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Relative Character Graphs: Some Properties and Examples

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Abstract

Let G be a finite group and H be a subgroup of G. T. Gnanaseelan and C. Selvaraj defined a graph $\Gamma(G, H)$ by using the irreducible characters of G and the restrictions of these characters to H. In this paper we continue the study of the connected components of these graphs. We also study examples of these graphs for some specific groups.

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1 Introduction

In this paper, all groups are finite unless otherwise stated. All characters of a group are assumed to be complex-valued. For a group G, let $Irr(G)$ denote the set of all irreducible characters of G. Let H be a subgroup of G and χ be a character of G. χ_H is its restriction to H. Let θ be a character of H. θ^G is the induced character of G. In [2], a graph called *relative character graph* is defined as follows:

Definition 1.1 *If G is a finite group and H is a subgroup of G, then the relative character graph denoted by $\Gamma(G, H)$ has the vertex set $V = Irr(G)$. Two vertices χ and ψ are joined by an edge if χ_H and ψ_H have at least one irreducible character of H as a common constituent.*

Some properties of this graph were derived in [2]. Examples of the relative character graphs of some groups were investigated in [2] and [4]. In this paper, we will continue the investigation about the properties of this graph.

2 Some properties of $\Gamma(G, H)$

If N is a normal subgroup of G, G acts on $Irr(N)$ by conjugation. That is, $g.\theta = \theta^g$ where $\theta^g(h) = \theta(ghg^{-1})$ for $g \in G$ and $\theta \in Irr(N)$. We say that θ^g is *conjugate* to θ in G. Then

$$I_G(\theta) = \{g \in G \mid \theta^g = \theta\}$$

is called the *inertia group* in G. Note that $I_G(\theta)$ is the stabilizer of θ under the action of G on $Irr(N)$. It is also clear that $N \subseteq I_G(\theta)$.

Let k be the number of orbits of $Irr(N)$ under the action of G. By Lemma 2.3 in [2], the number of connected components in $\Gamma(G, N)$ is k. We pick one element in each orbit of $Irr(N)$ under the action of G and call them θ_1, θ_2,..., θ_k.

Definition 2.1 *([2]) If H is a subgroup of G and $\theta \in Irr(H)$, the set of all irreducible constituents of θ^G is called the induced cover of θ in G and is denoted by $I(\theta)$.*

For each $j = 1, 2, ..., k$, the subgraph in $\Gamma(G, N)$ spanned by the vertices in an induced cover $I(\theta_j) \subset Irr(G)$ is complete and is also a connected component of $\Gamma(G, N)$ by Lemma 2.3 in [2]. We denote each of these subgraphs by $\Gamma_j(G, N)$. The following lemma describes the structure of $\Gamma_j(G, N)$.

Lemma 2.2 *If N is a normal subgroup of G, let θ_1, θ_2,..., θ_k be representatives of each orbit in $Irr(N)$ under the action of G by conjugation. For each $j = 1, 2, ..., k$, let $\Gamma_j(G, N)$ be the subgraph in $\Gamma(G, N)$ spanned by the vertices in the induced cover $I(\theta_j)$ and let $T_j = I_G(\theta_j)$, where $I_G(\theta_j)$ is the inertia group of θ_j in G. Then $\Gamma_j(G, N)$ is isomorphic to $\Gamma_j(T_j, N)$.*

Proof. Let $\langle\ ,\ \rangle_G$ be the standard inner product defined on the space of characters of G (see Chapter 14 in [5]). For each $j = 1, 2, ..., k$, let

$$\mathrm{A}_j = \{\psi \in Irr(T_j) \mid \langle \psi_N, \theta_j\rangle_N \neq 0\}$$

and

$$\mathrm{B}_j = \{\chi \in Irr(G) \mid \langle \chi_N, \theta_j\rangle_N \neq 0\}.$$

By taking the elements in B_j as vertices, the complete graph spanned by the elements in B_j is the same as $\Gamma_j(G,N)$. By Clifford's Theorem (see Theorem 6.2, 6.5 in [3]), for any $\psi \in \mathrm{A}_j$,

$$\psi_N = e\theta_j$$

where $e = \langle \psi_N, \theta_j \rangle_N$. By taking the elements in A_j as vertices, the complete graph spanned by the elements in A_j is the same as $\Gamma_j(T_j, N)$.

By Theorem 6.11 in [3], the map $\psi \mapsto \psi^G$ is a bijection of A_j onto B_j. We conclude that $\Gamma_j(G,N)$ and $\Gamma_j(T_j,N)$ are isomorphic. Hence the result.

If H is a subgroup of G and $|G|/|H| = 2$, then H is normal in G. In this case, we have the following lemma.

Lemma 2.3 *Suppose that N is a normal subgroup of index 2 in G, then $\Gamma(G,N)$ is disconnected. Each connected component contains a single vertex or is a copy of K_2, a complete graph with two vertices.*

Proof. $\Gamma(G,N)$ is disconnected by Theorem 2.5 in [2]. For $\chi \in Irr(G)$, χ_N is either irreducible or a sum of two distinct elements in $Irr(N)$ by Proposition 20.9 in [5].

Let λ be the linear character of G obtained by lifting the non-trivial linear character of $G/H \cong C_2$ to G. If χ_N is irreducible, the induced cover $I(\chi_N)$ contains only χ and $\lambda\chi$ by Proposition 20.11 in [5]. Hence χ and $\lambda\chi$ are joined by an edge in this connected component.

If χ_H is a sum of two distinct elements in $Irr(N)$, then χ is the only element in $Irr(G)$ with restriction χ_H by Proposition 20.12 in [5]. The induced cover $I(\chi_H)$ has one single element χ and hence a single vertex as a connected component. Hence the result.

Remark 2.4 *In the special case that if G is D_{2n}, the dihedral group of order $2n$ and C_n is the cyclic normal subgroup of order n in D_{2n}, the structure of $\Gamma(D_{2n}, C_n)$ is described in Theorem 3.1 [4].*

3 Examples of $\Gamma(G,H)$

3.1 G is a pq-group where p, q are distinct primes

If G is a pq-group where p, q are distinct primes and $p > q$, it is well-known that G is either abelian or q divides $p-1$ (for example, see Chapter 18 in [1]). In the former case, the structure of the relative character graph is described by Theorem 2.6 in [2]. In the latter case, G is isomorphic to the non-abelian group $F_{p,q}$ where $F_{p,q}$ is a group of order pq with presentation

$$F_{p,q} = \langle a, b \mid a^p = b^q = 1, b^{-1}ab = a^u \rangle \tag{1}$$

where u is an element of order q in $\mathbb{Z}_p^*$. The complete character table of $F_{p,q}$ is described by Theorem 25.10 in [5]:

Theorem 3.1 *([5]) Let p be a prime number, $q|p-1$, $r=(p-1)/q$ and $F_{p,q}$ be the group of order pq defined by (1). Then $F_{p,q}$ has $q+r$ irreducible characters. Of these, q have degree 1 and are given by*

$$\chi_n(a^x b^y) = e^{2\pi i n y/q} \ (0 \leq n \leq q-1)$$

and r have degree q and are given by

$$\begin{aligned} \phi_j(a^x b^y) &= 0 \text{ if } 1 \leq y \leq q-1, \\ \phi_j(a^x) &= \sum_{s \in S} e^{2\pi i v_j s x/p}, \end{aligned}$$

for $1 \leq j \leq r$, where v_1S, v_2S,...,v_rS are the cosets in $\mathbb{Z}_p^*$ of the subgroup S generated by u.

Let K_q be the complete graph with q vertices. We are in the position to describe the graph $\Gamma(F_{p,q}, \langle a \rangle)$:

Theorem 3.2 *Let $F_{p,q}$ be the group of order pq defined by (1). $\langle a \rangle$ is the cyclic normal subgroup of order p in $F_{p,q}$. Then $\Gamma(F_{p,q}, \langle a \rangle)$ has $q+r$ vertices and $r+1$ connected components. Of these components, one is K_q and r contains one single vertex.*

Proof. Let $H = \langle a \rangle$. By Theorem 3.1, $F_{p,q}$ has $q+r$ irreducible characters and $(\chi_n)_H$, the restriction of χ_n to H, is given by $(\chi_n)_H(a^x) = 1$ for all $n = 0, 1, ..., q-1$. Hence

$$(\chi_n)_H = 1_H$$

for all $n = 0, 1, ..., q-1$. It accounts for a copy of K_q in $\Gamma(F_{p,q}, H)$.

There are p irreducible characters in H, namely,

$$\psi_k(a^x) = e^{2\pi i k x/p}$$

for $k = 0, 1, ..., p-1$. By the formula of $\phi_j(a^x)$ in Theorem 3.1, $(\phi_j)_H$, the restriction of ϕ_j to H has the following formula:

$$(\phi_j)_H = \sum_{s \in S} \psi_{v_j s}$$

for $j = 1, 2, ..., r$. Since v_1S, v_2S,..., v_rS are distinct cosets in $\mathbb{Z}_p^*$, none of $(\phi_j)_H$ has common irreducible constituent in $Irr(H)$, each ϕ_j is an isolated vertex in $\Gamma(F_{p,q}, H)$. Hence the result.

Remark 3.3 *If n is an odd prime, Theorem 3.1 in [4] is just a special case of Theorem 3.2.*

3.2 G is a group of order p^3

The abelian groups G of order p^3 are C_{p^3}, $C_{p^2} \times C_p$ and $C_p \times C_p \times C_p$. The structure of $\Gamma(G, H)$, where H is any subgroup of G, is completely described by Theorem 2.6 in [2].

If G is non-abelian of order p^3, it is well-known that G has a normal subgroup Z of order p and $G/Z \cong C_p \times C_p$ (for example, see [6]). Choose aZ, bZ such that $G/Z = \langle aZ, bZ\rangle$ and let z be a generator of $Z \cong C_p$. Then every element of G is of the form $a^r b^s z^t$ for some r, s, t with $0 \le r, s, t \le p-1$. The complete character table of G is described by Theorem 26.6 in [5]:

Theorem 3.4 *([5]) Let* $G = \{a^r b^s z^t \mid 0 \le r, s, t \le p-1\}$ *be a non-abelian group of order* p^3 *described as above. Write* $\epsilon = e^{2\pi i/p}$. *Then the irreducible characters of* G *are* $\chi_{u,v}$ *for* $0 \le u, v \le p-1$ *where*

$$\chi_{u,v}(a^r b^s z^t) = \epsilon^{ru+sv}$$

and ϕ_u *for* $1 \le u \le p-1$ *where*

$$\phi_u(a^r b^s z^t) = p\epsilon^{ut}, \text{ if } r = s = 0$$

and ϕ_u *is equal to 0 otherwise.*

We are in the position to describe the graph $\Gamma(G, Z)$.

Theorem 3.5 *Let* G *be a non-abelian group of order* p^3 *described as above. Let* Z *be a normal subgroup of order* p *in* G. *Then* $\Gamma(G, Z)$ *has a copy of* K_{2p} *and* $p-1$ *isolated vertices.*

Proof. Write $\epsilon = e^{2\pi i/p}$. There are p irreducible characters of $Z \cong C_p$, namely

$$\psi_u(z^t) = \epsilon^{ut}$$

for $0 \le u, t \le p-1$. By Theorem 3.4, for $0 \le u, v \le p-1$, we have

$$(\chi_{u,v})_Z = 1_Z$$

It accounts for a copy of K_{2p} in $\Gamma(G, Z)$. Similarly, for $1 \le u \le p-1$, we have

$$(\phi_u)_Z = p\psi_u.$$

Hence there are p isolated vertices in $\Gamma(G, Z)$. Hence the result.

3.3 $G = GL(2,q)$ where q is an odd prime

Before looking at the character table of $G = GL(2,q)$, we need to know the conjugacy classes of G. For each $s \in \mathbb{Z}_q^*$, let

$$sI = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}, \; u_s = \begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix}.$$

For $s, t \in \mathbb{Z}_q^*$ where $s \neq t$, let

$$d_{s,t} = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}.$$

Let $\mathbb{Z}_{q^2}$ be the finite field of order q^2 and let $S = \{s \in \mathbb{Z}_{q^2} \mid s^q = s\}$. Then S is a subfield of $\mathbb{Z}_{q^2}$ of order q and hence is isomorphic to $\mathbb{Z}_q$. From now on we shall identify the subfield S of $\mathbb{Z}_{q^2}$ with the field $\mathbb{Z}_q$. Note that if $r \in \mathbb{Z}_{q^2}$, then r^{1+q} and $r + r^q$ are both in the subfield $\mathbb{Z}_q$ of $\mathbb{Z}_{q^2}$, see [1] for further reference.

Take $r \in \mathbb{Z}_{q^2}^* \backslash \mathbb{Z}_q^*$, let

$$v_r = \begin{pmatrix} 0 & 1 \\ -r^{1+q} & r + r^q \end{pmatrix}.$$

The conjugacy classes of G can be described by Proposition 28.4 in [5]:

Proposition 3.6 *([5]) There are $q^2 - 1$ conjugacy classes in $GL(2,q)$, described as follows:*

class rep. g	sI	u_s	$d_{s,t}$	v_r
$\lvert C_G(g)\rvert$	$(q^2-1)(q^2-q)$	$(q-1)q$	$(q-1)^2$	q^2-1
No. of classes	$q-1$	$q-1$	$(q-1)(q-2)/2$	$(q^2-q)/2$

The family of conjugacy class representatives sI and u_s are indexed by the elements $s \in \mathbb{Z}_q^*$. The family of conjugacy class representatives $d_{s,t}$ is indexed by unordered pairs $\{s,t\}$ of distinct elements of $\mathbb{Z}_q^*$. The family of conjugacy class representatives v_r is indexed by unordered pairs $\{r, r^q\}$ of elements in $\mathbb{Z}_{q^2}^* \backslash \mathbb{Z}_q^*$.

Let ϵ be a generator of the cyclic group $\mathbb{Z}_{q^2}^*$ and let $\omega = e^{2\pi i/(q^2-1)}$. Suppose that $r \in \mathbb{Z}_{q^2}^*$ and $r = \epsilon^m$ for some m. Let $\overline{r} = \omega^m$. Then $r \mapsto \overline{r}$ is an irreducible character of $\mathbb{Z}_{q^2}^*$. Note that every irreducible character of $\mathbb{Z}_{q^2}^*$ is of the form $r \mapsto \overline{r}^j$ for some j.

The character table of $GL(2,q)$ is described by Theorem 28.5 in [5].

Theorem 3.7 *([5]) Label the conjugacy classes in $GL(2,q)$ as in Proposition 3.6, and let $r \mapsto \overline{r}$ be the irreducible character from $\mathbb{Z}_{q^2}^*$ to* C *described as above. Then the irreducible characters of $GL(2,q)$ are given by λ_i, ψ_i, $\psi_{i,j}$, χ_i as follows.*

	sI	u_s	$d_{s,t}$	v_r
λ_i	$\overline{s}^{2i}$	$\overline{s}^{2i}$	$(\overline{st})^i$	$\overline{r}^{i(1+q)}$
ψ_i	$q\overline{s}^{2i}$	0	$(\overline{st})^i$	$-\overline{r}^{i(1+q)}$
$\psi_{i,j}$	$(q+1)\overline{s}^{i+j}$	$\overline{s}^{i+j}$	$\overline{s}^i\overline{t}^j + \overline{s}^j\overline{t}^i$	0
χ_i	$(q-1)\overline{s}^i$	$-\overline{s}^i$	0	$-(\overline{r}^i + \overline{r}^{i(1+q)})$

For λ_i, ψ_i, we have $0 \leq i \leq q-2$. For $\psi_{i,j}$, we have $0 \leq i < j \leq q-2$. For χ_i, we first consider the set of integers j with $0 \leq j \leq q^2-1$ such that $q+1$ does not divide j. If j_1, j_2 belong to this set and $j_1 \equiv j_2 q \mod (q^2-1)$, we choose precisely one of j_1 and j_2 to belong to the indexing set for the character χ_i. Hence there are $(q^2-q)/2$ characters χ_i.

Let

$$D = \{\begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} \mid s, t \in \mathbb{Z}_q^*\}$$

We call D the *diagonal subgroup* of $GL(2,q)$. Note that D is an abelian subgroup of $GL(2,q)$ and is of order $(q-1)^2$. All conjugacy classes are of size 1 and, by the notations above, are given by sI, $d_{s,t}$ and $d_{t,s}$ where $1 \leq s, t \leq q-1$ and $s \neq t$. There are $(q-1)^2$ irreducible linear characters of D which are given by $f_{i,j}$ as follows:

	sI	$d_{s,t}$	$d_{t,s}$
$f_{i,j}$	$\overline{s}^{i+j}$	$\overline{s}^i\overline{t}^j$	$\overline{s}^j\overline{t}^i$

where $0 \leq i, j \leq q-2$.

Lemma 3.8 *Let D be the diagonal subgroup of $GL(2,q)$. By the same notation as in Theorem 3.7, for $0 \leq i \leq q-2$, we have*

$$(\lambda_i)_D = f_{i,i}$$

Proof. By Theorem 3.7, the restriction of λ_i to $D \subset GL(2,q)$ takes the following values:

	sI	$d_{s,t}$	$d_{t,s}$
$(\lambda_i)_D$	$\overline{s}^{2i}$	$(\overline{st})^i$	$(\overline{st})^i$

The lemma is now obvious by inspection on the values of $f_{i,i}$ based on the character table of D given above. Hence the lemma.

If $k < 0$, then we define $f_{i,k}$ as a character of D by

$$f_{i,k} \equiv f_{i,k'}$$

where $0 \leq k' \leq q-2$ and $k' \equiv k \bmod (q-1)$.

If $k > q-2$, then we define $f_{i,k}$ as a character of D by

$$f_{i,k} \equiv f_{i,k''}$$

where $0 \leq k'' \leq q-2$ and $k'' \equiv k \bmod (q-1)$.

Lemma 3.9 *Let D be the diagonal subgroup of $GL(2,q)$. By the same notation as in Theorem 3.7, for $0 \leq i \leq q-2$, we have*

$$(\psi_i)_D = f_{i,i} + \sum_{x=0}^{q-2} f_{x,2i-x}.$$

Proof. By Theorem 3.7, the restriction of ψ_i to $D \subset GL(2,q)$ takes the following values:

	sI	$d_{s,t}$	$d_{t,s}$
$(\psi_i)_D$	$q\overline{s}^{2i}$	$(\overline{st})^i$	$(\overline{st})^i$

For $f_{i,i} + \sum_{x=0}^{q-2} f_{x,2i-x}$, on the conjugacy class sI, it takes the value

$$\begin{aligned} \overline{s}^{i+i} + \sum_{x=0}^{q-2} \overline{s}^{x+(2i-x)} &= \overline{s}^{2i} + \sum_{x=0}^{q-2} \overline{s}^{2i} \\ &= \overline{s}^{2i} + (q-1)\overline{s}^{2i} \\ &= q\overline{s}^{2i} \end{aligned}$$

On the conjugacy class $d_{s,t}$, it takes the value

$$\overline{s}^i\overline{t}^i + \sum_{x=0}^{q-2} \overline{s}^x\overline{t}^{2i-x} \tag{2}$$

We need to show that the sum of all terms starting from the second term is equal to 0.

$$\begin{aligned} \sum_{x=0}^{q-2} \overline{s}^x\overline{t}^{2i-x} &= \overline{s}^0\overline{t}^{2i} + \overline{s}\overline{t}^{2i-1} + ... + \overline{s}^{q-2}\overline{t}^{2i-(q-2)} \\ &= \overline{t}^{2i}(1 + (\frac{\overline{s}}{\overline{t}}) + (\frac{\overline{s}}{\overline{t}})^2 + ... + (\frac{\overline{s}}{\overline{t}})^{q-2}) \\ &= \overline{t}^{2i}(\frac{(\frac{\overline{s}}{\overline{t}})^{q-1} - 1}{\frac{\overline{s}}{\overline{t}} - 1}) \end{aligned} \tag{3}$$

The last equation is valid since $\overline{s} \neq \overline{t}$. Since $\frac{\overline{s}}{\overline{t}} \in \mathbb{Z}_q^*$, $(\frac{\overline{s}}{\overline{t}})^{q-1} = 1$. (3) is equal to 0 and hence, by (2), $f_{i,i} + \sum_{x=0}^{q-2} f_{x,2i-x}$ takes the value $\overline{s}^i\overline{t}^i$ in the conjugacy class $d_{s,t}$.

Similarly, $f_{i,i} + \sum_{x=0}^{q-2} f_{x,2i-x}$ also takes the value $\overline{s}^i\overline{t}^i$ in the conjugacy class $d_{t,s}$. Hence the lemma.

The following two lemma can be shown by the same method as in Lemma 3.9.

Lemma 3.10 *Let D be the diagonal subgroup of $GL(2,q)$. For $0 \leq i,j \leq q-2$, $f_{i,j}$ are the irreducible characters of D defined above in the character table of D. By the same notation as in Theorem 3.7, for $0 \leq i < j \leq q-2$,*

$$(\psi_{i,j})_D = f_{i,j} + f_{j,i} + \sum_{x=0}^{q-2} f_{x,(i+j)-x}$$

Lemma 3.11 *Let D be the diagonal subgroup of $GL(2,q)$. For $0 \leq i,j \leq q-2$, $f_{i,j}$ are the irreducible characters of D defined above in the character table of D. By the same notation as in Theorem 3.7, for all i that are in the index set of χ_i,*

$$(\chi_i)_D = \sum_{x=0}^{q-2} f_{x,i-x}$$

In general, very little can be said about the relative character graph $\Gamma(G,D)$. The main difficulty comes from the restrictions of the characters χ_i of $G = GL(2,q)$ to D. Instead, we shall describe certain subgraph of $\Gamma(G,D)$. Let

$$S = Irr(G) \backslash \{\chi_i \text{ , } \forall \text{i in the index set}\} \tag{4}$$

Then let $\Gamma_S(G,D)$ be the subgraph of $\Gamma(G,D)$ spanned by the vertices in S. We shall look at the structure of the relative character graph $\Gamma_S(G,D)$. Note that D contains a non-trivial normal subgroup of G that is generated by the elements sI where s is any element in $\mathbb{Z}_q^*$. Hence, by Theorem 2.5 in [2], $\Gamma_S(G,D)$ is disconnected. We are interested in the structure of its connected components.

Let K'_p be the graph obtained by removing an edge from K_p, the complete graph with p vertices. It is clear that, up to isomorphism, K'_p is independent of the choice of the removed edge.

Theorem 3.12 *Let $G = GL(2,q)$ and D be the diagonal subgroup of G. Let S be the set defined in (4). Let $\Gamma_S(G,D)$ be the subgraph of $\Gamma(G,D)$ spanned by the vertices in S. Then $\Gamma_S(G,D)$ has $(q-1)$ connected components. Of these, $(q-1)/2$ are isomorphic to $K_{(q-1)/2}$ and $(q-1)/2$ are isomorphic to $K'_{(q+5)/2}$, where $K'_{(q+5/2)}$ is a graph obtained by removing an edge from $K_{(q+5)/2}$.*

Proof. Let $S_\lambda \subset Irr(G)$ be

$$S_\lambda = \{\lambda_i \mid 0 \le i \le q-2\}$$

Let $\Gamma_{S_\lambda}(G,D)$ be the subgraph of $\Gamma(G,D)$ spanned by the vertices in S_λ. By Lemma 3.8, $\Gamma_{S_\lambda}(G,D)$ contains $(q-1)$ isolated vertices.

Let $S_{\lambda,\psi} \subset Irr(G)$ be

$$S_{\lambda,\psi} = \{\lambda_i, \psi_i \mid 0 \le i \le q-2\}$$

By Lemma 3.9, for $0 \le i \le (q-3)/2$, $(\psi_i)_D$ and $(\psi_{((q-1)/2)+i})_D$ have common irreducible constituents $f_{i,i}$ and $f_{((q-1)/2)+i,((q-1)/2)+i}$. Hence, $(\psi_i)_D$, $(\psi_{((q-1)/2)+i})_D$, $(\lambda_i)_D$ and $(\lambda_{((q-1)/2)+i})_D$ form a connected component in $\Gamma(G,D)$ but there is no edge between $(\lambda_i)_D$ and $(\lambda_{((q-1)/2)+i})_D$. So the subgraph $\Gamma_{S_{\lambda,\psi}}(G,D)$ of $\Gamma(G,D)$ spanned by the vertices in $S_{\lambda,\psi}$ has $(q-1)/2$ connected component, each of which is isomorphic to K_4', the graph obtained by removing an edge in K_4.

We consider $\Gamma_S(G,D)$. There are $(q-1)(q-2)/2$ pairs of $\{i,j\}$ such that $0 \le i < j \le q-2$. We call a pair *odd* if $i+j$ is an odd number and *even* otherwise. A simple combinatorial argument shows that there are $(q-1)^2/4$ odd pairs and $(q-1)(q-3)/4$ even pairs. If $\{i,j\}$ is an odd pair and $\{i',j'\}$ is an even pair, then there is no common irreducible consituent between $(\psi_{i,j})_D$ and $(\psi_{i',j'})_D$.

For any odd pair $\{i,j\}$, $(\psi_{i,j})_D$ has no common irreducible constituent with any vertex in the subgraph $\Gamma_{S_{\lambda,\psi}}(G,D)$. By Lemma 3.10, $(\psi_{i,j})_D$ and $(\psi_{i',j'})_D$ have common irreducible constituents if $i+j = i'+j'$. Furthermore, $(\psi_{i,j})_D$ and $(\psi_{i'',j''})_D$ have common irreducible constituents if $i+j=k$ and $i''+j'' = k+(q-1)$ where k is odd and $1 \le k \le (q-4)$. The sum of the number of pairs $\{i,j\}$ such that $i+j=k$ and the number of pairs $\{i',j'\}$ such that $i'+j' = k+(q-1)$ is equal to $(q-1)/2$. Hence there are $k = (q-3)/2$ connected components in $\Gamma_S(G,D)$ isomorphic to $K_{(q-1)/2}$. There are $(q-1)/2$ pairs of $\{i,j\}$ such that $i+j = q-2$ and all of these $(\psi_{i,j})_D$ have common irreducible constituents. Hence we have another copy of $K_{(q-1)/2}$. To conclude, the set of all odd pairs $\psi_{i,j}$ accounts for $(q-1)/2$ connected components in $\Gamma_S(G,D)$. Each of them is isomorphic to $K_{(q-1)/2}$.

Label the $(q-1)/2$ disjoint copies of K_4' in $\Gamma_{S_{\lambda,\psi}}(G,D)$ by $T_0, T_2, ..., T_{(q-3)/2}$. We further assume that, for $0 \le k \le (q-3)/2$, T_k contains the vertices ψ_k and $\psi_{((q-1)/2)+k}$. For an even pair $\{i,j\}$ such that $i+j = 2k$ or $i+j = 2k+(q-1)$, $(\psi_{i,j})_D$, $(\psi_k)_D$ and $(\psi_{((q-1)/2)+k})_D$ have common constituents, namely $f_{k,k}$ and $f_{((q-1)/2+k),((q-1)/2+k)}$. Also, $(\psi_{i,j})_D$ has common consitituent with $(\lambda_k)_D$ and $(\lambda_{((q-1)/2+k)})_D$ respectively. Hence $\psi_{i,j}$ is joined to every vertex in T_k if $i+j = 2k$ or $i+j = 2k+(q-1)$. There are $(q-3)/2$ pairs of $\{i,j\}$ such that $i+j = 2k$ or $i+j = 2k+(q-1)$ for any fixed k, $0 \le k \le (q-3)/2$. Hence, in $\Gamma_S(G,D)$, we have $(q-1)/2$ disjoint copies of $K'_{((q-3)/2)+4} = K'_{(q+5)/2}$. Hence the theorem.

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