Congruence in Identity Difference Full Transformation Semigroup

A. O. Adeniji

Department of Mathematics
Faculty of Science
University of Abuja, Abuja, Nigeria
adeniji4love@yahoo.com

S. O. Makanjuola

Department of Mathematics
Faculty of Science
University of Ilorin, Ilorin, Nigeria
somakanjuola@unilorin.edu.ng

Copyright © 2013 A. O. Adeniji and S. O. Makanjuola. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

A certain semigroup named identity difference full transformation semigroup, $IDT_n$, in between full transformation semigroup $T_n$ is examined for congruence.
Mathematics Subject Classification: 20M20

Keywords: Congruence, Identity Difference Full Transformation Semigroup and Green’s relations

1. Introduction

A binary relation $\sigma$ on a semigroup $S$ is called left compatible provided that $x\sigma y$ implies $ax \sigma ay$ for all $a, x, y \in S$. $\sigma$ is called right compatible provided that $x \sigma y$ implies $xa \sigma ya$ for all $a, x, y \in S$ and $\sigma$ is called compatible provided that it is both left and right compatible.

A left compatible equivalence relation is called a left congruence. A right compatible equivalence relation is called a right congruence. A compatible equivalence relation is called a congruence.

The five equivalences, known as Green’s relations [4] are defined, on a semigroup $S$ by defining $S'$ to be the semigroup $S$ with a unity element 1 adjoined if necessary.

Let $S$ be a semigroup denoted by $S'$. If $P, Q \in S$, then

$PQ = \{pq : p \in P, q \in Q\}$. A subset $P \subset S'$ is called a left ideal provided that for all $s \in S'$ and $p \in P$, we have $sp \in P$ (in other words $S'P \in P$). If $PS' \subset P$, then $P$ is called a right ideal. The idea of ideal naturally leads to the five important Green’s equivalence relations on $S'$. Left and right ideals are also called one-sided ideals. A left [resp. right or two-sided] ideal $P$ of $S$ is called principal provided that there exists $x \in S$ such that $P = S'x[resp. P = xS', P = S'xS']$. It should be noted that $x \in S'x, x \in xS'$ and $x \in S'xS'$. For all $x, y \in S'$, we define

1. $xRy$ if and only if $xS' = yS'$$

2. $xLy$ if and only if $S'x = S'y$$

3. $xJy$ if and only if $S'xS' = S'yS'$
4. $\mathcal{H} = \mathcal{L} \wedge \mathcal{R}$ (intersection of $\mathcal{L}$ and $\mathcal{R}$)

5. $\mathcal{D} = \mathcal{L} \lor \mathcal{R}$ (the smallest equivalence containing both $\mathcal{L}$ and $\mathcal{R}$).

Many Semigroups have been considered in recent years. The semigroup of all order-preserving singular selfmaps ($\text{Sing}_n$) of $X_n$ defined as

$$O_n = \{ \alpha \in \text{Sing}_n : (\forall x, y \in X_n) x \leq y \Rightarrow x\alpha \leq y\alpha \} \text{ where}$$

$\text{Sing}_n = \{ \alpha \in T_n : |\text{Im}\alpha| \leq n - 1 \}$ was studied by Howie\cite{5} and it was shown that $|O_n| = \binom{2n-1}{n-1} - 1$.

Howie and McFadden\cite{6} considered the semigroup $K(n, r) = \{ \alpha \in T_n : |\text{Im}\alpha| \leq r \}$ ($2 \leq r \leq n - 1$) and showed that both the rank and the idempotent rank are equal to $\mathcal{S}(n, r)$, the Stirling number of the second kind. Garba\cite{2} considered the Semigroup $P_n$ of all partial transformations of $X_n$ and showed that for the semigroup $K'(n, r) = \{ \alpha \in P_n : |\text{Im}\alpha| \leq r \}$ both the rank and idempotent rank are equal to $\mathcal{S}(n + 1, r + 1)$.

Gomes and Howie\cite{3} examined the symmetric inverse semigroup $I_n$ consisting of all partial one-one maps of $X_n$ and showed that the rank (as an inverse semigroup) of the inverse semigroup $SP_n = \{ \alpha \in I_n : |\text{Im}\alpha| \leq n - 1 \}$ is $n + 1$. These semigroups are just to mention few. The semigroup in this paper is called the identity difference full transformation semigroup, $IDT_n$.

The word identity in identity difference transformation semigroup is coined from the usual additive($0$) and multiplicative($1$) identities. The identities were made use of, in the difference between the images of each transformation. The right[left] waist of $\alpha$ is $W^+(\alpha) = \max(\text{Im}\alpha)[W^-(\alpha) = \min(\text{Im}\alpha)]$ as defined by Umar\cite{7}. $\text{Dom}(\alpha)$ is the domain of $\alpha$ while $\text{Im}(\alpha)$ stands for the image of $\alpha$. Hence identity difference transformation is defined when $W^+(\alpha) - W^-(\alpha) \leq 1$.

Let $S$ be a semigroup. The rank of $S$ (denoted by rank $S$) is defined as the minimal number of elements of a generating set of $S$. The generating
set of each element in $S$ can as well be defined as follow: Let $S$ be a semigroup and $A \subseteq S$ be a set. An element $s \in S$ is said to be generated by $A$ provided that $s$ can be written as a finite product of elements from $A$. The set of all elements from $S$ generated by $A$ is usually denoted by $\langle A \rangle$.

2. Congruence on $L$ and $R$ Green’s Relations

The following lemmas and examples are sufficient to show that Green’s relations $L$ is a right congruence and Green’s relation $R$ is a left congruence. $IDT_n$ is used as a case study.

**Lemma 2.1**: Let $S' = IDT_n$ and $a \in S'$. If $xLy$ then $xaLy$.

**Proof**: Given that $xLy$, then $S'x = S'y$ for any $(x, y) \in L$. That is, $x$ and $y$ are in the same Green’s equivalence relation.

Also, $xaLy \Rightarrow S'xa = S'ya \Rightarrow S'(xa) = S'(ya)$. From right compatibility of a binary relation $\sigma$ on a semigroup $S'$, Green’s relation $L$ is a right congruence. For $a, x, y \in S'$, $x \sigma y$ implies $xa \sigma ya$ where Green’s relation $L$ replaces the binary relation $\sigma$.

**Lemma 2.2**: If $xRy$, then $axRy$.

The proof of this can be shown as in Lemma 2.1.

**Example 2.1**

Let $S'$ be the identity difference full transformation semigroup denoted by $IDT_n$. Show that Green’s relation $L$ is a right congruence and Green’s relation $R$ is a left congruence.

For $n = 3$, $IDT_3$ has the following elements

$$
\begin{align*}
\{a &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, c = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}, d = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \\
e &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}, e = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}, f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix}, g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}, h = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix},
\end{align*}
$$

A. O. Adeniji and S. O. Makanjuola
\[ i = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}, \quad j = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix}, \quad k = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}, \quad l = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix}, \]

\[ m = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix}, \quad n = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}, \quad o = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}. \]

**Table 2.1** Illustration of Right and Left Ideals using IDT$_3$

<table>
<thead>
<tr>
<th>.</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
<th>i</th>
<th>j</th>
<th>k</th>
<th>l</th>
<th>m</th>
<th>n</th>
<th>o</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>h</td>
<td>h</td>
<td>h</td>
<td>h</td>
<td>h</td>
<td>h</td>
<td>h</td>
<td>h</td>
<td>h</td>
<td>o</td>
<td>o</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>a</td>
<td>b</td>
<td>b</td>
<td>g</td>
<td>g</td>
<td>h</td>
<td>h</td>
<td>i</td>
<td>i</td>
<td>n</td>
<td>n</td>
<td>n</td>
<td>o</td>
<td>o</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>a</td>
<td>c</td>
<td>c</td>
<td>f</td>
<td>f</td>
<td>h</td>
<td>h</td>
<td>h</td>
<td>j</td>
<td>j</td>
<td>m</td>
<td>m</td>
<td>o</td>
<td>o</td>
</tr>
<tr>
<td>d</td>
<td>a</td>
<td>a</td>
<td>d</td>
<td>d</td>
<td>e</td>
<td>e</td>
<td>h</td>
<td>h</td>
<td>h</td>
<td>k</td>
<td>k</td>
<td>l</td>
<td>l</td>
<td>o</td>
<td>o</td>
</tr>
<tr>
<td>e</td>
<td>a</td>
<td>a</td>
<td>e</td>
<td>e</td>
<td>d</td>
<td>d</td>
<td>h</td>
<td>h</td>
<td>h</td>
<td>l</td>
<td>l</td>
<td>k</td>
<td>k</td>
<td>o</td>
<td>o</td>
</tr>
<tr>
<td>f</td>
<td>a</td>
<td>a</td>
<td>f</td>
<td>f</td>
<td>c</td>
<td>c</td>
<td>h</td>
<td>h</td>
<td>h</td>
<td>m</td>
<td>m</td>
<td>j</td>
<td>j</td>
<td>o</td>
<td>o</td>
</tr>
<tr>
<td>g</td>
<td>a</td>
<td>a</td>
<td>g</td>
<td>g</td>
<td>b</td>
<td>b</td>
<td>h</td>
<td>h</td>
<td>h</td>
<td>n</td>
<td>n</td>
<td>i</td>
<td>i</td>
<td>o</td>
<td>o</td>
</tr>
<tr>
<td>h</td>
<td>a</td>
<td>a</td>
<td>h</td>
<td>h</td>
<td>a</td>
<td>a</td>
<td>h</td>
<td>h</td>
<td>h</td>
<td>o</td>
<td>o</td>
<td>h</td>
<td>h</td>
<td>o</td>
<td>o</td>
</tr>
<tr>
<td>i</td>
<td>a</td>
<td>b</td>
<td>g</td>
<td>h</td>
<td>a</td>
<td>b</td>
<td>g</td>
<td>h</td>
<td>i</td>
<td>n</td>
<td>o</td>
<td>h</td>
<td>i</td>
<td>n</td>
<td>o</td>
</tr>
<tr>
<td>j</td>
<td>a</td>
<td>c</td>
<td>f</td>
<td>h</td>
<td>a</td>
<td>c</td>
<td>f</td>
<td>h</td>
<td>n</td>
<td>j</td>
<td>m</td>
<td>h</td>
<td>j</td>
<td>m</td>
<td>o</td>
</tr>
<tr>
<td>k</td>
<td>a</td>
<td>d</td>
<td>e</td>
<td>h</td>
<td>a</td>
<td>d</td>
<td>e</td>
<td>h</td>
<td>k</td>
<td>l</td>
<td>o</td>
<td>h</td>
<td>k</td>
<td>l</td>
<td>o</td>
</tr>
<tr>
<td>l</td>
<td>a</td>
<td>e</td>
<td>d</td>
<td>h</td>
<td>a</td>
<td>e</td>
<td>d</td>
<td>h</td>
<td>l</td>
<td>k</td>
<td>o</td>
<td>h</td>
<td>l</td>
<td>k</td>
<td>o</td>
</tr>
<tr>
<td>m</td>
<td>a</td>
<td>f</td>
<td>c</td>
<td>h</td>
<td>a</td>
<td>f</td>
<td>c</td>
<td>h</td>
<td>m</td>
<td>j</td>
<td>o</td>
<td>h</td>
<td>m</td>
<td>j</td>
<td>o</td>
</tr>
<tr>
<td>n</td>
<td>a</td>
<td>g</td>
<td>b</td>
<td>h</td>
<td>a</td>
<td>g</td>
<td>b</td>
<td>h</td>
<td>n</td>
<td>i</td>
<td>o</td>
<td>h</td>
<td>n</td>
<td>i</td>
<td>o</td>
</tr>
<tr>
<td>o</td>
<td>a</td>
<td>h</td>
<td>a</td>
<td>h</td>
<td>a</td>
<td>h</td>
<td>a</td>
<td>h</td>
<td>o</td>
<td>h</td>
<td>o</td>
<td>h</td>
<td>o</td>
<td>h</td>
<td>o</td>
</tr>
</tbody>
</table>

**Example 2.2**

Let \((x, y) \in \mathcal{L}, S' = IDT_3\) and \(a \in S'\) where \(x = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \)

\[ y = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} \text{ and } a = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix}. \]

\[ S'x = S'y = \{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix} \}. \]
Also, $S'xa = S'ya = S'(xa) = S'(ya) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix}, \\
\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}.$

**Theorem 2.1:** Let $S$ be the identity difference full transformation and $\sigma$ a congruence defined by Greeen’s relations $L$ and $R$ on the semigroup $S$. Then $x\sigma y$ forms an equivalence class.

**Proof:** Let $x, y, z \in S$ and $\sigma$ a relation on $S$. Choosing $x \in S$ and $(x, x) \in \sigma$.

From lemma 2.1, $(xa, xa) \in \sigma, a \in S$ to show that $L$ is a right congruence. Also from lemma 2.2, $(ax, ax) \in \sigma$. Since $\sigma$ is a congruence and $(x, x) \in \sigma$ for all $x \in S$, then $\sigma$ is reflexive.

Given that $(x, y) \in \sigma$. From Green’s relations $L$ and $R$, if $xLy$ then $S'x = S'y$ and if $xRy$ then $xS' = yS'$. Thus, $S'x = S'y = S'x$ and $xS' = yS' \Rightarrow yS' = xS'$. Then $\sigma$ is symmetric since $(x, y) \in \sigma$ implies $(y, x) \in \sigma$.

If $(x, y) \in \sigma$ and $(y, z) \in \sigma$. From Green’s relations,

$$xLy \Rightarrow S'x = S'y,$$

$$yLz \Rightarrow S'y = S'z.$$

$$\Rightarrow S'x = S'y = S'z,$$

$$\Rightarrow S'x = S'z.$$ 

Also, $xRy \Rightarrow xS' = yS',

$$yRz \Rightarrow yS' = zS'.$$

$$\Rightarrow xS' = yS' = zS',$$

$$\Rightarrow xS' = zS'.$$ Thus $\sigma$ is transitive since $(x, z) \in \sigma$. This implies that $\sigma$ is an equivalence relation. This equivalence relation $\sigma$ splits $S$ into disjoint subsets but their union is whole $S$. For $x \in S$, let $\sigma_x = \{y \in S : (x, y) \in S\}$. $\sigma_x$ is called the equivalence class of $x$. Thus $x\sigma y$ is an equivalence class with respect to Green’s left relation since they generate the same principal left
ideal. Equivalently, \((x, y) \in \sigma\) is also an equivalence class with respect to Green’s right relation if they generate the same principal right ideal.

**Theorem 2.2:** Let \(\alpha : \text{dom}(\alpha) \subseteq X_n \mapsto \text{Im}(\alpha) \subseteq X_n\) and \(S\) be the identity difference full transformation semigroup \((IDT_n)\). If \(x \mathcal{L} y\), then \(x^n \tilde{y} y^n\), that is \(x^n\) is not \(\mathcal{L}\) related to \(y^n\), for some \(a \in \text{dom}(xory)\).

**Proof:** Let \(X_n = \{1, 2, 3 \ldots n\}\) be the natural ordering of numbers, \(x, y \in S\) and \(x \mathcal{L} y\). The maximum length of image of \(\alpha\) is 2, that is \(a\) and \(a + 1\).

Let \(a, b, c \in \text{dom}(xory)\). If \(x_1 = \begin{pmatrix} a & b & c \\ a & a & b \end{pmatrix}\), \(x_2 = \begin{pmatrix} a & b & c \\ b & a & a \end{pmatrix}\), \(x_1^2 = \begin{pmatrix} a & b & c \\ a & a & a \end{pmatrix}\) \(\notin \mathcal{L}\); \(x_2^2 = \begin{pmatrix} a & b & c \\ b & a & b \end{pmatrix}\) \(\in \mathcal{L}\); \(y = \begin{pmatrix} a & b & c \\ c & c & b \end{pmatrix}\), \(y^2 = \begin{pmatrix} a & b & c \\ b & b & c \end{pmatrix}\) \(\in \mathcal{L}\); \(z = \begin{pmatrix} a & b & c \\ b & c & c \end{pmatrix}\), \(z^2 = \begin{pmatrix} a & b & c \\ c & c & c \end{pmatrix}\) \(\notin \mathcal{L}\).

If \(\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{L}\); \(\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2; \mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset\); \(x_1, x_2 \in \mathcal{L}_1\) and \(y, z \in \mathcal{L}_2\).

It can be seen that the exponent of each element in \(\mathcal{L}_1\) is such that \(x_1 \mathcal{L} x_2\) but \(x_1^n \tilde{\mathcal{L}} x_2^n\). This is because for \(b = a + 1\) then \(a \alpha = a\) and \((a + 1)\alpha = a\). Also, if \(c = b + 1\) and \(b = a + 1\) then \(c = a + 2\). In \(x_1, a \alpha = a, (a + 1)\alpha = a, (a + 2)\alpha = b\) thus \(x_1^2 \notin \mathcal{L}\). Also, in \(x_2, a \alpha = b, b \alpha = a, c \alpha = a\) thus \(x_2^2 \in \mathcal{L}\) \(\Rightarrow x_1^2 \tilde{\mathcal{L}} x_2^2\).

The same is applicable to \(y\) and \(z, y \mathcal{L} z\). In \(y, a \alpha = c, b \alpha = c\) and \(c \alpha = b\). The transformation is such that \(b \alpha = c\) and \(c \alpha = b\), that is the image of a point becomes the domain of another point and vice versa. It is also noted that other points in the image assume one of such points that interchange. The exponents of elements that observe this property are \(\mathcal{L}\) related in \(IDT_n\).

For instance, in \(z\), had it been that \(a \alpha = b\) and \(b \alpha = a\) then \(c \alpha \neq c\) as is expected in identity difference transformation. This means that either \(c \alpha = a\) or \(b\) then the exponent of \(z\) can be in \(\mathcal{L}_2\). As it were, in \(z\), \(a \alpha = b, b \alpha = c\) and \(c \alpha = c\) then \(z^n \notin \mathcal{L}_2\). Thus, \(x^n \mathcal{L} y^n\) whenever for \(a, b, c \in X_n\) such that if \(a \alpha = b\) and \(b \alpha = a\) or \(b \alpha = c\) and \(c \alpha = b\) where \(\text{Im} \alpha = (i, i + 1), i = a, b, c \ldots n \in X_n\).
In $IDT_n$, there are $(2^n - 2)$ elements in $(n - 1)$ ways sharing right compatible equivalence relation.

**Theorem 2.3:** Let $\alpha, \beta \in IDT_n$ and $\alpha L \beta$.

Then $|S\alpha| = 2^n$.

**Proof:** Since $\alpha L \beta$, then $S\alpha = S\beta$. Also the maximum length, image of $\alpha$ can assume is 2, which for each $n$ makes $|S\alpha| = |S\beta| = 2^n = \sum_{r=0}^{n} \binom{n}{r}$, $\alpha \neq \beta$. $L$ related elements in $IDT_n$ have $(n-1)$groups and each group consists of $(2^n - 2)$ elements. Also the number of $\alpha, \beta \in IDT_n$ having the property that $\text{Im}(\alpha) = \text{Im}(\beta)$ (images with the same outlook) is $|\alpha| = 2^n - 2 = |\beta|$, $n \geq 2$.

The table below shows the values of elements in $IDT_n$,

for $n = 1, 2, 3$ that are $L$-related.

<table>
<thead>
<tr>
<th>n</th>
<th>$\text{Im}\alpha = {1}$</th>
<th>$\text{Im}\alpha = {2}$</th>
<th>$\text{Im}\alpha = {3}$</th>
<th>$\text{Im}\alpha = {1, 2}$</th>
<th>$\text{Im}\alpha = {2, 3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>-</td>
<td>2</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

$h = |\text{Im}\alpha|$

From the table, $|IDT_n|$ can be known. There are $n$ such elements having the property $|\text{Im}\alpha| = 1$ and $(n - 1)(2^n - 2)$ elements for $|\text{Im}\alpha| = 2$.

Thus, if $S = IDT_n$, then

$|S| = n + (n - 1)(2^n - 2)$. Hence the following table of values for $IDT_n$:

**Table 2.3:** Elements in $IDT_n$(A188716) by Adeniji and Makanjuola[1].
Theorem 2.4: Let $X_n = \{1, 2, 3 \ldots n\}$, $n \in N$ and $S = IDT_n$, then the rank of $IDT_n$ is $2^{(n-1)}$, for each $n$.

Proof: Let $A$ be a set of generators of $IDT_n$ then $\langle A \rangle = IDT_n$ and no proper subset of $A$ generates $IDT_n$, i.e. $A$ is irreducible. As defined earlier, one of the features of Identity Difference is that $\text{max}(\text{Im} \alpha) - \text{min}(\text{Im} \alpha) \leq 1$. Also $\text{Im}(\alpha) = \{i, i+1\}$, $i = 1, 2, 3 \ldots n - 1$. When $n = 1$, $\text{Im}(\alpha) = \{i\}$, since $0 \notin N$ and $|A| = 1 = |S|$. That is, the single element generates itself. When $n = 2$, $|S| = 4$ but $|A| = 2$ and the structure of elements in $A$ is such that one of the generators has its $\text{Im}(\alpha) = \{i\}$ and the other $\{i, i+1\}$, (not an identity), $i = 1, 2, \ldots n - 1$. In the case where $n \geq 2$, the L-related elements are considered as in Thm. 2.3, where $|S\alpha| = 2^n$. Hence the generators for $IDT_n$, $n \geq 2$ is calculated using $\frac{2^n}{2}$ and each point in $X$ is well represented for each $n$.

REFERENCES


Received: February 10, 2013