemma 2.2** Let $A$ be a ring, $E$ be an $A$-module, $F(\neq 0)$ be a sub-module of $E$ such that $Z(F) \neq 0$ and $R := A \propto E$ be a trivial ring extension of $A$ by $E$. Then $0 \propto F$ is not a $P$-flat $R$-module.

**Proof.** Let $(0, f) (\neq (0, 0))$ and $(0, e) (\neq (0, 0))$ two elements of $(0 \propto F)$. Then, $(0, f)(0, e) := (0, 0)$ and $(0 : (0, e)) := (0 \propto E)$ since $Z(F) = 0$. Then $(0, f) \notin (0 : (0, e))(0 \propto F) := (0 \propto E)(0 \propto F) := 0$. Thus $0 \propto F$ is not a $P$-flat $R$-module.
Proof of Theorem 2.1.
1) Assume that $R$ is a $P$-semihereditary ring and let $Q := \sum_{i=1}^{n} Ab_i$ be a nonzero finitely generated prime ideal of $A$, where $b_i \in Q$ for each $i = 1, \ldots, n$. Set $P := Q \propto E$. Then, $P \in \text{Spec}(R)$ by [4, Theorem 25.1] and $P = \sum_{i=1}^{n} R(b_i, 0)$ since $E$ is a divisible $A$-module. Hence, $P$ is a projective ideal of $R$ and so $Q$ is a projective ideal of $A$ since $P \otimes_R (R/(O \propto E)) = P/P(O \propto E) = (Q \propto E)/(0 \propto E) \cong Q$, as desired.

Conversely, assume that $A$ is a $P$-semihereditary ring and let $P := \sum_{i=1}^{n} R(b_i, e_i)$ be a nonzero finitely generated prime ideal of $R$, where $(b_i, e_i) \in P$ for each $i = 1, \ldots, n$. Then $P$ has the form $P := Q \propto E$ by [4, Theorem 25.1], where $Q$ is a prime ideal of $A$. Since $0 \propto E$ is not a finitely generated ideal of $R$ (since $E$ is not finitely generated $A$-modules), then $Q \neq 0$. But $Q = \sum_{i=1}^{n} Ab_i$ which is a nonzero finitely generated prime ideal of $A$, so $Q$ is a projective ideal of $A$ since $A$ is a $P$-semihereditary ring. Therefore, $P(\cong Q \otimes R)$ is a projective ideal of $R$ which means that $R$ is a $P$-semihereditary ring, as desired.

2) Let $F(\neq 0)$ be a finitely generated sub-module of $E$, such that $Z(F) = 0$ and $I = 0 \propto F$ be a finitely generated ideal of $R$. Then is not a projective ideal of $R$ by Lemma 2.2. Therefore, $R$ is not semihereditary and this completes the proof of Theorem 2.1.

Corollary 2.3 Let $A$ be a domain, $K = qf(A)$, $E$ be a $K$-vector space, and $R := A \propto E$ be the trivial ring extension of $A$ by $E$. Then:
1) $R$ is a $P$-semihereditary ring if and only if so is $A$.
2) $R$ is a never semihereditary ring.

Now, we are able to construct a non-semihereditary ring which is a $P$-semihereditary ring.

Example 2.4 Let $A$ be a Prüfer domain (for example, a polynomial ring in one indeterminate over any field), $K := qf(A)$, $E$ be a $K$-vector space, and let $R = A \propto E$. Then:
1) $R$ is a $P$-semihereditary ring by Theorem 2.1 since $A$ is it too.
2) $R$ is not a semihereditary ring by Theorem 2.1.

Next, we explore a different context; namely, the trivial ring extension of a local domain $(A, M)$ by an $A$-module $E$ such that $ME = 0$.

Theorem 2.5 Let $(A, M)$ be a local ring, $E(\neq 0)$ an $A$-module with $ME = 0$, and let $R := A \propto E$ be the trivial ring extension of $A$ by $E$. Then:
1) \( R \) is a \( P \)-semihereditary ring if and only if \( E \) is an \((A/M)\)-vector space of infinite rank.

2) \( R \) is never a semihereditary ring.

We need the following Lemma before proving Theorem 2.5.

\textbf{Lemma 2.6} Let \((A, M)\) be a local ring, \( E(\neq 0) \) an \( A \)-module with \( ME = 0 \) and let \( R := A \times E \) be the trivial ring extension of \( A \) by \( E \). Then, \( R \) is a \( P \)-semihereditary ring provided \( E \) is an \((A/M)\)-vector space of infinite rank.

\textbf{Proof.} Assume that \( E \) is an \((A/M)\)-vector space of infinite rank. Our aim is to show that there exists no proper finitely generated prime ideal of \( R \). Deny. Let \( Q := P \times E = \sum_{i=1}^{n} R(a_i, e_i) \) be a finitely generated prime ideal of \( R \), where \((a_i, e_i) \in Q\) for each \( i = 1, \ldots, n \). Hence, \( E = \sum_{i=1}^{n} Ae_i \) since \( a_iE = 0 \) (since \( a_i \in M \) for each \( i = 1, \ldots, n \), a contradiction since \( E \) is an \((A/M)\)-vector space of infinite rank. Therefore, \( R \) is a \( P \)-semihereditary ring, as desired.

\textbf{Proof of Theorem 2.5.}

1) If \( E \) is an \((A/M)\)-vector space of infinite rank, then \( R \) is a \( P \)-semihereditary ring by Lemma 2.6. Conversely, assume that \( R \) is a \( P \)-semihereditary ring. We claim that \( E \) is an \((A/M)\)-vector space of infinite rank. Deny. Then \( E \) is an \((A/M)\)-vector space of finite rank and let \((x_i)_{i=1}^{m} \) be its basis. Then \( P := 0 \times E = \sum_{i=1}^{m} R(0, x_i) \) is a proper finitely generated prime ideal of \( R \), thus \( P(= R(0, x)) \) for some regular element \((0, x) \in R \) since \( P \) is projective (since \( P \)-semihereditary) and \( R \) is a local ring, a contradiction since \((0, x)(0, x) = (0, 0)\). Hence, \( E \) is an \((A/M)\)-vector space of infinite rank, as desired.

2) We claim that \( R \) is never a semihereditary ring. Deny. Let \( e \in E - \{0\} \), and set \( J := R(0, e) \). Then, \( J \) is a generated by regular element since \( R \) is a local semihereditary ring, a contradiction since \((0, e)J = 0\). Hence, \( R \) is never a semihereditary ring.

\textbf{Remark 2.7} In Theorem 2.5, the surprise is that the \( P \)-semihereditary property holds for a trivial ring extension of a local ring \((A, M)\) by an \((A/M)\)-vector space without any hypothesis on the basic ring \( A \).
Now, we are able to construct a non-semihereditary ring which is a $P$-semihereditary ring.

**Example 2.8** Let $(A,M)$ be a local domain, $E(\neq 0)$ an $(A/M)$-vector space of infinite rank and let $R := A \otimes E$ be the trivial ring extension of $A$ by $E$. Then:
1) $R$ is a $P$-semihereditary ring by Theorem 2.5.
3) $R$ is not a semihereditary ring by Theorem 2.5.

We close this section with a result establish the transfer of $P$-semihereditary property to a particular homomorphic image.

**Proposition 2.9** Let $R$ be a ring and let $I$ be a finitely generated pure ideal of $R$. Then, $R/I$ is a $P$-semihereditary ring, if so is $R$.

**Proof of Proposition 2.9.**
Let $P$ be a finitely generated prime ideal of $R/I$, then there exist a prime ideal $p$ of $R$ containing $I$, such that $P = p/I$. Since $I$ and $P$ are finitely generated $R$-modules, then $p$ is finitely generated prime ideal of $R$. Hence $p$ is projective ideal of $R$ and so $(p \otimes R/I)$ is projective $R/I$-module. On the other hand $(p \otimes R/I) = p/pI = p/I = P$ since $I$ is pure ideal of $R$. Therefore $R/I$ is $P$-semihereditary.

**References**


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