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Some Classification of Prehomogeneous Vector Spaces Associated with Dynkin Quivers of Exceptional Type

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Abstract

In this paper, we give a complete list of tilting Λ-modules where Λ is the tensor algebra of an oriented K-modulation $(\mathfrak{M}, \Omega)$ of a valued graph of exceptional type. Our result gives some kind of classification of prehomogeneous vector spaces associated with $(\mathfrak{M}, \Omega)$.

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1 Introduction

Let $(\Gamma, \boldsymbol{v})$ be a valued graph, i.e., $\Gamma = \{1, 2, \ldots, n\}$ is a finite set of vertices, and $\boldsymbol{v} = (v_{ij})_{i,j \in \Gamma}$ the set of nonnegative integers which satisfy the following condition: $v_{ii} = 0$ for $i \in \Gamma$, and there exist strictly positive integers f_i $(i \in \Gamma)$ satisfying $v_{ij} f_j = v_{ji} f_i$ for each $i, j \in \Gamma$ with $i \neq j$. In the case of $v_{ij} \neq 0$ which is equivalent to $v_{ji} \neq 0$, we use the symbol $\underset{i}{\circ}\overset{(v_{ij},\ v_{ji})}{\text{———}}\underset{j}{\circ}$ and we call it an edge of $(\Gamma, \boldsymbol{v})$. If $v_{ij} = v_{ji} = 1$, we simply write $\underset{i}{\circ}\text{———}\underset{j}{\circ}$. In this paper, we deal with the exceptional types, i.e., $\mathbb{G}_2$: $\underset{1}{\circ}\overset{(1,\ 3)}{\text{———}}\underset{2}{\circ}$, $\mathbb{F}_4$: $\underset{1}{\circ}\text{———}\underset{2}{\circ}\overset{(1,\ 2)}{\text{———}}\underset{3}{\circ}\text{———}\underset{4}{\circ}$, and $\mathbb{E}_l$ $(l = 6, 7, 8)$.

An orientation Ω of $(\Gamma, \boldsymbol{v})$ is given by prescribing an ordering for every edge, indicated by an arrow $\underset{i}{\circ}\xrightarrow{(v_{ij},\ v_{ji})}\underset{j}{\circ}$ or $\underset{i}{\circ}\xleftarrow{(v_{ij},\ v_{ji})}\underset{j}{\circ}$. Sometimes we write $i \to j$ instead of $\underset{i}{\circ}\xrightarrow{(v_{ij},\ v_{ji})}\underset{j}{\circ}$ for simplicity. A valued graph $(\Gamma, \boldsymbol{v})$ with an orientation Ω is called a valued quiver $Q = (\Gamma, \boldsymbol{v}; \Omega)$. For an oriented K-modulation $(\mathfrak{M}, \Omega)$ of a valued graph $(\Gamma, \boldsymbol{v})$ with $\mathfrak{M} = ((F_i)_{i\in\Gamma}, ({}_iM_j)_{i,j\in\Gamma})$ (see Definition 2.1) and a dimension vector $\boldsymbol{d} = (d_i) \in \mathbb{Z}^n_{\geq 0}$, we can define a pair $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}})$ with $G_{\boldsymbol{d}} = \prod_{i\in\Gamma} GL(d_i; F_i)$ and $R_{\boldsymbol{d}} = \bigoplus_{i\to j} \mathrm{Hom}_{F_j}(F_i^{d_i} \otimes_{F_i} {}_iM_j, F_j^{d_j})$ where $G_{\boldsymbol{d}}$ acts on $R_{\boldsymbol{d}}$ (see Definition 2.3). A point $x \in R_{\boldsymbol{d}}$ is called a generic point of $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}})$ if $\dim_K(G_{\boldsymbol{d}})_x = \dim_K G_{\boldsymbol{d}} - \dim_K R_{\boldsymbol{d}}$ where $(G_{\boldsymbol{d}})_x = \{g \in G_{\boldsymbol{d}}; gx = x\}$ is the isotropy subgroup of $G_{\boldsymbol{d}}$ at x. In our case, $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}})$ is a prehomogeneous vector space (abbrev. PV) in the sense that it has generic points (cf. [K]). If characteristic of K is 0, by Proposition 2.7, $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}})$ has a dense orbit. Moreover we show that for these type of PVs $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}})$, some partial tilting $\Lambda(\mathfrak{M}, \Omega)$-modules correspond. For the definition of partial tilting modules, see [ASS; p. 192].

Note that although $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}})$ depends on the choice of the orientation and the modulation, the number of the isomorphism classes of basic tilting modules does not depend on the choice of the orientation (see from the below of Proposition 2.12) and the choice of modulation (see Remark 5.5).

In this paper, we give a complete list of tilting Λ-modules of type $\mathbb{G}_2$ and $\mathbb{F}_4$ (see Sections 3 and 4); and these results give a classification (see Theorems 3.2 and 4.2) of PVs $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}})$ according to corresponding partial tilting modules.

For $\mathbb{E}_n$-type ($n = 6, 7, 8$), by using a computer, we count the number of isomorphism classes of (partial) tilting modules (see Sections 5,6,7).

This paper consists of the following 7 sections.

Section 1. Introduction
Section 2. Preliminaries
Section 3. The Case for exceptional type $\mathbb{G}_2$
Section 4. The Case for exceptional type $\mathbb{F}_4$
Section 5. The Case for exceptional type $\mathbb{E}_6$
Section 6. The Case for exceptional type $\mathbb{E}_7$
Section 7. The Case for exceptional type $\mathbb{E}_8$

2 Preliminaries

In the following, we shall always assume that a valued graph $(\Gamma, \boldsymbol{v})$ is connected, i.e., for any $i, j \in \Gamma$, there is a sequence of vertices $k_1 = i, k_2, \ldots, k_t = j$ such that $v_{k_s k_{s+1}} \neq 0$ for all $s = 1, 2, \ldots, t-1$. An orientation Ω is called admissible if $Q = (\Gamma, \boldsymbol{v}; \Omega)$ has no oriented cycle. In this paper, we assume that an orientation is admissible.

Definition 2.1. (An oriented K-modulation $(\mathfrak{M}, \Omega)$)
We say that $\mathfrak{M} = ((F_i)_{i\in\Gamma}, ({}_iM_j)_{i,j\in\Gamma})$ *is a K-modulation of a valued graph* $(\Gamma, \boldsymbol{v})$ *with* $\boldsymbol{v} = (v_{ij})$ *when it satisfies the following conditions (see [D; p. 32]).*

1. *Each F_i ($i \in \Gamma$) is a finite-dimensional division algebra over a commutative field K.*
2. *Each ${}_iM_j$ is a F_i-F_j-bimodule on which K acts centrally, and satisfies the following conditions.*
   (a) *The dimension over F_j of ${}_iM_j$ as a right F_j-module is v_{ij}, while the dimension over F_i of ${}_iM_j$ as a left F_i-module is v_{ji}.*
   (b) ${}_jM_i \cong \mathrm{Hom}_{F_i}({}_iM_j, F_i) \cong \mathrm{Hom}_{F_j}({}_iM_j, F_j)$ *as a F_j-F_i-bimodule.*

Note that if we put $f_i = \dim_K F_i$, then we have $v_{ij}f_j = v_{ji}f_i$ from ${}_K({}_iM_j) = ({}_iM_j)_K$. For a valued quiver $(\Gamma, \boldsymbol{v}; \Omega)$, if we give a F_i-F_j-module ${}_iM_j$ satisfying the condition (a) for each $\underset{i}{\circ}\xrightarrow{(v_{ij},\ v_{ji})}\underset{j}{\circ}$ *, then we can define ${}_jM_i$ by the condition (b), so that we obtain a K-modulation (see [D; p. 32]).*

Now let $(\mathfrak{M}, \Omega)$ be a pair of a K-modulation of a valued graph $(\Gamma, \boldsymbol{v})$ and an orientation Ω of $(\Gamma, \boldsymbol{v})$ which we call an oriented K-modulation of a valued graph $(\Gamma, \boldsymbol{v})$.

Definition 2.2. (The abelian category $\mathrm{rep}(\mathfrak{M}, \Omega)$)
A representation $W = ({}_j\varphi_i, W_i)$ of an oriented K-modulation $(\mathfrak{M}, \Omega)$ is a pair of finite-dimensional right F_i-vector spaces W_i for $i \in \Gamma$ and F_j-linear mappings ${}_j\varphi_i : W_i \otimes_{F_i} {}_iM_j \to W_j$ for each arrow $i \to j$. A morphism from $W = ({}_j\varphi_i, W_i)$ to $W' = ({}_j\varphi'_i, W'_i)$ is a set $\alpha = (\alpha_i)_{i\in\Gamma}$ of F_i-linear mappings $\alpha_i : W_i \to W'_i$ ($i \in \Gamma$) such that the following diagram is commutative for every arrow $i \to j$.

$$\begin{array}{ccc} W_i \otimes_{F_i} {}_iM_j & \xrightarrow{{}_j\varphi_i} & W_j \\ {\scriptstyle \alpha_i\otimes 1}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_j} \\ W'_i \otimes_{F_i} {}_iM_j & \xrightarrow{{}_j\varphi'_i} & W'_j \end{array}$$

We denote by $\mathrm{Hom}(W, W')$ *the totality of all morphisms from W to W'. Thus we have an abelian category* $\mathrm{rep}(\mathfrak{M}, \Omega)$ *(see [D]). The category* $\mathrm{rep}(\mathfrak{M}, \Omega)$ *is said to be of finite (representation) type if it possesses only a finite number of non-isomorphic indecomposable representations. It is known that* $\mathrm{rep}(\mathfrak{M}, \Omega)$ *is of finite representation type if and only if* $(\Gamma, \boldsymbol{v})$ *is a Dynkin graph (see [D; Theorem 3.1]).*

Definition 2.3. (A representation $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}})$ associated with $(\mathfrak{M}, \Omega)$)

Let $(\mathfrak{M}, \Omega)$ be an oriented K-modulation of a valued graph $(\Gamma, \boldsymbol{v})$ with $\mathrm{Card}(\Gamma) = n$ and $\mathfrak{M} = (F_i, {}_iM_j)$. For a dimension vector $\boldsymbol{d} = (d_i) \in \mathbb{Z}^n_{\geq 0}$, the direct product $G_{\boldsymbol{d}} = \prod_{i \in \Gamma} GL(d_i; F_i)$ acts on the space $R_{\boldsymbol{d}} = \bigoplus_{i \to j} \mathrm{Hom}_{F_j}(F_i^{d_i} \otimes_{F_i} {}_iM_j, F_j^{d_j})$ of representations $({}_jx_i, F_i^{d_i})$ of $(\mathfrak{M}, \Omega)$ by $R_{\boldsymbol{d}} \ni x = ({}_jx_i)_{i \to j} \mapsto x' = ({}_jx'_i)_{i \to j} \in R_{\boldsymbol{d}}$ with ${}_jx'_i = g_j \cdot {}_jx_i \cdot (g_i \otimes 1)^{-1}$ for each $i \to j$.

$$\begin{array}{ccc} F_i^{d_i} \otimes_{F_i} {}_iM_j & \xrightarrow{\;{}_jx_i\;} & F_j^{d_j} \\ {\scriptstyle g_i \otimes 1}\downarrow & & \downarrow{\scriptstyle g_j} \\ F_i^{d_i} \otimes_{F_i} {}_iM_j & \xrightarrow{\;{}_jx'_i\;} & F_j^{d_j} \end{array}$$

By fixing a basis of $F_i^{d_i} \otimes_{F_i} {}_iM_j$ over F_j, we can identify $\mathrm{Hom}_{F_j}(F_i^{d_i} \otimes_{F_i} {}_iM_j, F_j^{d_j})$ with $M(d_j, v_{ij}d_i; F_j)$ and hence we have $R_{\boldsymbol{d}} = \bigoplus_{i \to j} M(d_j, v_{ij}d_i; F_j)$.

We call $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}})$ a representation associated with an oriented K-modulation $(\mathfrak{M}, \Omega)$. In the case where $(\Gamma, \boldsymbol{v})$ is a Dynkin graph, such a representation $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}})$ is a finite PV, i.e., $R_{\boldsymbol{d}}$ has only a finitely many $G_{\boldsymbol{d}}$-orbits. Note that $\dim_K G_{\boldsymbol{d}} = \sum_{i \in \Gamma} f_i d_i^2$ and $\dim_K R_{\boldsymbol{d}} = \sum_{i \to j} v_{ij} f_j d_i d_j$.

Definition 2.4. (The tensor algebra Λ of an oriented K-modulation $(\mathfrak{M}, \Omega)$)

We define the tensor algebra of an oriented K-modulation $(\mathfrak{M}, \Omega)$ by $\Lambda = \Lambda(\mathfrak{M}, \Omega) = \bigoplus_{t \geq 0} \mathfrak{M}^{(t)}$ where $\mathfrak{M}^{(0)} = F_1 \times \cdots \times F_n$, $\mathfrak{M}^{(1)} = \bigoplus_{i \to j} {}_iM_j$, and inductively $\mathfrak{M}^{(t+1)} = \mathfrak{M}^{(t)} \otimes_{\mathfrak{M}^{(0)}} \mathfrak{M}^{(1)}$ where $\mathfrak{M}^{(1)}$ has the structure of the $\mathfrak{M}^{(0)}$–$\mathfrak{M}^{(0)}$ bimodule induced by the projections $\mathfrak{M}^{(0)} \to F_i$. Note that if $j \neq k$, then we have ${}_iM_j \otimes {}_kM_l = 0$ since ${}_iM_j \otimes {}_kM_l = {}_iM_j \cdot e_j \otimes e_k \cdot {}_kM_l = {}_iM_j \otimes (e_j e_k) \cdot {}_kM_l = 0$ where $e_j = (0, \dots, 0, 1_{F_j}, 0, \dots, 0) \in \mathfrak{M}^{(0)}$. Hence we have $\mathfrak{M}^{(t)} = \bigoplus {}_{k_1}M_{k_2} \otimes {}_{k_2}M_{k_3} \otimes \cdots \otimes {}_{k_t}M_{k_{t+1}}$ where the summation runs over the paths $k_1 \to k_2 \to \cdots \to k_{t+1}$. Therefore if the maximal length of the paths in the valued quiver $Q = (\Gamma, \boldsymbol{v}, \Omega)$ is s, we have $\Lambda = \mathfrak{M}^{(0)} \oplus \mathfrak{M}^{(1)} \oplus \cdots \oplus \mathfrak{M}^{(s)}$. Multiplications in Λ is defined by the tensor product by identifying $\mathfrak{M}^{(r)} \otimes \mathfrak{M}^{(s)} = \mathfrak{M}^{(r+s)}$ and distributively. We have a category mod_{Λ} of right Λ-module of finite length.

Remark 2.5. (Equivalence of categories between $\mathrm{rep}(\mathfrak{M}, \Omega)$ and $\mathrm{mod}_{\Lambda(\mathfrak{M}, \Omega)}$)

It is well-known (see [D; Proposition 1.2]) that there is an equivalence between the category $\mathrm{rep}(\mathfrak{M}, \Omega)$ of representations of an oriented K-modulation $(\mathfrak{M}, \Omega)$ and the category mod_{Λ} of right Λ-modules of finite length with $\Lambda = \Lambda(\mathfrak{M}, \Omega)$.

For an object $W = ({}_j\varphi_i, W_i)$ of $\mathrm{rep}(\mathfrak{M}, \Omega)$, we construct a right Λ-module $\tilde{W}$ as follows. Put $\tilde{W} = \bigoplus_{i \in \Gamma} W_i$ which is naturally a $\mathbb{Z}$-module. We define the right action of $\Lambda = \bigoplus_{t \geq 0} \mathfrak{M}^{(t)}$ on $\tilde{W}$ inductively. For $(f_i) \in \mathfrak{M}^{(0)} = F_1 \times \cdots \times F_n$ and $(w_i)_{i \in \Gamma} \in \tilde{W}$, define the right action $\tilde{W} \times \mathfrak{M}^{(0)} \to \tilde{W}$ by $((w_i)_{i \in \Gamma}, (f_i)_{i \in \Gamma}) \mapsto (w_i f_i)_{i \in \Gamma}$.

For $m_{ij} \in {}_iM_j \subset \mathfrak{M}^{(1)} = \bigoplus_{i\to j} {}_iM_j$ and $(w_i)_{i\in\Gamma} \in \tilde{W}$, define the right action $\tilde{W} \times {}_iM_j \to \tilde{W}$ by $((w_k)_{k\in\Gamma}, m_{ij}) \mapsto (w'_t)_{t\in\Gamma}$ where $w'_j = {}_j\varphi_i(w_i \otimes m_{ij})$ if $t = j$ and $w'_t = 0$ otherwise. Then inductively by $w_i(m_{ij} \otimes \cdots \otimes m_{pq} \otimes m_{qs}) = {}_s\varphi_q(w_i(m_{ij} \otimes \cdots \otimes m_{pq}) \otimes m_{qs})$, we obtain the right action of $\mathfrak{M}^{(t)} = \mathfrak{M}^{(t-1)} \otimes_{\mathfrak{M}^{(0)}} \mathfrak{M}^{(1)}$, and hence we have the right action of $\Lambda = \bigoplus_{t\geq 0} \mathfrak{M}^{(t)}$ on $\tilde{W}$, i.e., $\tilde{W}$ is a right Λ-module of finite length. For $\alpha = (\alpha_i)_{i\in\Gamma} : W \to W'$, the map $\tilde{\alpha} : \tilde{W} \to \tilde{W}'$ defined by $\tilde{\alpha}((w_i)_i) = (\alpha_i(w_i))_i$ is clearly Λ-linear.

Conversely for a right Λ-module X of finite length, i.e., an object of mod_{Λ}, put $X_i = XF_i$ which is a finite dimensional right F_i-vector space for $i \in \Gamma$. Note that $F_i \subset \Lambda$. Then the action of ${}_iM_j (\subset \Lambda)$ on each X_i induces the F_j-linear maps ${}_j\varphi_i : X_i \otimes_{F_i} {}_iM_j \to X_j$ for each arrow $i \to j$. Thus we have an object $({}_j\varphi_i, X_i)$ of the category $\mathrm{rep}(\mathfrak{M}, \Omega)$. We define the dimension vector of a right Λ-module X by $\dim X = (\dim_{F_i} X_i)_{i\in\Gamma} \in \mathbb{Z}^n_{\geq 0}$ with $n = \mathrm{Card}(\Gamma)$.

By this correspondence, we can regard each point $x = ({}_jx_i)_{i\to j} \in R_{\boldsymbol{d}} = \bigoplus_{i\to j} M(d_j, v_{ij}d_i; F_j)$ as a Λ-module $X = F_1^{d_1} \oplus \cdots \oplus F_n^{d_n}$ on which $(m_{ij})_{i\to j} \in \mathfrak{M}^{(1)} = \bigoplus_{i\to j} {}_iM_j (\subset \Lambda)$ acts as $X \ni \tilde{x} = (x_1, \ldots, x_n) \mapsto \sum_{i\to j} (\overbrace{0, \ldots, 0}^{j-1}, {}_jx_i(x_i \otimes m_{ij}), 0, \ldots, 0) \in X$. The orbit $G_{\boldsymbol{d}}$-orbit $G_{\boldsymbol{d}} \cdot x$ corresponds to the isomorphism class of Λ-module X.

Definition 2.6. *Let $\langle -, - \rangle$ be the Ringel form of an oriented K-modulation $(\mathfrak{M}, \Omega)$ of a valued diagram $(\Gamma, \boldsymbol{v})$, i.e., for vectors $\boldsymbol{x} = (x_i)_{i\in\Gamma}, \boldsymbol{y} = (y_i)_{i\in\Gamma} \in \mathbb{Z}^n_{\geq 0}, \langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x} R^t \boldsymbol{y}$, where $R = (r_{ij})_{i,j\in\Gamma}$ with $r_{ii} = f_i$ $(i \in \Gamma), r_{ij} = -\dim_K {}_iM_j (= -v_{ij}f_j)$ if $\underset{i}{\circ} \xrightarrow{(v_{ij},\ \ v_{ji})} \underset{j}{\circ}$; and $r_{ij} = 0$ otherwise. Then we have $\langle \dim X, \dim Y \rangle = \dim_K \mathrm{Hom}_{\Lambda}(X, Y) - \dim_K \mathrm{Ext}^1_{\Lambda}(X, Y)$ for each Λ-modules X, Y (see [D; Proposition 2.2]).*

Proposition 2.7. *Let K be a field of characteristic 0. Let G be a connected algebraic group defined over K acting on the vector space V with a K-structure V_K. Then the K-rational points G_K of G acts on V_K. For a point $x \in V$, let $G_x = \{g \in G; g \cdot x = x\}$ be the isotropy subgroup of G at x. If a point $x \in V_K$ is a generic point, i.e., it satisfies $\dim_K (G_K)_x = \dim_K G_K - \dim_K V_K$, then we have $\overline{G_K \cdot x} = V_K$ with the Zariski topology in V_K.*

Proof. Let Ω be a universal field. Since $G = G_K \otimes_K \Omega$ and $V = V_K \otimes_K \Omega$, we have $\dim G_x = \dim G - \dim V$, and hence we have $\overline{G \cdot x} = V$ (see [K]). By [B; p. 220], we have $\overline{G_K} = G$. Since the map $f : G \to V$ defined by $f(g) = g \cdot x$ is continuous with respect to the Zariski topology, we have $G \cdot x = f(G) = f(\overline{G_K}) \subset \overline{f(G_K)} = \overline{G_K \cdot x}$. Since $\overline{G \cdot x} = V$, we have $\overline{G_K \cdot x} = V$ with the Zariski topology in V. Since $G_K \cdot x \subset V_K$ and the Zariski topology in V_K is weaker than the induced topology on V_K from the Zariski topology in V, we obtain our result. ∎

Definition 2.8. *A non-zero rational function $f(x)$ on V_K is called a relative invariant of (G_K, V_K) if there exists a rational character χ of G_K satisfying $f(g \cdot x) = \chi(g)f(x)$ as a rational function for all $g \in G$. By Proposition 2.7, if (G_K, V_K) is a prehomogeneous vector space, i.e., there exists a generic point, then a relative invariant $f(x)$ is uniquely determined by its corresponding character χ up to a constant multiple (cf. [K]). Let $X(G_K)$ be the group of rational characters of G_K.*

Proposition 2.9. *Let (G_K, V_K) be a prehomogeneous vector space with a generic point x_0. Let $f_1, \dots, f_l$ be relative invariants corresponding to characters $\chi_1, \dots, \chi_l$ respectively. Assume that these characters $\chi_1, \dots, \chi_l$ generate the subgroup $X_1(G_K) = \{\chi \in X(G_K); \chi|_{(G_K)_{x_0}} = 1\}$. Then these $f_1, \dots, f_l$ generate all relative invariants of (G_K, V_K).*

Proof. Let $f(x)$ be any relative invariant of (G_K, V_K) corresponding to a character χ. For any $g \in (G_K)_{x_0}$, we have $f(g \cdot x_0) = f(x_0) = \chi(g)f(x_0)$. By Proposition 2.7, we have $f(x_0) \neq 0, \infty$, and hence $\chi(g) = 1$, i.e., $\chi \in X_1(G_K)$. Hence we can express $\chi = \chi_1^{m_1} \cdots \chi_l^{m_l}$ with some $(m_1, \dots, m_l) \in \mathbb{Z}^l$, we have $f(x) = cf_1^{m_1} \cdots f_l^{m_l}$ with some constant multiple. ∎

Lemma 2.10. *Let $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}})$ be a PV associated with an oriented K-modulation $(\mathfrak{M}, \Omega)$ of a valued graph $(\Gamma, \boldsymbol{v})$, and X a Λ-module corresponding to a point $x \in R_{\boldsymbol{d}}$ where $\Lambda = \Lambda(\mathfrak{M}, \Omega)$. Then x is a generic point of the PV if and only if $\operatorname{Ext}^1_\Lambda(X, X) = 0$, i.e., X is a partial tilting Λ-module.*

Proof. Since Λ is hereditary, we have $\dim_K G_{\boldsymbol{d}} - \dim_K R_{\boldsymbol{d}} = \dim_K \operatorname{Hom}_\Lambda(X, X) - \dim_K \operatorname{Ext}^1_\Lambda(X, X) = \dim_K (G_{\boldsymbol{d}})_x - \dim_K \operatorname{Ext}^1_\Lambda(X, X)$. Hence, x is a generic point if and only if $\operatorname{Ext}^1_\Lambda(X, X) = 0$. ∎

By Proposition 2.7 and Lemma 2.10, we see that (the dense orbit of) a PV $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}})$ corresponds to the isomorphism class of some partial tilting Λ-module.

Definition 2.11. (Positive roots and indecomposable representations)

Let $(\Gamma, \boldsymbol{v})$ be a valued graph with $\Gamma = \{1, 2, \dots, n\}$. For each vertex $k \in \Gamma$, define the reflection $r_k : \mathbb{Q}^n \to \mathbb{Q}^n$ by $\boldsymbol{x} = (x_i) \mapsto (y_i)$ where $y_k = -x_k + \sum_{i \in \Gamma} v_{ik} x_i$ and $y_i = x_i$ $(i \neq k)$. The group W generated by the reflections is called the Weyl group of $(\Gamma, \boldsymbol{v})$. An element $\boldsymbol{x} \in \mathbb{Q}^n$ is called a root of $(\Gamma, \boldsymbol{v})$ if there exists $w \in W$ and $i \in \Gamma$ satisfying $\boldsymbol{x} = w\boldsymbol{e}_i$ where $\boldsymbol{e}_i = (0, \dots, 0, \overset{i}{\check{1}}, 0, \dots, 0)$. An element $\boldsymbol{x} = (x_i) \in \mathbb{Q}^n$ is called positive if all $x_i \geq 0$, and we write $\boldsymbol{x} \geq 0$. Note that positive roots belong to $\mathbb{Z}^n_{\geq 0}$. It is known that, in our case, the map $\operatorname{rep}(\mathfrak{M}, \Omega) \to \mathbb{Z}^n_{\geq 0}$ defined by $W \mapsto \dim W$ induces the bijection between the isomorphism classes of indecomposable representation *and the positive roots of* $(\Gamma, \boldsymbol{v})$ *(see [DR; p. 2]). On the other hand, by Lemma 2.10, for a positive root* $\boldsymbol{d}$*, the corresponding indecomposable representations* $x \in R_{\boldsymbol{d}}$ *are generic points of* $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}})$ *since the corresponding* Λ*-module* X *of an indecomposable representation* x *has the property* $\mathrm{Ext}^1_\Lambda(X, X) = 0$.

By Theorem of Krull-Schmidt, any representation is uniquely decomposed to the direct sum of indecomposable representations.

Let us decompose a representation $x \in R_{\boldsymbol{d}}$ into the direct sum $x = m_1 x_1 \oplus \cdots \oplus m_s x_s$ of indecomposable representations $x_1, \ldots, x_s$ which are not isomorphic of each other and $m_k x_k$ denotes the m_k-copies of x_k. Then x is a generic point if and only if the corresponding Λ-module $m_1 X_1 \oplus \cdots \oplus m_s X_s$ is a partial tilting module. When $m_1 = \cdots = m_s = 1$, such a module is called basic. It is known that Λ-module $m_1 X_1 \oplus \cdots \oplus m_s X_s$ is a partial tilting module if and only if a basic Λ-module $X_1 \oplus \cdots \oplus X_s$ is a partial tilting modules. Hence $x' = x_1 \oplus \cdots \oplus x_s \in R_{\boldsymbol{d}'}$ is a generic point of a PV $(G_{\boldsymbol{d}'}, R_{\boldsymbol{d}'})$ if and only if $x = m_1 x_1 \oplus \cdots \oplus m_s x_s \in R_{\boldsymbol{d}}$ is a generic point of a PV $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}})$ for any $m_1 \geq 1, \ldots, m_s \geq 1$.

In general, for a partial tilting Λ-module X, there exists a Λ-module Y such that $X \oplus Y$ is a tilting module which is, by definition, the number of its non-isomorphic direct summand is equal to $n = \mathrm{Card}(\Gamma)$ (see [ASS; p. 196]) where $\Lambda = \Lambda(\mathfrak{M}, \Omega)$ and $(\mathfrak{M}, \Omega)$ is a K-modulation of a valued graph $(\Gamma, \boldsymbol{v})$.

Therefore, to find a generic point of a PV $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}})$ associated with an oriented K-modulation $(\mathfrak{M}, \Omega)$ of a valued graph $(\Gamma, \boldsymbol{v})$, it is sufficient to give a complete list of basic tilting $\Lambda(\mathfrak{M}, \Omega)$-modules.

Proposition 2.12. *For* $x = ({}_j x_i) \in R_{\boldsymbol{d}} = \bigoplus_{i \to j} \mathrm{Hom}_{F_j}(F_i^{d_i} \otimes_{F_i} {}_i M_j, F_j^{d_j})$ *and* $x' = ({}_j x'_i) \in R_{\boldsymbol{d}'} = \bigoplus_{i \to j} \mathrm{Hom}_{F_j}(F_i^{d'_i} \otimes_{F_i} {}_i M_j, F_j^{d'_j})$*, let* $X = F_1^{d_1} \oplus \cdots \oplus F_n^{d_n}$ *and* $X' = F_1^{d'_1} \oplus \cdots \oplus F_n^{d'_n}$ *be corresponding* Λ*-modules. Then a linear map* $\alpha : X \to X'$ *with* $\alpha = (\alpha_1, \ldots, \alpha_n) \in M(d'_1, d_1; F_1) \oplus \cdots \oplus M(d'_n, d_n; F_n)$ *is a* Λ*-homomorphism if and only if* $\alpha_j \cdot {}_j x_i = {}_j x'_i \cdot (\alpha_i \otimes 1)$ *for each arrow* $i \to j$.

Proof. Since Λ is generated by ${}_i M_j$, it is enough to show that $\alpha(m_{ij}\tilde{x}) = m_{ij}\alpha(\tilde{x})$ for $m_{ij} \in {}_i M_j$ and $\tilde{x} = (x_1, \ldots, x_n) \in X$. Since $\alpha(m_{ij}\tilde{x}) = (0, \ldots, 0, \alpha_j({}_j x_i(x_i \otimes m_{ij})), 0, \ldots, 0)$ and $m_{ij}\alpha(\tilde{x}) = m_{ij}(\alpha_1 x_1, \ldots, \alpha_n x_n) = (0, \ldots, 0, {}_j x'_i((\alpha_i x_i) \otimes m_{ij}), 0, \ldots, 0)$, we obtain our result. ∎

Now we shall consider the relation with the case of another orientation Ω'.

Definition 2.13. *Let* $Q = (\Gamma, \boldsymbol{v}; \Omega)$ *be a valued quiver with* $\Gamma = \{1, 2, \ldots, n\}$*. We call* $k \in \Gamma$ *a sink (resp. source) if* $i \neq k$ *(resp.* $j \neq k$*) for any arrow* $i \to j$*. For an orientation* Ω *and* $k \in \Gamma$*, we define the new orientation* $s_k\Omega$ *by changing the direction of all arrows which contain* k*, and keep the arrows which do not contain* k*. An ordered set* $(k_1, \ldots, k_n)$ *of all vertices is called an* *admissible sequence of sinks if* k_1 *is a sink with respect to* Ω *and* k_t *is a sink with respect to* $s_{k_{t-1}} \cdots s_{k_1}\Omega$ *for all* t *with* $2 \le t \le n$. *Then the following facts hold (see [D; p. 30]).*

1. $s_{k_n} s_{k_{n-1}} \cdots s_{k_1}\Omega = \Omega$
2. $s_{k_1} s_{k_2} \cdots s_{k_n}\Omega = \Omega$
3. *Each* k_t *is a source with respect to* $s_{k_{t+1}} \cdots s_{k_n}\Omega$ *for* $1 \le t \le n$.

Proposition 2.14. *1. For any sink* $k \in \Gamma$, *the reflection functor* $\Delta_k^+ : \mathrm{rep}(\mathfrak{M}, \Omega) \to \mathrm{rep}(\mathfrak{M}, s_k\Omega)$ *induces the categorical equivalence* $\mathrm{rep}^{(k)}(\mathfrak{M}, \Omega) \cong \mathrm{rep}^{(k)}(\mathfrak{M}, s_k\Omega)$ *where* $\mathrm{rep}^{(k)}(\mathfrak{M}, \Omega)$ *(resp.* $\mathrm{rep}^{(k)}(\mathfrak{M}, s_k\Omega)$*) is a full subcategory of* $\mathrm{rep}(\mathfrak{M}, \Omega)$ *(resp.* $\mathrm{rep}(\mathfrak{M}, s_k\Omega)$*) consisting of representations which do not contain, as their direct summands, the representation* $\underline{\boldsymbol{e}_k} = ({}_j\varphi_i, W_i)$ *with the dimension vector* $\boldsymbol{e}_k = (0, \dots, 0, \overset{k}{\check{1}}, 0, \dots, 0)$, *i.e.,* $W_k = F_k, W_i = 0$ *for all* $i \neq k$ *and all* ${}_j\varphi_i = 0$.

2. *Similarly for any source* $k \in \Gamma$, *the reflection functor* $\Delta_k^- : \mathrm{rep}(\mathfrak{M}, \Omega) \to \mathrm{rep}(\mathfrak{M}, s_k\Omega)$ *is defined. For an admissible sequence of sinks* $(k_1, \dots, k_n)$, *the covariant functor* $\Delta^- : \mathrm{rep}(\mathfrak{M}, \Omega) \to \mathrm{rep}(\mathfrak{M}, \Omega)$ *called the Coxeter functor defined by* $\Delta^- = \Delta_{k_1}^- \Delta_{k_2}^- \cdots \Delta_{k_n}^-$ *induces the categorical equivalence* $\mathrm{rep}_*(\mathfrak{M}, \Omega) \cong \mathrm{rep}^*(\mathfrak{M}, \Omega)$ *where* $\mathrm{rep}_*(\mathfrak{M}, \Omega)$ *(resp.* $\mathrm{rep}^*(\mathfrak{M}, \Omega)$*) is a full subcategory of* $\mathrm{rep}(\mathfrak{M}, \Omega)$ *consisting of representations without injective (resp. projective) direct summand. For* $X, X' \in \mathrm{Ob}(\mathrm{rep}_*(\mathfrak{M}, \Omega))$, *we have a* K*-linear isomorphism*

$$\mathrm{Hom}(X, X') \cong \mathrm{Hom}(\Delta^- X, \Delta^- X').$$

Proof. For 1, see [D; p. 62]. For 2, see [D; Proposition 2.10]. ∎

Proposition 2.15. ([D; Lemma 2.5, Corollary 2.7 (i)])
Let $k \in \Gamma$ *be a sink with respect to* Ω.

1. *If* $X \in \mathrm{Ob}(\mathrm{rep}^{(k)}(\mathfrak{M}, \Omega))$ *is indecomposable, then* $\Delta_k^+ X \in \mathrm{Ob}(\mathrm{rep}^{(k)}(\mathfrak{M}, s_k\Omega))$ *is also indecomposable, and we have* $\dim \Delta_k^+ X = r_k(\dim X)$.
2. *For* $X, X' \in \mathrm{Ob}(\mathrm{rep}^{(k)}(\mathfrak{M}, \Omega))$, *we have* $\mathrm{Ext}^1(X, X') \cong \mathrm{Ext}^1(\Delta_k^+ X, \Delta_k^+ X')$.

Proposition 2.16. *Let* $k \in \Gamma$ *be a sink with respect to* Ω. *Then we have* $\mathrm{Ext}^1_{\Lambda(\mathfrak{M}, s_k\Omega)}(X, \underline{\boldsymbol{e}_k}) = 0$ *for any* $\Lambda(\mathfrak{M}, s_k\Omega)$*-module* X.

Proof. Since $k \in \Gamma$ is a source of $s_k\Omega$, and it is known that there exists an admissible sequence of sinks $(i_1, \dots, i_n)$ with $i_n = k$ with respect to $s_k\Omega$ (see [ASS; p. 279]), we have $\underline{\boldsymbol{e}_k} \in \mathrm{Ob}(\mathrm{rep}(\mathfrak{M}, s_k\Omega))$ is injective by [D; Lemma 2.1]. Hence we obtain our result. ∎

Proposition 2.17. ([D; Corollary 2.4 (ii), p. 62])

Let $k \in \Gamma$ *be a sink with respect to* Ω. *Let* $Y \in \mathrm{Ob}(\mathrm{rep}(\mathfrak{M}, s_k\Omega))$ *be a representation without* $\underline{\boldsymbol{e}_k}$ *as its direct summand. Then we have* $\dim_{F_k} \mathrm{Ext}^1_{\Lambda(\mathfrak{M},s_k\Omega)}(\underline{\boldsymbol{e}_k}, Y) = [r_k(\dim Y)]_k$ *where* $[(d_1, \dots, d_n)]_k = d_k$. *Note that* $\dim_K \mathrm{Ext}^1_{\Lambda(\mathfrak{M},s_k\Omega)}(\underline{\boldsymbol{e}_k}, Y) = f_k \dim_{F_k} \mathrm{Ext}^1_{\Lambda(\mathfrak{M},s_k\Omega)}(\underline{\boldsymbol{e}_k}, Y)$ *where* $f_k = \dim_K F_k$.

Theorem 2.18. *If we obtain the table of* $(\dim_K \mathrm{Ext}^1_\Lambda(X_i, X_j))$ *with respect to some orientation* Ω, *then we can obtain the similar table* $(\dim_K \mathrm{Ext}^1_{\Lambda'}(Y_i, Y_j))$ *with respect to another orientation* Ω' *from* $(\dim_K \mathrm{Ext}^1_\Lambda(X_i, X_j))$.

Proof. By Propositions 2.15-2.17, we obtain our result. ∎

Hence it is essentially enough to calculate the table of $(\dim_K \mathrm{Ext}^1_\Lambda(X_i, X_j))$ with respect to any one of orientations Ω.

Our results show that the number of (partial) tilting Λ-modules does not depend on the choice of an orientation of arrows.

The following proposition is well-known.

Proposition 2.19. *In general, if* $\mathrm{End}_R(M)$ *is a local ring for an* R*-module* M, *then* M *is indecomposable. The converse also holds if* M *has the composition series.*

3 The Case for exceptional type $\mathbb{G}_2$

In this section, we consider the valued graph $(\Gamma, \boldsymbol{v})$ with $\Gamma = \{1, 2\}$ and $\boldsymbol{v} = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$, with an orientation Ω: $\underset{1}{\circ} \xrightarrow{(1,\,3)} \underset{2}{\circ}$, i.e., of type $\mathbb{G}_2$. We take a K-modulation $\mathfrak{M} = \{F_1 = K,\ F_2 = L,\ {}_1M_2 = {}_KL_L\}$ where L/K is a cubic extension obtained by adding α to K satisfying $\alpha, \alpha^2 \notin K$ and $\alpha^3 \in K$. We denote L as a K-L-bimodule by ${}_KL_L$. Then, for a dimension vector $\boldsymbol{d} = (d_1, d_2) \in \mathbb{Z}^2_{\geq 0}$, we have the representation $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}})$ with $G_{\boldsymbol{d}} = GL(d_1; K) \times GL(d_2; L)$ and $R_{\boldsymbol{d}} = M(d_2, d_1; L)$. The action of $G_{\boldsymbol{d}}$ on $R_{\boldsymbol{d}}$ is given by $R_{\boldsymbol{d}} \ni x \mapsto BxA^{-1} \in R_{\boldsymbol{d}}$ for $(A, B) \in G_{\boldsymbol{d}}$.

For a positive root $\alpha = (d_1, d_2)$, let $x = {}_2\varphi_1 : K^{d_1} \otimes_K L \to L^{d_2}$ be the corresponding indecomposable representation. If we identify $K^{d_1} \otimes_K L$ with L^{d_1}, we may regard $x = {}_2\varphi_1$ as an element of $M(d_2, d_1; L)$. Then the Λ-action of the corresponding Λ-module $X = K^{d_1} \oplus L^{d_2}$ is given by

$$X \ni \left(\begin{pmatrix} x_1 \\ \vdots \\ x_{d_1} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_{d_2} \end{pmatrix} \right) \mapsto \left(\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, {}_2\varphi_1\left(\begin{pmatrix} x_1 l \\ \vdots \\ x_{d_1} l \end{pmatrix} \right) \right)$$

for $l \in L = {}_1M_2 = \mathfrak{M}^{(1)} \subset \Lambda$. There are 6 positive roots α_k $(1 \le k \le 6)$ of ($\Gamma = \{1,2\}, \boldsymbol{v} = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$), which is listed below. We denote by x_k (resp. X_k) an indecomposable representation (resp. Λ-module) corresponding to a positive root α_k. Note that x_k is a generic point of $(G_{\alpha_k}, R_{\alpha_k})$ (see Definition 2.11). An indecomposable representation x_k and the Λ-module X_k corresponding to a positive root α_k is given as follows where $\Lambda = \Lambda(\mathfrak{M}, \Omega)$ (see Definition 2.4).

$\alpha_1 = (0,1) \longleftrightarrow x_1 = 0 : \{0\} \to L \longleftrightarrow X_1 = L$

$\alpha_2 = (1,0) \longleftrightarrow x_2 = 0 : L \to \{0\} \longleftrightarrow X_2 = K$

$\alpha_3 = (1,1) \longleftrightarrow x_3 = 1 : L \to L \longleftrightarrow X_3 = K \oplus L$

$\alpha_4 = (2,1) \longleftrightarrow x_4 = (1\ \alpha) : L^2 \to L \longleftrightarrow X_4 = K^2 \oplus L$

$\alpha_5 = (3,1) \longleftrightarrow x_5 = (1\ \alpha\ \alpha^2) : L^3 \to L \longleftrightarrow X_5 = K^3 \oplus L$

$\alpha_6 = (3,2) \longleftrightarrow x_6 = \begin{pmatrix} \alpha & 0 & 1 \\ 0 & \alpha^2 & 1 \end{pmatrix} : L^3 \to L^2 \longleftrightarrow X_6 = K^3 \oplus L^2$

Theorem 3.1. (The basic tilting $\Lambda(\mathfrak{M}, \Omega)$-modules of type $\mathbb{G}_2$)

For the tensor algebra Λ of the oriented K-modulation $(\mathfrak{M}, \Omega)$, there exist exactly 5 isomorphism classes of basic tilting Λ-modules whose complete list is given by the following $1 \sim 5$.

1. $X_1 \oplus X_3 = L \oplus (K \oplus L)$

2. $X_2 \oplus X_5 = K \oplus (K^3 \oplus L)$

3. $X_3 \oplus X_6 = (K \oplus L) \oplus (K^3 \oplus L^2)$

4. $X_4 \oplus X_5 = (K^2 \oplus L) \oplus (K^3 \oplus L)$

5. $X_4 \oplus X_6 = (K^2 \oplus L) \oplus (K^3 \oplus L^2)$

Proof. In our case, we have $r_{11} = \dim_K K = 1, r_{22} = \dim_K L = 3, r_{12} = -\dim_K {}_1M_2 = -3$, and hence the Ringel form is given by $\langle (x_1, x_2), (y_1, y_2) \rangle = (x_1, x_2) \begin{pmatrix} 1 & -3 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ (see Definition 2.6). Since $\dim X_k = \alpha_k$ $(k = 1, \ldots, 6)$, we have the following matrix (3.1) of $\langle \dim X_i, \dim X_j \rangle$.

$$\begin{pmatrix} 3 & 0 & 3 & 3 & 3 & 6 \\ -3 & 1 & -2 & -1 & 0 & -3 \\ 0 & 1 & 1 & 2 & 3 & 3 \\ -3 & 2 & -1 & 1 & 3 & 0 \\ -6 & 3 & -3 & 0 & 3 & -3 \\ -3 & 3 & 0 & 3 & 6 & 3 \end{pmatrix} \tag{3.1}$$

We can calculate $\dim_K \mathrm{Hom}_\Lambda(X_i, X_j)$ by Proposition 2.12, and its matrix is given by

$$\begin{pmatrix} 3 & 0 & 3 & 3 & 3 & 6 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 3 & 3 \\ 0 & 2 & 0 & 1 & 3 & 0 \\ 0 & 3 & 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 3 & 6 & 3 \end{pmatrix} \tag{3.2}$$

Since $\dim_K \mathrm{Ext}^1_\Lambda(X_i, X_j) = \dim_K \mathrm{Hom}_\Lambda(X_i, X_j) - \langle \dim X_i, \dim X_j \rangle$, we have the following matrix (3.3) of $\dim_K \mathrm{Ext}^1_\Lambda(X_i, X_j)$ from (3.2).

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ 6 & 0 & 3 & 0 & 0 & 3 \\ 3 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.3}$$

Since $X = X_i \oplus X_j$ is a tilting module if and only if $\dim_K \mathrm{Ext}^1_\Lambda(X, X) = \dim_K \mathrm{Ext}^1_\Lambda(X_i, X_i) + \dim_K \mathrm{Ext}^1_\Lambda(X_i, X_j) + \dim_K \mathrm{Ext}^1_\Lambda(X_j, X_i) + \dim_K \mathrm{Ext}^1_\Lambda(X_j, X_j) = 0$, we obtain our result by (3.3). ∎

Theorem 3.2. (PVs corresponding to (partial) tilting modules of type $\mathbb{G}_2$) *For a dimension vector $\boldsymbol{d} = (d_1, d_2) \in \mathbb{Z}^2_{>0}$, a (partial) tilting module corresponding to a prehomogeneous vector space $(GL(d_1; K) \times GL(d_2; L), M(d_2, d_1; L))$ is given as follows. Note that if it corresponds to $a_1 X_{i1} \oplus a_2 X_{i2}$ (resp. $b_j X_j$), then the basic tilting module $X_{i1} \oplus X_{i2}$ (resp. the basic partial tilting module X_j) is its invariant.*

1. *If $d_2 > d_1$, it corresponds to $(d_2 - d_1)X_1 \oplus d_1 X_3$.*
2. *If $d_1 > 3d_2$, it corresponds to $(d_1 - 3d_2)X_2 \oplus d_2 X_5$.*
3. *If $d_1 > d_2 > \frac{2}{3}d_1$, it corresponds to $(3d_2 - 2d_1)X_3 \oplus (d_1 - d_2)X_6$.*
4. *If $3d_2 > d_1 > 2d_2$, it corresponds to $(3d_2 - d_1)X_4 \oplus (d_1 - 2d_2)X_5$.*
5. *If $2d_2 > d_1 > \frac{2}{3}d_1 > d_2$, it corresponds to $(2d_1 - 3d_2)X_4 \oplus (2d_2 - d_1)X_6$.*
6. *If $d_1 = d_2$, it corresponds to $d_1 X_3$.*
7. *If $d_1 = 3d_2$, it corresponds to $d_2 X_5$.*
8. *If $d_1 = \frac{3}{2}d_2$ (and hence d_2 is even), it corresponds to $\frac{d_2}{2}X_6$.*

9. If $d_1 = 2d_2$, it corresponds to d_2X_4.

Proof. Since $m_iX_i \oplus m_jX_j$ $(i \neq j)$ is a partial tilting module if and only if $X_i \oplus X_j$ is a partial tilting module, we obtain all partial tilting modules by Theorem 3.1. For example, a tilting module $m_1X_1 \oplus m_3X_3$ corresponds to $\boldsymbol{d} = (d_1, d_2) = m_1\alpha_1 + m_3\alpha_3 = (m_3, m_1 + m_3)$. Hence we have $d_1 = m_3 < d_2 = m_1 + m_3$ and $X = (d_2 - d_1)X_1 \oplus d_1X_3$, i.e., we obtain *1*. The remaining parts are obtained similarly. ∎

Remark 3.3. (The other orientation)

Since the vertex 2 is the sink for the orientation Ω, we can apply Proposition 2.15 for $k = 2$. Now we shall consider the other orientation Ω' : $\underset{1}{\circ} \xleftarrow{(1,\ 3)} \underset{2}{\circ}$, *and put $\Lambda' = \Lambda(\mathfrak{M}, s_2\Omega)$. Let $Y_1, \dots, Y_6$ be indecomposable Λ'-modules corresponding to positive roots $\alpha_1, \dots, \alpha_6$. Since $s_2((d_1, d_2)) = (d_1, -d_2 + d_1)$, we have the correspondence: $s_2((1,0)) = (1,1)$, $\Delta_2^+X_2 \cong Y_3$; $s_2((1,1)) = (1,0)$, $\Delta_2^+X_3 \cong Y_2$; $s_2((2,1)) = (2,1)$, $\Delta_2^+X_4 \cong Y_4$; $s_2((3,1)) = (3,2)$, $\Delta_2^+X_5 \cong Y_6$; $s_2((3,2)) = (3,1)$, $\Delta_2^+X_6 \cong Y_5$. Hence by Proposition 2.15, we have $\mathrm{Ext}^1_{\Lambda'}(Y_i, Y_j)$ $(i, j \geq 2)$. By Proposition 2.16, we have $\mathrm{Ext}^1_{\Lambda'}(Y_i, Y_1) = 0$ $(i \geq 1)$. By Proposition 2.17, we have $\dim_{F_2} \mathrm{Ext}^1_{\Lambda'}(Y_1, Y_2) = [s_2(\dim Y_2)]_2 = [s_2(1,0)]_2 = [(1,1)]_2 = 1$. Since $F_2 = L$ and $[L : K] = 3$, we have $\dim_K \mathrm{Ext}^1_{\Lambda'}(Y_1, Y_2) = 3$. We can calculate similarly $\dim_K \mathrm{Ext}^1_{\Lambda'}(Y_1, Y_j)$ $(j \geq 3)$, and hence we obtain the table of $\dim_K \mathrm{Ext}^1_{\Lambda(\mathfrak{M},\Omega')}(Y_i, Y_j)$, which is actually given by the transposed matrix of (2.3). In particular, the number of (partial) tilting Λ-modules does not depend on the choice of an orientation of arrows.*

Remark 3.4. (The relative invariants)

We define the injection $\varphi : L \to M(3; K)$ by $a \cdot (1, \alpha, \alpha^2) = (1, \alpha, \alpha^2)\varphi(a)$, i.e., $\varphi(a) = \begin{pmatrix} p & r\alpha^3 & q\alpha^3 \\ q & p & r\alpha^3 \\ r & q & p \end{pmatrix}$ *for $a = p + q\alpha + r\alpha^2 \in L = K + K\alpha + K\alpha^2$. Define the injective homomorphism $\Phi : GL(d_2; L) \hookrightarrow GL(3d_2; K)$ by $\Phi((z_{ij}) = (\varphi(z_{ij}))$.*

We also define the K-isomorphism $\Psi : M(d_2, d_1; L) \cong M(3d_2, d_1; K)$ by

$$\Psi((z_{ij})) = (\psi(z_{ij})) \text{ with } \psi : L \to K^3 \text{ given by } \psi(s + t\alpha + u\alpha^2) = \begin{pmatrix} s \\ t \\ u \end{pmatrix}.$$

Now let $\boldsymbol{d} = (d_1, d_2)$ be the one of the positive roots $\alpha_3, \alpha_4, \alpha_5, \alpha_6$. Then we define the pair $(G^K_{\boldsymbol{d}}, R^K_{\boldsymbol{d}})$ by

$G^K_{\boldsymbol{d}} = GL(d_1; K) \times \Phi(GL(d_2; L))$ $(\subset GL(d_1; K) \times GL(3d_2; K))$ *and*
$R^K_{\boldsymbol{d}} = \Psi(M(d_2, d_1; L)) = M(3d_2, d_1; K)$.

The action of $G^K_{\boldsymbol{d}}$ on $R^K_{\boldsymbol{d}}$ is given by $(g_1, \Phi(g_2)) \cdot \Psi(x) = \Psi(g_2 x g_1^{-1})$ for $(g_1, g_2) \in$

$G_{\boldsymbol{d}}$ and $x \in R_{\boldsymbol{d}}$. Since $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}})$ is a PV, by Proposition 2.7, there exists a dense $G_{\boldsymbol{d}}^{K}$-orbit. If $\boldsymbol{d} = \alpha_i$, then $\Psi(x_i)$ is a generic point $(i = 3, 4, 5, 6)$. To obtain the all relative invariants (see Definition 2.8), it is enough to obtain the relative invariants whose corresponding characters generate the group $X_1(G_K)$ by Proposition 2.9.

Actually we can construct them similarly to [S]. As an example, we show the case $\boldsymbol{d} = (3, 2)$. In this case, the relative invariant $f(\Psi(X))$ with $X = (x_{ij} + y_{ij}\alpha + z_{ij}\alpha^2) \in M(2, 3; L)$ is given by

$$\det \begin{pmatrix} X_{11} & X_{12} & X_{13} & O & O & O & A_{11} & O & B_{11} & O & C_{11} & O \\ X_{21} & X_{22} & X_{23} & O & O & O & O & A_{11} & O & B_{11} & O & C_{11} \\ O & O & O & X_{11} & X_{12} & X_{13} & A_{12} & O & B_{12} & O & C_{12} & O \\ O & O & O & X_{21} & X_{22} & X_{23} & O & A_{12} & O & B_{12} & O & C_{12} \end{pmatrix}$$

where

$X_{ij} = \psi(x_{ij} + y_{ij}\alpha + z_{ij}\alpha^2) \in M(3, 1)$,
$A_{ij} = \psi(a_{ij} + b_{ij}\alpha + c_{ij}\alpha^2) \in M(3, 1; K)$,
$B_{ij} = \psi(\alpha(a_{ij} + b_{ij}\alpha + c_{i,j}\alpha^2)) \in M(3, 1; K)$,
$C_{ij} = \psi(\alpha^2(a_{ij} + b_{ij}\alpha + c_{ij}\alpha^2)) \in M(3, 1; K)$.

We should choose constants so that $f(\Psi(X))$ is not identically zero. As far as it satisfies this condition, the choice of constant gives just a constant multiple of the relative invariant ([S]). For example, we may choose as $a_{11} + b_{11}\alpha + c_{11}\alpha^2 = 1$ and $a_{12} + b_{12}\alpha + c_{12}\alpha^2 = \alpha$.

4 The Case for exceptional type $\mathbb{F}_4$

In this section, we consider the valued graph $(\Gamma, \boldsymbol{v})$ with $\Gamma = \{1, 2, 3, 4\}$ and $\boldsymbol{v} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ with an orientation $\Omega : \underset{1}{\circ} \longrightarrow \underset{2}{\circ} \xrightarrow{(1,\ 2)} \underset{3}{\circ} \longrightarrow \underset{4}{\circ}$, i.e., of type $\mathbb{F}_4$. We take a K-modulation $\mathfrak{M} = \{F_1 = F_2 = K,\ F_3 = F_4 = L,\ {}_1M_2 = {}_KK_K,\ {}_2M_3 = {}_KL_L, {}_3M_4 = {}_LL_L\}$ where L/K is a quadratic extension obtained by adding a square root α of an element of K. Then, for a dimension vector $\boldsymbol{d} = (d_1, d_2, d_3, d_4) \in \mathbb{Z}_{\geq 0}^4$, we have the representation $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}})$ with $G_{\boldsymbol{d}} = GL(d_1; K) \times GL(d_2; K) \times GL(d_3; L) \times GL(d_4; L)$ and $R_{\boldsymbol{d}} = M(d_2, d_1; K) \oplus M(d_3, d_2; L) \oplus M(d_4, d_3; L)$. The action of $G_{\boldsymbol{d}}$ on $R_{\boldsymbol{d}}$ is given by $R_{\boldsymbol{d}} \ni \tilde{x} = (x, y, z) \mapsto g\tilde{x} = (BxA^{-1}, CyB^{-1}, DzC^{-1}) \in R_{\boldsymbol{d}}$ for $g = (A, B, C, D) \in G_{\boldsymbol{d}}$.

For a positive root $\alpha = (d_1, d_2, d_3, d_4)$, the corresponding indecomposable representation $x = ({}_2\varphi_1, {}_3\varphi_2, {}_4\varphi_3)$ is given by maps ${}_2\varphi_1 : K^{d_1} \otimes_K K (\cong K^{d_1}) \to K^{d_2}$, ${}_3\varphi_2 : K^{d_2} \otimes_K L (\cong L^{d_2}) \to L^{d_3}$ and ${}_4\varphi_3 : L^{d_3} \otimes_L L (\cong L^{d_3}) \to L^{d_4}$. We may regard x as an element of $M(d_2,d_1;K)\oplus M(d_3,d_2;L)\oplus M(d_4,d_3;L)$. The Λ-action on the corresponding Λ-module $X=K^{d_1}\oplus K^{d_2}\oplus L^{d_3}\oplus L^{d_4}$ is given by

$$\left(\begin{pmatrix}x_1\\ \vdots\\ x_{d_1}\end{pmatrix},\begin{pmatrix}y_1\\ \vdots\\ y_{d_2}\end{pmatrix},\begin{pmatrix}z_1\\ \vdots\\ z_{d_3}\end{pmatrix},\begin{pmatrix}w_1\\ \vdots\\ w_{d_4}\end{pmatrix}\right)\mapsto\left(\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix},\ {}_2\varphi_1\begin{pmatrix}x_1k\\ \vdots\\ x_{d_1}k\end{pmatrix},\ {}_3\varphi_2\begin{pmatrix}y_1l\\ \vdots\\ y_{d_2}l\end{pmatrix},\ {}_4\varphi_3\begin{pmatrix}z_1l'\\ \vdots\\ z_{d_3}l'\end{pmatrix}\right)$$

for $(k,l,l')\in K\oplus L\oplus L={}_1M_2\oplus{}_2M_3\oplus{}_3M_4=\mathfrak{M}^{(1)}\subset\Lambda$.

There are 24 positive roots α_k $(1\le k\le 24)$ of the valued graph $(\Gamma,\boldsymbol{v})$ of type $\mathbb{F}_4$, which is listed below. We denote by x_k (resp. X_k) an indecomposable representation (resp. Λ-module) corresponding to a positive root α_k. Note that x_k is a generic point of a PV $(G_{\alpha_k},R_{\alpha_k})$. An indecomposable representation x_k corresponding to a positive root α_k is given as follows.

$\alpha_1=(0,0,0,1)\longleftrightarrow x_1=(0,0,0)\longleftrightarrow X_1=\{0\}\oplus\{0\}\oplus\{0\}\oplus L$
$\alpha_2=(0,0,1,0)\longleftrightarrow x_2=(0,0,0)\longleftrightarrow X_2=\{0\}\oplus\{0\}\oplus L\oplus\{0\}$
$\alpha_3=(0,0,1,1)\longleftrightarrow x_3=(0,0,1)\longleftrightarrow X_3=\{0\}\oplus\{0\}\oplus L\oplus L$
$\alpha_4=(0,1,0,0)\longleftrightarrow x_4=(0,0,0)\longleftrightarrow X_4=\{0\}\oplus K\oplus\{0\}\oplus\{0\}$
$\alpha_5=(0,1,1,0)\longleftrightarrow x_5=(0,1,0)\longleftrightarrow X_5=\{0\}\oplus K\oplus L\oplus\{0\}$
$\alpha_6=(0,1,1,1)\longleftrightarrow x_6=(0,1,1)\longleftrightarrow X_6=\{0\}\oplus K\oplus L\oplus L$
$\alpha_7=(0,2,1,0)\longleftrightarrow x_7=(0,\begin{pmatrix}1&\alpha\end{pmatrix},0)\longleftrightarrow X_7=\{0\}\oplus K^2\oplus L\oplus\{0\}$
$\alpha_8=(0,2,1,1)\longleftrightarrow x_8=(0,\begin{pmatrix}1&\alpha\end{pmatrix},1)\longleftrightarrow X_8=\{0\}\oplus K^2\oplus L\oplus L$
$\alpha_9=(0,2,2,1)\longleftrightarrow x_9=(0,I_2,\begin{pmatrix}1&\alpha\end{pmatrix})\longleftrightarrow X_9=\{0\}\oplus K^2\oplus L^2\oplus L$
$\alpha_{10}=(1,0,0,0)\longleftrightarrow x_{10}=(0,0,0)\longleftrightarrow X_{10}=K\oplus\{0\}\oplus\{0\}\oplus\{0\}$
$\alpha_{11}=(1,1,0,0)\longleftrightarrow x_{11}=(1,0,0)\longleftrightarrow X_{11}=K\oplus K\oplus\{0\}\oplus\{0\}$
$\alpha_{12}=(1,1,1,0)\longleftrightarrow x_{12}=(1,1,0)\longleftrightarrow X_{12}=K\oplus K\oplus L\oplus\{0\}$
$\alpha_{13}=(1,1,1,1)\longleftrightarrow x_{13}=(1,1,1)\longleftrightarrow X_{13}=K\oplus K\oplus L\oplus L$
$\alpha_{14}=(1,2,1,0)\longleftrightarrow x_{14}=(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1&\alpha\end{pmatrix},0)\longleftrightarrow X_{14}=K\oplus K^2\oplus L\oplus\{0\}$
$\alpha_{15}=(1,2,1,1)\longleftrightarrow x_{15}=(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1&\alpha\end{pmatrix},1)\longleftrightarrow X_{15}=K\oplus K^2\oplus L\oplus L$
$\alpha_{16}=(1,2,2,1)\longleftrightarrow x_{16}=(\begin{pmatrix}1\\0\end{pmatrix},I_2,\begin{pmatrix}1&\alpha\end{pmatrix})\longleftrightarrow X_{16}=K\oplus K^2\oplus L^2\oplus L$
$\alpha_{17}=(1,3,2,1)\longleftrightarrow x_{17}=(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}\alpha&0&0\\0&1&\alpha\end{pmatrix},\begin{pmatrix}0&1\end{pmatrix})$
$\longleftrightarrow X_{17}=K\oplus K^3\oplus L^2\oplus L$
$\alpha_{18}=(2,2,1,0)\longleftrightarrow x_{18}=(I_2,\begin{pmatrix}1&\alpha\end{pmatrix},0)\longleftrightarrow X_{18}=K^2\oplus K^2\oplus L\oplus\{0\}$
$\alpha_{19}=(2,2,1,1)\longleftrightarrow x_{19}=(I_2,\begin{pmatrix}1&\alpha\end{pmatrix},1)\longleftrightarrow X_{19}=K^2\oplus K^2\oplus L\oplus L$
$\alpha_{20}=(2,2,2,1)\longleftrightarrow x_{20}=(I_2,I_2,\begin{pmatrix}1&\alpha\end{pmatrix})\longleftrightarrow X_{20}=K^2\oplus K^2\oplus L^2\oplus L$
$\alpha_{21}=(2,3,2,1)\longleftrightarrow x_{21}=(\begin{pmatrix}1&0\\0&0\\0&1\end{pmatrix},\begin{pmatrix}\alpha&0&0\\0&1&\alpha\end{pmatrix},\begin{pmatrix}1&\alpha\end{pmatrix})$

$\longleftrightarrow X_{21} = K^2 \oplus K^3 \oplus L^2 \oplus L$

$$\alpha_{22} = (2,4,2,1) \longleftrightarrow x_{22} = (\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \alpha & 0 & 0 \\ 0 & 0 & 1 & \alpha \end{pmatrix}, \begin{pmatrix} 1 & 1 \end{pmatrix})$$

$\longleftrightarrow X_{22} = K^2 \oplus K^4 \oplus L^2 \oplus L$

$$\alpha_{23} = (2,4,3,1) \longleftrightarrow x_{23} = (\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 & \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \end{pmatrix})$$

$\longleftrightarrow X_{23} = K^2 \oplus K^4 \oplus L^3 \oplus L$

$$\alpha_{24} = (2,4,3,2) \longleftrightarrow x_{24} = (\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 & \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix})$$

$\longleftrightarrow X_{24} = K^2 \oplus K^4 \oplus L^3 \oplus L^2$

Theorem 4.1. (The basic tilting $\Lambda(\mathfrak{M}, \Omega)$-modules of type $\mathbb{F}_4$)

For the tensor algebra Λ *of the oriented* K*-modulation* $(\mathfrak{M}, \Omega)$ *of type* $\mathbb{F}_4$*, there exist exactly 66 isomorphism classes of basic tilting* Λ*-modules. They are given by the following list. For example,* $(1,3,6,13)$ *indicates the tilting module* $X_1 \oplus X_3 \oplus X_6 \oplus X_{13} = L \oplus (L \oplus L) \oplus (K \oplus L \oplus L) \oplus (K \oplus K \oplus L \oplus L)$.

(1)	(1, 3, 6, 13)	(2)	(1, 3, 10, 13)	(3)	(1, 4, 8, 15)	(4)	(1, 4, 11, 19)
(5)	(1, 4, 15, 19)	(6)	(1, 6, 8, 13)	(7)	(1, 8, 13, 15)	(8)	(1, 10, 11, 19)
(9)	(1, 10, 13, 19)	(10)	(1, 13, 15, 19)	(11)	(2, 3, 6, 13)	(12)	(2, 3, 10, 13)
(13)	(2, 5, 9, 16)	(14)	(2, 5, 12, 20)	(15)	(2, 5, 16, 20)	(16)	(2, 6, 9, 13)
(17)	(2, 9, 13, 16)	(18)	(2, 10, 12, 20)	(19)	(2, 10, 13, 20)	(20)	(2, 13, 16, 20)
(21)	(4, 7, 8, 15)	(22)	(4, 7, 14, 22)	(23)	(4, 7, 15, 22)	(24)	(4, 11, 18, 19)
(25)	(4, 14, 18, 19)	(26)	(4, 14, 19, 22)	(27)	(4, 15, 19, 22)	(28)	(5, 7, 8, 17)
(29)	(5, 7, 12, 23)	(30)	(5, 7, 17, 23)	(31)	(5, 8, 9, 16)	(32)	(5, 8, 16, 24)
(33)	(5, 8, 17, 24)	(34)	(5, 12, 20, 23)	(35)	(5, 16, 20, 24)	(36)	(5, 17, 20, 23)
(37)	(5, 17, 20, 24)	(38)	(6, 8, 9, 13)	(39)	(7, 8, 15, 17)	(40)	(7, 12, 14, 22)
(41)	(7, 12, 21, 22)	(42)	(7, 12, 21, 23)	(43)	(7, 15, 17, 23)	(44)	(7, 15, 21, 22)
(45)	(7, 15, 21, 23)	(46)	(8, 9, 13, 16)	(47)	(8, 13, 15, 24)	(48)	(8, 13, 16, 24)
(49)	(8, 15, 17, 24)	(50)	(10, 11, 18, 19)	(51)	(10, 12, 18, 19)	(52)	(10, 12, 19, 20)
(53)	(10, 13, 19, 20)	(54)	(12, 14, 18, 19)	(55)	(12, 14, 19, 22)	(56)	(12, 19, 20, 21)
(57)	(12, 19, 21, 22)	(58)	(12, 20, 21, 23)	(59)	(13, 15, 19, 20)	(60)	(13, 15, 20, 24)
(61)	(13, 16, 20, 24)	(62)	(15, 17, 20, 23)	(63)	(15, 17, 20, 24)	(64)	(15, 19, 20, 21)
(65)	(15, 19, 21, 22)	(66)	(15, 20, 21, 23)				

Proof. To express the matrix $(\dim_K \mathrm{Hom}_\Lambda(X_i, X_j))$ simply, we introduce Y_i $(1 \le i \le 24)$ as follows (cf. §5). $Y_1 = X_{13}, Y_2 = X_6, Y_3 = X_3, Y_4 = X_1, Y_5 = X_5, Y_6 = X_{16}, Y_7 = X_9, Y_8 = X_2, Y_9 = X_{15}, Y_{10} = X_{17}, Y_{11} = X_{24}, Y_{12} = X_8, Y_{13} = X_{12}, Y_{14} = X_{21}, Y_{15} = X_{23}, Y_{16} = X_{20}, Y_{17} = X_4, Y_{18} = X_{14}, Y_{19} = X_{22}, Y_{20} = X_7, Y_{21} = X_{10}, Y_{22} = X_{11}, Y_{23} = X_{18}, Y_{24} = X_{19}$.

Then we have the following table (4.1) of $\dim_K \mathrm{Hom}_\Lambda(Y_i, Y_j)$.

$$\begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ 0 & A_1 & A_2 & A_3 & A_4 & A_5 \\ 0 & 0 & A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & 0 & A_1 & A_2 & A_3 \\ 0 & 0 & 0 & 0 & A_1 & A_2 \\ 0 & 0 & 0 & 0 & 0 & A_1 \end{pmatrix} \tag{4.1}$$

and each 4×4 matrices $A_1 \ldots A_6$ is given by

$$A_1 = \begin{pmatrix} 1&0&0&0 \\ 1&1&0&0 \\ 2&2&2&0 \\ 2&2&2&2 \end{pmatrix}, A_2 = \begin{pmatrix} 0&1&0&0 \\ 1&2&2&0 \\ 2&4&4&2 \\ 0&2&2&0 \end{pmatrix}, A_3 = \begin{pmatrix} 1&1&2&0 \\ 2&3&4&2 \\ 2&4&6&2 \\ 2&2&4&2 \end{pmatrix}, A_4 = \begin{pmatrix} 1&2&2&2 \\ 1&3&4&2 \\ 2&4&6&4 \\ 0&2&2&2 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 0&1&2&0 \\ 1&2&4&2 \\ 0&2&4&2 \\ 0&0&2&0 \end{pmatrix}, A_6 = \begin{pmatrix} 1&1&2&2 \\ 0&1&2&2 \\ 0&0&2&2 \\ 0&0&0&2 \end{pmatrix}.$$

The table of the Ringel form $\langle Y_i, Y_j \rangle$ is given by (4.2).

$$\begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ -{}^tA_1 & A_1 & A_2 & A_3 & A_4 & A_5 \\ -{}^tA_2 & -{}^tA_1 & A_1 & A_2 & A_3 & A_4 \\ -{}^tA_3 & -{}^tA_2 & -{}^tA_1 & A_1 & A_2 & A_3 \\ -{}^tA_4 & -{}^tA_3 & -{}^tA_2 & -{}^tA_1 & A_1 & A_2 \\ -{}^tA_5 & -{}^tA_4 & -{}^tA_3 & -{}^tA_2 & -{}^tA_1 & A_1 \end{pmatrix} \tag{4.2}$$

Since $\dim_K \mathrm{Ext}^1_\Lambda(Y_i, Y_j) = \dim_K \mathrm{Hom}_\Lambda(Y_i, Y_j) - \langle \dim Y_i, \dim Y_j \rangle$, we have the following table (4.3) of $\dim_K \mathrm{Ext}^1_\Lambda(Y_i, Y_j)$ from (4.1), (4.2).

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ {}^tA_1 & 0 & 0 & 0 & 0 & 0 \\ {}^tA_2 & {}^tA_1 & 0 & 0 & 0 & 0 \\ {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 & 0 \\ {}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 \\ {}^tA_5 & {}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 \end{pmatrix} \tag{4.3}$$

Thus we obtain our result.

∎

From Theorem 4.1, we have immediately a complete list of PVs associated with the $\mathbb{F}_4$-type quiver $\circ \longrightarrow \circ \xrightarrow{(1,\,2)} \circ \longrightarrow \circ$. Although $(G_{\boldsymbol{d}}, R_{\boldsymbol{d}})$ is a finite PV for any dimension $\boldsymbol{d}$ and also a trivial PV in the sense of [K], our list can be regarded as a finer classification of such PVs.

Theorem 4.2. (PVs corresponding to tilting modules of type $\mathbb{F}_4$)
For a dimension vector $\boldsymbol{d} = (d_1, d_2, d_3, d_4) \in \mathbb{Z}^4_{>0}$*, a tilting module corresponding to a prehomogeneous vector space* $(GL(d_1; K) \times GL(d_2; K) \times GL(d_3; L) \times GL(d_4; L), M(d_2, d_1; K) \oplus M(d_3, d_2; L) \oplus M(d_4, d_3; L))$ *is given as follows. Note that if it corresponds to* $a_1X_{i1} \oplus a_2X_{i2} \oplus a_3X_{i3} \oplus a_4X_{i4}$*, then the basic tilting module* $X_{i1} \oplus X_{i2} \oplus X_{i3} \oplus X_{i4}$ *is its invariant.*

1. *If* $d_1 < d_2 < d_3 < d_4$*, it corresponds to* $(d_4 - d_3)X_1 \oplus (d_3 - d_2)X_3 \oplus (d_2 - d_1)X_6 \oplus d_1X_{13}$.

2. *If* $d_2 < d_1, d_2 < d_3 < d_4$*, it corresponds to* $(d_4 - d_3)X_1 \oplus (d_3 - d_2)X_3 \oplus (d_1 - d_2)X_{10} \oplus d_2X_{13}$.

3. *If* $2d_3 < d_2, d_1 < d_3 < d_4$*, it corresponds to* $(d_4 - d_3)X_1 \oplus (d_2 - 2d_3)X_4 \oplus (d_3 - d_1)X_8 \oplus d_1X_{15}$.

4. *If* $2d_3 < d_1 < d_2, d_3 < d_4$*, it corresponds to* $(d_4 - d_3)X_1 \oplus (d_2 - d_1)X_4 \oplus (d_1 - 2d_3)X_{11} \oplus d_3X_{19}$.

5. *If* $d_3 < d_1 < 2d_3 < d_2, d_3 < d_4$*, it corresponds to* $(d_4 - d_3)X_1 \oplus (d_2 - 2d_3)X_4 \oplus (2d_3 - d_1)X_{15} \oplus (d_1 - d_3)X_{19}$.

6. *If* $d_3 < d_2, d_1 + d_2 < 2d_3, d_3 < d_4$*, it corresponds to* $(d_4 - d_3)X_1 \oplus (2d_3 - d_1 - d_2)X_6 \oplus (d_2 - d_3)X_8 \oplus d_1X_{13}$.

7. *If* $d_2 < 2d_3 < d_1 + d_2, d_1 < d_3 < d_4$*, it corresponds to* $(d_4 - d_3)X_1 \oplus (d_3 - d_1)X_8 \oplus (2d_3 - d_2)X_{13} \oplus (d_1 + d_2 - 2d_3)X_{15}$.

8. *If* $2d_3 < d_2 < d_1, d_3 < d_4$*, it corresponds to* $(d_4 - d_3)X_1 \oplus (d_1 - d_2)X_{10} \oplus (d_2 - 2d_3)X_{11} \oplus d_3X_{19}$.

9. *If* $d_2 < d_1, d_3 < d_2 < 2d_3, d_3 < d_4$*, it corresponds to* $(d_4 - d_3)X_1 \oplus (d_1 - d_2)X_{10} \oplus (2d_3 - d_2)X_{13} \oplus (d_2 - d_3)X_{19}$.

10. *If* $d_3 < d_1 < d_2 < 2d_3, d_3 < d_4$*, it corresponds to* $(d_4 - d_3)X_1 \oplus (2d_3 - d_2)X_{13} \oplus (d_2 - d_1)X_{15} \oplus (d_1 - d_3)X_{19}$.

11. *If* $d_1 < d_2 < d_4 < d_3$*, it corresponds to* $(d_3 - d_4)X_2 \oplus (d_4 - d_2)X_3 \oplus (d_2 - d_1)X_6 \oplus d_1X_{13}$.

12. *If* $d_2 < d_1, d_2 < d_4 < d_3$*, it corresponds to* $(d_3 - d_4)X_2 \oplus (d_4 - d_2)X_3 \oplus (d_1 - d_2)X_{10} \oplus d_2X_{13}$.

13. *If* $d_1 < d_4, 2d_4 < d_2 < d_3$, *it corresponds to* $(d_3 - d_2)X_2 \oplus (d_2 - 2d_4)X_5 \oplus (d_4 - d_1)X_9 \oplus d_1X_{16}$.

14. *If* $2d_4 < d_1 < d_2 < d_3$, *it corresponds to* $(d_3 - d_2)X_2 \oplus (d_2 - d_1)X_5 \oplus (d_1 - 2d_4)X_{12} \oplus d_4X_{20}$.

15. *If* $d_4 < d_1 < 2d_4 < d_2 < d_3$, *it corresponds to* $(d_3 - d_2)X_2 \oplus (d_2 - 2d_4)X_5 \oplus (2d_4 - d_1)X_{16} \oplus (d_1 - d_4)X_{20}$.

16. *If* $d_1 + d_2 < 2d_4, d_4 < d_2 < d_3$, *it corresponds to* $(d_3 - d_2)X_2 \oplus (2d_4 - d_1 - d_2)X_6 \oplus (d_2 - d_4)X_9 \oplus d_1X_{13}$.

17. *If* $d_1 < d_4, d_2 < 2d_4 < d_1 + d_2, d_2 < d_3$, *it corresponds to* $(d_3 - d_2)X_2 \oplus (d_4 - d_1)X_9 \oplus (2d_4 - d_2)X_{13} \oplus (d_1 + d_2 - 2d_4)X_{16}$.

18. *If* $2d_4 < d_2 < d_1, d_2 < d_3$, *it corresponds to* $(d_3 - d_2)X_2 \oplus (d_1 - d_2)X_{10} \oplus (d_2 - 2d_4)X_{12} \oplus d_4X_{20}$.

19. *If* $d_4 < d_2 < 2d_4, d_2 < d_1, d_2 < d_3$, *it corresponds to* $(d_3 - d_2)X_2 \oplus (d_1 - d_2)X_{10} \oplus (2d_4 - d_2)X_{13} \oplus (d_2 - d_4)X_{20}$.

20. *If* $d_4 < d_1 < d_2 < 2d_4, d_2 < d_3$, *it corresponds to* $(d_3 - d_2)X_2 \oplus (2d_4 - d_2)X_{13} \oplus (d_2 - d_1)X_{16} \oplus (d_1 - d_4)X_{20}$.

21. *If* $d_1 < d_4 < d_3, 2d_3 < d_2$, *it corresponds to* $(d_2 - 2d_3)X_4 \oplus (d_3 - d_4)X_7 \oplus (d_4 - d_1)X_8 \oplus d_1X_{15}$.

22. *If* $2d_4 < d_1 < d_3, 2d_3 < d_2$, *it corresponds to* $(d_2 - 2d_3)X_4 \oplus (d_3 - d_1)X_7 \oplus (d_1 - 2d_4)X_{14} \oplus d_4X_{22}$.

23. *If* $d_4 < d_1 < 2d_4, d_1 < d_3, 2d_3 < d_2$, *it corresponds to* $(d_2 - 2d_3)X_4 \oplus (d_3 - d_1)X_7 \oplus (2d_4 - d_1)X_{14} \oplus (d_1 - d_4)X_{22}$.

24. *If* $d_4 < d_3, 2d_3 < d_1 < d_2$, *it corresponds to* $(d_2 - d_1)X_4 \oplus (d_1 - 2d_3)X_{11} \oplus (d_3 - d_4)X_{18} \oplus d_4X_{19}$.

25. *If* $d_3 + d_4 < d_1 < 2d_3 < d_2$, *it corresponds to* $(d_2 - 2d_3)X_4 \oplus (2d_3 - d_1)X_{14} \oplus (d_1 - d_3 - d_4)X_{18} \oplus d_4X_{19}$.

26. *If* $d_3 < d_1, 2d_4 < d_1 < d_3 + d_4, 2d_3 < d_2$, *it corresponds to* $(d_2 - 2d_3)X_4 \oplus (d_1 - 2d_4)X_{14} \oplus (d_1 - d_3)X_{19} \oplus (d_3 + d_4 - d_1)X_{22}$.

27. *If* $d_4 < d_3 < d_1 < 2d_4, 2d_3 < d_2$, *it corresponds to* $(d_2 - 2d_3)X_4 \oplus (2d_4 - d_1)X_{15} \oplus (d_1 - d_3)X_{19} \oplus (d_3 - d_4)X_{22}$.

28. *If* $d_1 < d_4, d_3 + d_4 < d_2, d_1 + d_2 < 2d_3$, *it corresponds to* $(2d_3 - d_1 - d_2)X_5 \oplus (d_2 - d_3 - d_4)X_7 \oplus (d_4 - d_1)X_8 \oplus d_1X_{17}$.

29. *If* $2d_4 < d_1, d_3 + d_4 < d_2, d_1 + d_2 < 2d_3$*, it corresponds to* $(2d_3 - d_1 - d_2)X_5 \oplus (d_2 - d_3 - d_4)X_7 \oplus (d_1 - 2d_4)X_{12} \oplus d_4X_{23}$.

30. *If* $d_4 < d_1 < 2d_4, d_3 + d_4 < d_2, d_1 + d_2 < 2d_3$*, it corresponds to* $(2d_3 - d_1 - d_2)X_5 \oplus (d_2 - d_3 - d_4)X_7 \oplus (2d_4 - d_1)X_{17} \oplus (d_1 - d_4)X_{23}$.

31. *If* $2d_4 < d_2, d_3 < d_2, d_1 + d_2 < d_3 + d_4$*, it corresponds to* $(d_2 - 2d_4)X_5 \oplus (d_2 - d_3)X_8 \oplus (d_3 + d_4 - d_1 - d_2)X_9 \oplus d_1X_{16}$.

32. *If* $d_1 < d_4, 2d_4 < d_2, d_1 + 2d_2 < 2d_3 + 2d_4, d_3 + d_4 < d_1 + d_2$*, it corresponds to* $(d_2 - 2d_4)X_5 \oplus (d_4 - d_1)X_8 \oplus (2d_3 + 2d_4 - d_1 - 2d_2)X_{16} \oplus (d_1 + d_2 - d_3 - d_4)X_{24}$.

33. *If* $d_1 < d_4, 2d_3 + 2d_4 < d_1 + 2d_2, d_2 < d_3 + d_4, d_1 + d_2 < 2d_3$*, it corresponds to* $(2d_3 - d_1 - d_2)X_5 \oplus (d_4 - d_1)X_8 \oplus (d_1 + 2d_2 - 2d_3 - 2d_4)X_{17} \oplus (d_3 + d_4 - d_2)X_{24}$.

34. *If* $2d_4 < d_1, d_3 < d_2 < d_3 + d_4, d_1 + d_2 < 2d_3$*, it corresponds to* $(2d_3 - d_1 - d_2)X_5 \oplus (d_1 - 2d_4)X_{12} \oplus (d_3 + d_4 - d_2)X_{20} \oplus (d_2 - d_3)X_{23}$.

35. *If* $d_1 < d_4, 2d_4 < d_2, d_3 < d_2, d_1 + 2d_2 < 2d_3 + 2d_4$*, it corresponds to* $(d_2 - 2d_4)X_5 \oplus (2d_3 + 2d_4 - d_1 - 2d_2)X_{16} \oplus (d_1 - d_4)X_{20} \oplus (d_2 - d_3)X_{24}$.

36. *If* $d_1 < 2d_4, d_1 + d_2 < d_3 + 2d_4, d_2 < d_3 + d_4, d_1 + d_2 < 2d_3$*, it corresponds to* $(2d_3 - d_1 - d_2)X_5 \oplus (2d_4 - d_1)X_{17} \oplus (d_3 + d_4 - d_2)X_{20} \oplus (d_1 + d_2 - d_3 - 2d4)X_{23}$.

37. *If* $d_4 < d_1, d_1 + d_2 < d_3 + 2d_4, d_1 + 2d_2 < 2d_3 + 2d_4, d_1 + d_2 < 2d_3$*, it corresponds to* $(2d_3 - d_1 - d_2)X_5 \oplus (d_1 + 2d_2 - 2d_3 - 2d_4)X_{17} \oplus (d_1 - d_4)X_{20} \oplus (d_3 + 2d4 - d_1 - d_2)X_{24}$.

38. *If* $d_4 < d_3 < d_2, d_1 + d_2 < 2d_4$*, it corresponds to* $(2d_4 - d_1 - d_2)X_6 \oplus (d_2 - d_3)X_8 \oplus (d_3 - d_4)X_9 \oplus d_1X_{13}$.

39. *If* $d_1 < d_4, d_2 < 2d_3 < d_1 + d_2, d_3 + d_4 < d_2$*, it corresponds to* $(d_2 - d_3 - d_4)X_7 \oplus (d_4 - d_1)X_8 \oplus (d_1 + d_2 - 2d_3)X_{15} \oplus (2d_3 - d_2)X_{17}$.

40. *If* $d_2 < 2d_3, 2d_3 + 2d_4 < d_1 + d_2, d_1 < d_3$*, it corresponds to* $(d_3 - d_1)X_7 \oplus (2d_3 - d_2)X_{12} \oplus (d_1 + d_2 - 2d_3 - 2d_4)X_{14} \oplus d_4X_{22}$.

41. *If* $2d_3 + d_4 < d_1 + d_2, 2d_4 < d_1, d_1 + d_2 < 2d_3 + 2d_4, d_1 < d_3$*, it corresponds to* $(d_3 - d_1)X_7 \oplus (d_1 - 2d_4)X_{12} \oplus (2d_3 + 2d_4 - d_1 - d_2)X_{21} \oplus (d_1 + d_2 - 2d_3 - d_4)X_{22}$.

42. *If* $d_3 + d_4 < d_2, 2d_3 < d_1 + d_2, 2d_4 < d_1, d_1 + d_2 < 2d_3 + d_4$*, it corresponds to* $(d_2 - d_3 - d_4)X_7 \oplus (d_1 - 2d_4)X_{12} \oplus (d_1 + d_2 - 2d_3)X_{21} \oplus (2d_3 + d_4 - d_1 - d_2)X_{23}$.

43. *If* $d_3+d_4<d_2, 2d_3<d_1+d_2, d_4<d_1, 2d_1+d_2<2d_3+2d_4$, *it corresponds to* $(d_2-d_3-d_4)X_7 \oplus (d_1+d_2-2d_3)X_{15} \oplus (2d_3+2d_4-2d_1-d_2)X_{17} \oplus (d_1-d_4)X_{23}$.

44. *If* $d_2<2d_3, d_1<2d_4, 2d_3+d_4<d_1+d_2, d_1<d_3$, *it corresponds to* $(d_3-d_1)X_7 \oplus (2d_4-d_1)X_{15} \oplus (2d_3-d_2)X_{21} \oplus (d_1+d_2-2d_3-d_4)X_{22}$.

45. *If* $d_1+d_2<2d_3+d_4, d_3+d_4<d_2, d_1<2d_4, 2d_3+2d_4<2d_1+d_2$, *it corresponds to* $(d_2-d_3-d_4)X_7 \oplus (2d_4-d_1)X_{15} \oplus (2d_1+d_2-2d_3-2d_4)X_{21} \oplus (2d_3+d_4-d_1-d_2)X_{23}$.

46. *If* $d_3<d_2, d_1+d_2<d_3+d_4, d_2<2d_4<d_1+d_2$, *it corresponds to* $(d_2-d_3)X_8 \oplus (d_3+d_4-d_1-d_2)X_9 \oplus (2d_4-d_2)X_{13} \oplus (d_1+d_2-2d_4)X_{16}$.

47. *If* $d_1<d_4<d_3, d_2<2d_4, 2d_3<d_1+d_2$, *it corresponds to* $(d_4-d_1)X_8 \oplus (2d_4-d_2)X_{13} \oplus (d_1+d_2-2d_3)X_{15} \oplus (d_3-d_4)X_{24}$.

48. *If* $d_1<d_4, d_2<2d_4, d_3+d_4<d_1+d_2<2d_3$, *it corresponds to* $(d_4-d_1)X_8 \oplus (2d_4-d_2)X_{13} \oplus (2d_3-d_1-d_2)X_{16} \oplus (d_1+d_2-d_3-d_4)X_{24}$.

49. *If* $d_1<d_4, 2d_4<d_2<d_3+d_4, 2d_3<d_1+d_2$, *it corresponds to* $(d_4-d_1)X_8 \oplus (d_1+d_2-2d_3)X_{15} \oplus (d_2-2d_4)X_{17} \oplus (d_3+d_4-d_2)X_{24}$.

50. *If* $d_4<d_3, 2d_3<d_2<d_1$, *it corresponds to* $(d_1-d_2)X_{10} \oplus (d_2-2d_3)X_{11} \oplus (d_3-d_4)X_{18} \oplus d_4X_{19}$.

51. *If* $d_2<2d_3, d_3+d_4<d_2<d_1$, *it corresponds to* $(d_1-d_2)X_{10} \oplus (2d_3-d_2)X_{12} \oplus (d_2-d_3-d_4)X_{18} \oplus d_4X_{19}$.

52. *If* $d_3<d_2<d_3+d_4, 2d_4<d_2<d_1$, *it corresponds to* $(d_1-d_2)X_{10} \oplus (d_2-2d_4)X_{12} \oplus (d_2-d_3)X_{19} \oplus (d_3+d_4-d_2)X_{20}$.

53. *If* $d_4<d_3<d_2<d_1, d_2<2d_4$, *it corresponds to* $(d_1-d_2)X_{10} \oplus (2d_4-d_2)X_{13} \oplus (d_2-d_3)X_{19} \oplus (d_3-d_4)X_{20}$.

54. *If* $d_3+d_4<d_1<d_2<2d_3$, *it corresponds to* $(2d_3-d_2)X_{12} \oplus (d_2-d_1)X_{14} \oplus (d_1-d_3-d_4)X_{18} \oplus d_4X_{19}$.

55. *If* $2d_3+2d_4<d_1+d_2, d_3<d_1<d_3+d_4, d_2<2d_3$, *it corresponds to* $(2d_3-d_2)X_{12} \oplus (d_1+d_2-2d_3-2d_4)X_{14} \oplus (d_1-d_3)X_{19} \oplus (d_3+d_4-d_1)X_{22}$.

56. *If* $2d_4<d_1<d_2<d_3+d_4, d_3<d_1$, *it corresponds to* $(d_1-2d_4)X_{12} \oplus (d_1-d_3)X_{19} \oplus (d_3+d_4-d_2)X_{20} \oplus (d_2-d_1)X_{21}$.

57. *If* $2d_4<d_1, d_3<d_1, d_3+d_4<d_2, d_1+d_2<2d_3+2d_4$, *it corresponds to* $(d_1-2d_4)X_{12} \oplus (d_1-d_3)X_{19} \oplus (2d_3+2d_4-d_1-d_2)X_{21} \oplus (d_2-d_3-d_4)X_{22}$.

58. *If* $2d_4 < d_1, d_1 < d_3, d_2 < d_3 + d_4, 2d_3 < d_1 + d_2$, *it corresponds to* $(d_1 - 2d_4)X_{12} \oplus (d_3 + d_4 - d_2)X_{20} \oplus (d_1 + d_2 - 2d_3)X_{21} \oplus (d_3 - d_1)X_{23}$.

59. *If* $d_4 < d_3 < d_1 < d_2 < 2d_4$, *it corresponds to* $(2d_4 - d_2)X_{13} \oplus (d_2 - d_1)X_{15} \oplus (d_1 - d_3)X_{19} \oplus (d_3 - d_4)X_{20}$.

60. *If* $2d_3 < d_1 + d_2, d_4 < d_1 < d_3, d_2 < 2d_4$, *it corresponds to* $(2d_4 - d_2)X_{13} \oplus (d_1 + d_2 - 2d_3)X_{15} \oplus (d_1 - d_4)X_{20} \oplus (d_3 - d_1)X_{24}$.

61. *If* $d_1 + d_2 < 2d_3, d_4 < d_1, d_3 < d_2 < 2d_4$, *it corresponds to* $(2d_4 - d_2)X_{13} \oplus (2d_3 - d_1 - d_2)X_{16} \oplus (d_1 - d_4)X_{20} \oplus (d_2 - d_3)X_{24}$.

62. *If* $2d_1 + d_2 < 2d_3 + 2d_4, d_2 < d_3 + d_4, d_3 + 2d_4 < d_1 + d_2, 2d_3 < d_1 + d_2$, *it corresponds to* $(d_1 + d_2 - 2d_3)X_{15} \oplus (2d_3 + 2d_4 - 2d_1 - d_2)X_{17} \oplus (d_3 + d_4 - d_2)X_{20} \oplus (d_1 + d_2 - d_3 - 2d_4)X_{23}$.

63. *If* $2d_3 < d_1 + d_2 < d_3 + 2d_4, d_4 < d_1, 2d_4 < d_2$, *it corresponds to* $(d_1 + d_2 - 2d_3)X_{15} \oplus (d_2 - 2d_4)X_{17} \oplus (d_1 - d_4)X_{20} \oplus (d_3 + 2d_4 - d_1 - d_2)X_{24}$.

64. *If* $d_3 < d_1 < 2d_4 < d_2 < d_3 + d_4$, *it corresponds to* $(2d_4 - d_1)X_{15} \oplus (d_1 - d_3)X_{19} \oplus (d_3 + d_4 - d_2)X_{20} \oplus (d_2 - 2d_4)X_{21}$.

65. *If* $d_3 < d_1 < 2d_4, d_3 + d_4 < d_2 < 2d_3$, *it corresponds to* $(2d_4 - d_1)X_{15} \oplus (d_1 - d_3)X_{19} \oplus (2d_3 - d_2)X_{21} \oplus (d_2 - d_3 - d_4)X_{22}$.

66. *If* $d_1 < d_3, d_1 < 2d_4, d_2 < d_3 + d_4, 2d_3 + 2d_4 < 2d_1 + d_2$, *it corresponds to* $(2d_4 - d_1)X_{15} \oplus (d_3 + d_4 - d_2)X_{20} \oplus (2d_1 + d_2 - 2d_3 - 2d_4)X_{21} \oplus (d_3 - d_1)X_{23}$.

Proof. We can obtain the results similarly to Theorem 3.2. ∎

Remark 4.3. *PVs corresponding to a partial tilting module whose number of direct summands is less than 4 appears as the boundary of the above list. For example, the case of the boundary* $d_1 < d_2 < d_3 = d_4$ *(resp.* $d_1 < d_2 = d_3 = d_4$*) of 1. corresponds to a partial tilting module* $(d_3 - d_2)X_3 \oplus (d_2 - d_1)X_6 \oplus d_1X_{13}$ *(resp.* $(d_2 - d_1)X_6 \oplus d_1X_{13}$*). Since we assume that* $d \in \mathbb{Z}^4_{>0}$*, the partial tilting module* X_k *alone does not appear for* $k = 1, \ldots, 12, 14, 18$.

Remark 4.4. *Define the injective homomorphism* $\Phi_d : GL(d; L) \hookrightarrow GL(2d; K)$ *by* $(p_{ij} + q_{ij}\alpha)_{ij} \mapsto \begin{pmatrix} p_{ij} & q_{ij}\alpha^2 \\ q_{ij} & p_{ij} \end{pmatrix}_{ij}$ $(i, j = 1, \ldots, d)$ *and the* K*-isomorphism* $\Psi_{d,d'} : M(d, d'; L) \to M(2d, d'; K)$ *by* $(p_{st} + q_{st}\alpha)_{st} \mapsto \begin{pmatrix} p_{st} \\ q_{st} \end{pmatrix}_{st}$ $(s = 1, \ldots, d; t = 1, \ldots, d')$*. Then define the pair* $(G^K_{\boldsymbol{d}}, R^K_{\boldsymbol{d}})$ *by*

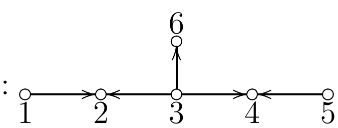
$$G_{\boldsymbol{d}}^{K} = GL(d_1;K) \times GL(d_2;K) \times \Phi_{d_3}(GL(d_3;L)) \times \Phi_{d_4}(GL(d_4;L))$$

$$R_{\boldsymbol{d}}^{K} = M(d_2,d_1;K) \oplus \Psi_{d_3d_2}(M(d_3,d_2;L)) \oplus \Psi_{d_4d_3}(M(d_4,d_3;L)).$$

Similarly to the case of $\mathbb{G}_2$, we can construct the relative invariants of type $\mathbb{F}_4$ of the above PV $(G_{\boldsymbol{d}}^K, R_{\boldsymbol{d}}^K)$.

5 The Case for exceptional type $\mathbb{E}_6$

In this section, we consider the valued graph $(\Gamma, \boldsymbol{v})$ with $\Gamma = \{1,2,3,4,5,6\}$ and $\boldsymbol{v} = \begin{pmatrix} 0&1&0&0&0&0\\ 1&0&1&0&0&0\\ 0&1&0&1&0&1\\ 0&0&1&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0 \end{pmatrix}$ with an orientation Ω :



,
i.e., of type $\mathbb{E}_6$. There exist 36 positive roots $\alpha_1, \ldots, \alpha_{36}$ given by
$\alpha_1 = (0,0,0,0,0,1)$, $\alpha_2 = (0,0,0,0,1,0)$, $\alpha_3 = (0,0,0,1,0,0)$, $\alpha_4 = (0,0,0,1,1,0)$,
$\alpha_5 = (0,0,1,0,0,0)$, $\alpha_6 = (0,0,1,0,0,1)$, $\alpha_7 = (0,0,1,1,0,0)$, $\alpha_8 = (0,0,1,1,0,1)$,
$\alpha_9 = (0,0,1,1,1,0)$, $\alpha_{10} = (0,0,1,1,1,1)$, $\alpha_{11} = (0,1,0,0,0,0)$, $\alpha_{12} = (0,1,1,0,0,0)$,
$\alpha_{13} = (0,1,1,0,0,1)$, $\alpha_{14} = (0,1,1,1,0,0)$, $\alpha_{15} = (0,1,1,1,0,1)$,
$\alpha_{16} = (0,1,1,1,1,0)$, $\alpha_{17} = (0,1,1,1,1,1)$, $\alpha_{18} = (0,1,2,1,0,1)$,
$\alpha_{19} = (0,1,2,1,1,1)$, $\alpha_{20} = (0,1,2,2,1,1)$, $\alpha_{21} = (1,0,0,0,0,0)$,
$\alpha_{22} = (1,1,0,0,0,0)$, $\alpha_{23} = (1,1,1,0,0,0)$, $\alpha_{24} = (1,1,1,0,0,1)$,
$\alpha_{25} = (1,1,1,1,0,0)$, $\alpha_{26} = (1,1,1,1,0,1)$, $\alpha_{27} = (1,1,1,1,1,0)$,
$\alpha_{28} = (1,1,1,1,1,1)$, $\alpha_{29} = (1,1,2,1,0,1)$, $\alpha_{30} = (1,1,2,1,1,1)$,
$\alpha_{31} = (1,1,2,2,1,1)$, $\alpha_{32} = (1,2,2,1,0,1)$, $\alpha_{33} = (1,2,2,1,1,1)$,
$\alpha_{34} = (1,2,2,2,1,1)$, $\alpha_{35} = (1,2,3,2,1,1)$, $\alpha_{36} = (1,2,3,2,1,2)$.

We take a K-modulation $\mathfrak{M} = \{F_1 = \cdots = F_6 = K,\ {}_1M_2 = {}_KK_K,\ {}_3M_2 = {}_KK_K,\ {}_3M_4 = {}_KK_K,\ {}_5M_4 = {}_KK_K,\ {}_3M_6 = {}_KK_K\}$ where ${}_KK_K$ is a commutative field K as a K-K-bimodule. Take an admissible sequence of sinks $(k_1, \ldots, k_6) = (6,4,5,2,1,3)$ with respect to Ω (see Definition 2.13). For any t satisfying $1 \leq t \leq 6$, let $\underline{\boldsymbol{e}_{k_t}} \in \mathrm{Ob}(\mathrm{rep}(\mathfrak{M}, s_{k_t} \cdots s_{k_6}\Omega))$ be the representation with the dimension vector $\boldsymbol{e}_{k_t} = (0, \ldots, 0, \overset{k_t}{\check{1}}, 0, \ldots, 0)$. Then define the representation $P_{k_t} \in \mathrm{Ob}(\mathrm{rep}(\mathfrak{M}, \Omega))$ by $\Delta_{k_1}^- \cdots \Delta_{k_{t-1}}^- \underline{\boldsymbol{e}_{k_t}}$. Let $\Delta^- = \Delta_{k_1}^- \cdots \Delta_{k_6}^- = \Delta_6^-\Delta_4^-\Delta_5^-\Delta_2^-\Delta_1^-\Delta_3^- : \mathrm{rep}(\mathfrak{M}, \Omega) \to \mathrm{rep}(\mathfrak{M}, \Omega)$ be the Coxeter functor. Take the Coxeter element $\delta = r_{k_6} \cdots r_{k_1}$ of the Weyl group (see Definition 2.11 and see [D, p. 44]).

Proposition 5.1. *The 36 representations* $\Delta^{-s}P_t$ $(1 \leq t \leq 6$ *and* $0 \leq s \leq 5)$ *are the complete representatives of isomorphism classes of indecomposable representations in* $\mathrm{rep}(\mathfrak{M}, \Omega)$*, and the 6 representations* $\Delta^{-5}P_t$ $(1 \leq t \leq 6)$ *are the complete representatives of isomorphism classes of indecomposable injective representations. Moreover we have* $\dim \Delta^{-s}P_t = \delta^{-s}(\dim P_t)$ $(1 \leq t \leq 6$ *and* $0 \leq s \leq 5)$ *and* $\dim P_{k_t} = r_{k_1} \cdots r_{k_{t-1}}(\boldsymbol{e}_{k_t})$.

Proof. For any indecomposable representation $X \in \mathrm{Ob}(\mathrm{rep}(\mathfrak{M}, \Omega))$, there exist t $(1 \leq t \leq 6)$ and a non-negative integer s satisfying $X \cong \Delta^{-s}P_t$ (see [D; p. 71 and p. 79 Theorem 2.19 (v)]). For any indecomposable representation $Y \in \mathrm{Ob}(\mathrm{rep}(\mathfrak{M}, \Omega))$, the following $(a), (b)$ hold.

(a) If $\Delta^{-}Y \neq 0$, then $\Delta^{-}Y$ is indecomposable and $\dim \Delta^{-}Y = \delta^{-1}(\dim Y)$ (see [D; p. 69, Lemma 2.9]).

(b) $\Delta^{-}Y = 0 \Leftrightarrow Y$ is injective $\Leftrightarrow \delta^{-1}(\dim Y) \notin \mathbb{Z}^6_{\geq 0}$(see [D; p. 73, Proposition 2.12])

By [D; Lemma 2.5], we have $\dim P_{k_t} = r_{k_1} \cdots r_{k_{t-1}}(\boldsymbol{e}_{k_t})$, and hence $\dim P_1 = \alpha_{22}$, $\dim P_2 = \alpha_{11}$, $\dim P_3 = \alpha_{15}$, $\dim P_4 = \alpha_3$, $\dim P_5 = \alpha_4, \dim P_6 = \alpha_1$. For example, $\dim \Delta^{-1}P_1 = \delta^{-1}\alpha_{22} = \alpha_8$, $\dim \Delta^{-2}P_1 = \delta^{-1}\alpha_8 = \alpha_{16}$, $\dim \Delta^{-3}P_1 = \delta^{-1}\alpha_{16} = \alpha_{24}$, $\dim \Delta^{-4}P_1 = \delta^{-1}\alpha_{24} = \alpha_7$, $\dim \Delta^{-5}P_1 = \delta^{-1}\alpha_7 = \alpha_2$, and $\delta^{-1}\alpha_2 = (0, 0, 0, -1, -1, 0) \notin \mathbb{Z}^6_{\geq 0}$. By similar direct calculation, we can show that the set of $\dim \Delta^{-s}P_t$ $(1 \leq t \leq 6$ and $0 \leq s \leq 5)$ coincides the set of all positive roots and $\delta^{-1}(\dim \Delta^{-5}P_t) \notin \mathbb{Z}^6_{\geq 0}$ $(1 \leq t \leq 6)$. Hence we obtain our result. ∎

Now we shall calculate $\dim_K \mathrm{Hom}(\Delta^{-s_1}P_{t_1}, \Delta^{-s_2}P_{t_2})$ $(1 \leq t_1, t_2 \leq 6$ and $0 \leq s_1, s_2 \leq 5)$.

Proposition 5.2. *The following assertions* $1, 2, 3$ *hold.*

1. *For* $1 \leq t_1, t_2 \leq 6$ *and* $0 \leq s \leq 5$*, we have*

$$\dim_K \mathrm{Hom}(P_{t_1}, P_{t_2}) = \dim_K \mathrm{Hom}(\Delta^{-s}P_{t_1}, \Delta^{-s}P_{t_2}).$$

2. *For* $1 \leq s_2 \leq 4,\ 1 \leq s \leq 5 - s_2$ *and* $1 \leq t_1, t_2 \leq 6$*, we have*

$$\dim_K \mathrm{Hom}(P_{t_1}, \Delta^{-s_2}P_{t_2}) = \dim_K \mathrm{Hom}(\Delta^{-s}P_{t_1}, \Delta^{-s_2-s}P_{t_2}).$$

3. *For* $1 \leq s_1 \leq 4,\ 1 \leq s \leq 5 - s_1$ *and* $1 \leq t_1, t_2 \leq 6$*, we have*

$$\dim_K \mathrm{Hom}(\Delta^{-s_1}P_{t_1}, P_{t_2}) = \dim_K \mathrm{Hom}(\Delta^{-s_1-s}P_{t_1}, \Delta^{-s}P_{t_2}).$$

Proof. By 2 of Proposition 2.14, we have our result. Note that $\{\Delta^{-5}P_t | 1 \leq t \leq 6\}$ is the set of complete representatives of isomorphism classes of indecomposable injective representations (see [D; Proposition 2.8]). ∎

Proposition 5.3. *For $1 \leq s_1 \leq 4$ and $1 \leq t_1, t_2 \leq 6$, we have*

$$\dim_K \operatorname{Hom}(\Delta^{-s_1} P_{t_1}, P_{t_2}) = 0.$$

Proof. Since $\Delta^{-s_1} P_{t_1}$ is a non-projective indecomposable representation, and P_{t_2} is an indecomposable projective representation, by [D; p. 75, Lemma 2.13], we obtain our result. ∎

Proposition 5.4. *For $1 \leq t \leq 6$ and $0 \leq s \leq 5$, we have*

$$(\dim_K \operatorname{Hom}(P_1, \Delta^{-s}P_t), \dim_K \operatorname{Hom}(P_2, \Delta^{-s}P_t), \ldots, \dim_K \operatorname{Hom}(P_6, \Delta^{-s}P_t)) = \delta^{-s}(\dim P_t).$$

Proof. Since the maximal length of paths is 1 in our orientation Ω, we have $\Lambda = \Lambda(\mathfrak{M}, \Omega) = \mathfrak{M}^{(0)} \oplus \mathfrak{M}^{(1)} = (F_1 \times \cdots \times F_6) \oplus ({}_1M_2 \oplus {}_3M_2 \oplus {}_3M_4 \oplus {}_3M_6 \oplus {}_5M_4)$ (see Definition 2.4). Put $e_t = (0, \ldots, 0, 1_{F_t}, 0, \ldots, 0) \in F_1 \times \cdots \times F_6 \subset \Lambda$. Then we have $1_\Lambda = e_1 + \cdots + e_6$, $e_t^2 = e_t$ $(1 \leq t \leq 6)$ and $e_{t_1} e_{t_2} = e_{t_2} e_{t_1} = 0$ $(1 \leq t_1 \neq t_2 \leq 6)$, and hence we obtain $\Lambda = e_1\Lambda + \cdots + e_6\Lambda$. By [ASS; p. 19, Lemma 4.2], we have $\operatorname{End}_\Lambda(e_t\Lambda) \cong e_t\Lambda e_t$ as K-algebras. Since $e_t\Lambda e_t = (0, \ldots, 0, F_t, 0, \ldots, 0) \cong F_t$, the endomorphism ring $\operatorname{End}_\Lambda(e_t\Lambda)$ is a field, and hence by Proposition 2.19, each $e_t\Lambda$ is indecomposable. By Remark 2.5, the dimension vector of the indecomposable representation corresponding to $e_t\Lambda \in \operatorname{Ob}(\bmod_\Lambda)$ $(1 \leq t \leq 6)$ is $(\dim_K e_t\Lambda e_1, \dim_K e_t\Lambda e_2, \ldots, \dim_K e_t\Lambda e_6)$. This coincides with $\dim P_t$. For example, we have $e_3\Lambda = F_3 \oplus ({}_3M_2 \oplus {}_3M_4 \oplus {}_3M_6)$ and hence $e_3\Lambda e_k = 0$ $(k = 1, 5)$, $e_3\Lambda e_k = {}_3M_k$ $(k = 2, 4, 6)$, $e_3\Lambda e_3 = F_3$. This implies that $(\dim_K e_3\Lambda e_1, \ldots, \dim_K e_3\Lambda e_6) = (0, 1, 1, 1, 0, 1) = \alpha_{15} = \dim P_3$. Hence the indecomposable representation corresponding to $e_t\Lambda$ $(1 \leq t \leq 6)$ coincides with P_t up to isomorphisms. The right Λ-module corresponding the representation $\Delta^{-s}P_t = ({}_j\varphi_i^{(s,t)}, W_i^{(s,t)})$ $(1 \leq t \leq 6$ and $0 \leq s \leq 5)$ is given by $M^{(s,t)} = \oplus_{i \in \Gamma} W_i^{(s,t)}$ (see Remark 1.5). By [ASS; p. 19, Lemma 4.2], we have $\dim_K \operatorname{Hom}_\Lambda(e_{t'}\Lambda, M^{(s,t)}) = \dim_K M^{(s,t)} e_{t'}$ $(1 \leq t, t' \leq 6$ and $0 \leq s \leq 5)$. Since $\dim_K M^{(s,t)} e_{t'} = \dim_K W_{t'}^{(s,t)} = [\dim \Delta^{-s}P_t]_{t'}$ where $[\dim \Delta^{-s}P_t]_{t'}$ stands for the t'-th component of $\dim \Delta^{-s}P_t$, we have

$$(\dim_K \operatorname{Hom}_\Lambda(e_1\Lambda, M^{(s,t)}), \ldots, \dim_K \operatorname{Hom}_\Lambda(e_6\Lambda, M^{(s,t)})) = \dim \Delta^{-s}P_t = \delta^{-s}(\dim P_t).$$

Since $e_t\Lambda$ (resp. $M^{(s,t)}$) corresponds to P_t (resp. $\Delta^{-s}P_t$), we obtain

$$(\dim_K \operatorname{Hom}(P_1, \Delta^{-s}P_t), \dim_K \operatorname{Hom}(P_2, \Delta^{-s}P_t), \ldots, \dim_K \operatorname{Hom}(P_6, \Delta^{-s}P_t)) = \delta^{-s}(\dim P_t)$$

for $1 \leq t \leq 6$ and $0 \leq s \leq 5$. ∎

By Propositions 5.1-5.4, we can calculate $\dim_K \mathrm{Hom}(\Delta^{-s_1}P_{t_1}, \Delta^{-s_2}P_{t_2})(1 \le t_1, t_2 \le 6$ and $0 \le s_1, s_2 \le 5)$. We denote by Y_{6s+t} the Λ-module corresponding to the representation $\Delta^{-s}P_t$ $(0 \le s \le 5$ and $1 \le t \le 6)$. Let X_k be the Λ-module corresponding to the positive root α_k $(1 \le k \le 36)$. Then we have $Y_1 = X_{22}, Y_2 = X_{11}, Y_3 = X_{15}, Y_4 = X_3, Y_5 = X_4, Y_6 = X_1, Y_7 = X_8, Y_8 = X_{26}, Y_9 = X_{34}, Y_{10} = X_{17}, Y_{11} = X_{13}, Y_{12} = X_{14}, Y_{13} = X_{16}, Y_{14} = X_{20}, Y_{15} = X_{36}, Y_{16} = X_{32}, Y_{17} = X_{25}, Y_{18} = X_{28}, Y_{19} = X_{24}, Y_{20} = X_{33}, Y_{21} = X_{35}, Y_{22} = X_{31}, Y_{23} = X_{10}, Y_{24} = X_{18}, Y_{25} = X_7, Y_{26} = X_{29}, Y_{27} = X_{30}, Y_{28} = X_{19}, Y_{29} = X_{12}, Y_{30} = X_{27}, Y_{31} = X_2, Y_{32} = X_9, Y_{33} = X_5, Y_{34} = X_{23}, Y_{35} = X_{21}, Y_{36} = X_6$.

Then we have the following table (5.1) of $\dim_K \mathrm{Hom}_\Lambda(Y_i, Y_j)$.

$$\begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ 0 & A_1 & A_2 & A_3 & A_4 & A_5 \\ 0 & 0 & A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & 0 & A_1 & A_2 & A_3 \\ 0 & 0 & 0 & 0 & A_1 & A_2 \\ 0 & 0 & 0 & 0 & 0 & A_1 \end{pmatrix} \tag{5.1}$$

and each 6×6 matrices $A_1, \ldots, A_6$ is given by

$$A_1 = \begin{pmatrix} 1&0&0&0&0&0 \\ 1&1&1&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&1&1&1&0 \\ 0&0&0&0&1&0 \\ 0&0&1&0&0&1 \end{pmatrix}, A_2 = \begin{pmatrix} 0&1&1&0&0&0 \\ 0&1&2&1&1&1 \\ 1&1&2&1&1&1 \\ 1&1&2&1&0&1 \\ 0&0&1&1&0&0 \\ 1&1&1&1&1&0 \end{pmatrix}, A_3 = \begin{pmatrix} 0&0&1&1&1&1 \\ 1&1&2&2&1&1 \\ 1&2&3&2&1&1 \\ 1&2&2&1&1&1 \\ 1&1&1&0&0&1 \\ 0&1&2&1&0&1 \end{pmatrix}, A_4 = \begin{pmatrix} 1&1&1&1&0&0 \\ 1&2&2&1&0&1 \\ 1&2&3&2&1&2 \\ 0&1&2&2&1&1 \\ 0&1&1&1&1&0 \\ 1&1&1&1&1&1 \end{pmatrix},$$
$$A_5 = \begin{pmatrix} 0&1&1&0&0&1 \\ 0&1&1&1&1&1 \\ 1&2&2&2&1&1 \\ 1&1&1&1&0&1 \\ 0&0&1&1&0&1 \\ 0&1&1&1&0&0 \end{pmatrix}, A_6 = \begin{pmatrix} 0&0&0&1&1&0 \\ 0&0&0&1&0&0 \\ 0&1&1&1&0&1 \\ 0&1&0&0&0&0 \\ 1&1&0&0&0&0 \\ 0&0&0&0&0&1 \end{pmatrix}.$$

The table of $\langle \dim Y_i, \dim Y_j \rangle$ is given by

$$\begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ -{}^tA_1 & A_1 & A_2 & A_3 & A_4 & A_5 \\ -{}^tA_2 & -{}^tA_1 & A_1 & A_2 & A_3 & A_4 \\ -{}^tA_3 & -{}^tA_2 & -{}^tA_1 & A_1 & A_2 & A_3 \\ -{}^tA_4 & -{}^tA_3 & -{}^tA_2 & -{}^tA_1 & A_1 & A_2 \\ -{}^tA_5 & -{}^tA_4 & -{}^tA_3 & -{}^tA_2 & -{}^tA_1 & A_1 \end{pmatrix} \tag{5.2}$$

Since $\dim_K \mathrm{Ext}^1_\Lambda(Y_i, Y_j) = \dim_K \mathrm{Hom}_\Lambda(Y_i, Y_j) - \langle \dim Y_i, \dim Y_j \rangle$, we have the following table (5.3) of $\dim_K \operatorname{Ext}^1_\Lambda(Y_i, Y_j)$ from (5.1) and (5.2).

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ {}^tA_1 & 0 & 0 & 0 & 0 & 0 \\ {}^tA_2 & {}^tA_1 & 0 & 0 & 0 & 0 \\ {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 & 0 \\ {}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 \\ {}^tA_5 & {}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 \end{pmatrix} \tag{5.3}$$

Remark 5.5. *By Proposition 5.4, each component of A_1 in (5.1), i.e., the value of $\dim_{F_i} \operatorname{Hom}_\Lambda(P_i, P_j)$, is determined by positive roots. Hence our results do not depend on the choice of the modulations.*

Remark 5.6. (How to make the list of tilting modules)

Although we have the table of $\dim_K \operatorname{Ext}^1_\Lambda(X_i, X_j)$, we use the computer to obtain the list of the tilting modules by the following procedure.

Let $X_1, \dots, X_r$ be the complete representatives of the isomorphism classes of indecomposable Λ-modules. Hence r is the cardinal number of positive roots. Then, for each X_i, put $\operatorname{per} X_i = \{j \in \Gamma; j > i$ and $\dim_K \operatorname{Ext}^1_\Lambda(X_i, X_j) = \dim_K \operatorname{Ext}^1_\Lambda(X_j, X_i) = 0\}$. As an example, we show the case to find the (partial) tilting modules with 4 indecomposable direct summands. So our aim is to find the quadruplets (i_1, i_2, i_3, i_4) with $i_1 < i_2 < i_3 < i_4$ such that $X_{i_1} \oplus X_{i_2} \oplus X_{i_3} \oplus X_{i_4}$ is the (partial) tilting modules.

1. *First run i_1 over $1, 2, \dots, r-3$.*
2. *For each i_1, run i_2 over $i_1+1, \dots, r-2$ with $i_2 \in \operatorname{per} X_{i_1}$.*
3. *For each (i_1, i_2), run i_3 over $i_2+1, \dots, r-1$ with $i_3 \in \operatorname{per} X_{i_1} \cap X_{i_2}$.*
4. *For each (i_1, i_2, i_3), run i_4 over $i_3+1, \dots, r$ with $i_4 \in \operatorname{per} X_{i_1} \cap \operatorname{per} X_{i_2} \cap X_{i_3}$.*

By this procedure, we obtain the (partial) tilting modules $X_{i_1} \oplus X_{i_2} \oplus X_{i_3} \oplus X_{i_4}$. In general, if we want to find the (partial) tilting modules $X_{i_1} \oplus \cdots \oplus X_{i_s}$, then for each obtained $(i_1, \dots, i_{p-1})$ $(p \geq 2)$, run i_p over $i_{p-1}+1, \dots, r-s+p$ with $i_p \in \operatorname{per} X_{i_1} \cap \cdots \cap \operatorname{per} X_{i_{p-1}}$.

For example, in the case of $\mathbb{F}_4$: $\underset{1}{\circ} \longrightarrow \underset{2}{\circ} \xrightarrow{(1,\ 2)} \underset{3}{\circ} \longrightarrow \underset{4}{\circ}$, let X_1 be the module corresponding to a positive root $\alpha_1 = (0,0,0,1)$. Then we have $\operatorname{per} X_1 = \{3, 4, 6, 8, 10, 11, 13, 15, 19\}$. If we take $i_2 = 3$, then X_3 corresponds to $\alpha_3 = (0,0,1,1)$ and we have $\operatorname{per} X_1 \cap \operatorname{per} X_3 = \{6, 10, 13\}$. If we take $i_3 = 6$, then X_6 corresponds to $\alpha_6 = (0,1,1,1)$, and we have $\operatorname{per} X_6 = \{8, 9, 13\}$. Since $\operatorname{per} X_1 \cap \operatorname{per} X_3 \cap \operatorname{per} X_6 = \{13\}$, we have $i_4 = 13$ and we obtain a tilting module $X_1 \oplus X_3 \oplus X_6 \oplus X_{13}$.

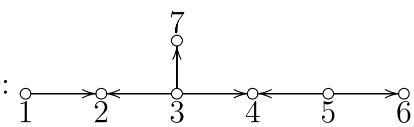
Since the number of partial tilting modules of type $\mathbb{E}_6$ (resp. $\mathbb{E}_7, \mathbb{E}_8$) is so large, we show only the number of tilting modules of each type.

Theorem 5.7. (The number of the basic partial tilting $\Lambda(\mathfrak{M}, \Omega)$-modules of type $\mathbb{E}_6$)

There exist 300 (resp. 1035, 1720, 1368, 418) isomorphism classes of basic partial tilting modules which are direct sums of 2 (resp. 3,4,5,6) indecomposable modules.

6 The Case for exceptional type $\mathbb{E}_7$

In this section, we consider the valued graph $(\Gamma, \boldsymbol{v})$ with $\Gamma = \{1, 2, 3, 4, 5, 6, 7\}$ and $\boldsymbol{v} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$ with an orientation Ω :



, i.e., of type $\mathbb{E}_7$. There exist 63 positive roots $\alpha_1, \dots, \alpha_{63}$ given by
$\alpha_1 = (0,0,0,0,0,0,1)$, $\alpha_2 = (0,0,0,0,0,1,0)$, $\alpha_3 = (0,0,0,0,1,0,0)$,
$\alpha_4 = (0,0,0,0,1,1,0)$, $\alpha_5 = (0,0,0,1,0,0,0)$, $\alpha_6 = (0,0,0,1,1,0,0)$,
$\alpha_7 = (0,0,0,1,1,1,0)$, $\alpha_8 = (0,0,1,0,0,0,0)$, $\alpha_9 = (0,0,1,0,0,0,1)$,
$\alpha_{10} = (0,0,1,1,0,0,0)$, $\alpha_{11} = (0,0,1,1,0,0,1)$, $\alpha_{12} = (0,0,1,1,1,0,0)$,
$\alpha_{13} = (0,0,1,1,1,0,1)$, $\alpha_{14} = (0,0,1,1,1,1,0)$, $\alpha_{15} = (0,0,1,1,1,1,1)$,
$\alpha_{16} = (0,1,0,0,0,0,0)$, $\alpha_{17} = (0,1,1,0,0,0,0)$, $\alpha_{18} = (0,1,1,0,0,0,1)$,
$\alpha_{19} = (0,1,1,1,0,0,0)$, $\alpha_{20} = (0,1,1,1,0,0,1)$, $\alpha_{21} = (0,1,1,1,1,0,0)$,
$\alpha_{22} = (0,1,1,1,1,0,1)$, $\alpha_{23} = (0,1,1,1,1,1,0)$, $\alpha_{24} = (0,1,1,1,1,1,1)$,
$\alpha_{25} = (0,1,2,1,0,0,1)$, $\alpha_{26} = (0,1,2,1,1,0,1)$, $\alpha_{27} = (0,1,2,1,1,1,1)$,
$\alpha_{28} = (0,1,2,2,1,0,1)$, $\alpha_{29} = (0,1,2,2,1,1,1)$, $\alpha_{30} = (0,1,2,2,2,1,1)$,
$\alpha_{31} = (1,0,0,0,0,0,0)$, $\alpha_{32} = (1,1,0,0,0,0,0)$, $\alpha_{33} = (1,1,1,0,0,0,0)$,
$\alpha_{34} = (1,1,1,0,0,0,1)$, $\alpha_{35} = (1,1,1,1,0,0,0)$, $\alpha_{36} = (1,1,1,1,0,0,1)$,
$\alpha_{37} = (1,1,1,1,1,0,0)$, $\alpha_{38} = (1,1,1,1,1,0,1)$, $\alpha_{39} = (1,1,1,1,1,1,0)$,
$\alpha_{40} = (1,1,1,1,1,1,1)$, $\alpha_{41} = (1,1,2,1,0,0,1)$, $\alpha_{42} = (1,1,2,1,1,0,1)$,
$\alpha_{43} = (1,1,2,1,1,1,1)$, $\alpha_{44} = (1,1,2,2,1,0,1)$, $\alpha_{45} = (1,1,2,2,1,1,1)$,
$\alpha_{46} = (1,1,2,2,2,1,1)$, $\alpha_{47} = (1,2,2,1,0,0,1)$, $\alpha_{48} = (1,2,2,1,1,0,1)$,
$\alpha_{49} = (1,2,2,1,1,1,1)$, $\alpha_{50} = (1,2,2,2,1,0,1)$, $\alpha_{51} = (1,2,2,2,1,1,1)$,
$\alpha_{52} = (1,2,2,2,2,1,1)$, $\alpha_{53} = (1,2,3,2,1,0,1)$, $\alpha_{54} = (1,2,3,2,1,0,2)$,
$\alpha_{55} = (1,2,3,2,1,1,1)$, $\alpha_{56} = (1,2,3,2,1,1,2)$, $\alpha_{57} = (1,2,3,2,2,1,1)$,
$\alpha_{58} = (1,2,3,2,2,1,2)$, $\alpha_{59} = (1,2,3,3,2,1,1)$, $\alpha_{60} = (1,2,3,3,2,1,2)$,
$\alpha_{61} = (1,2,4,3,2,1,2)$, $\alpha_{62} = (1,3,4,3,2,1,2)$, $\alpha_{63} = (2,3,4,3,2,1,2)$.

Let X_k be the Λ-module corresponding to the positive root α_k $(1 \le k \le 63)$. If we define Y_i similarly to the case $\mathbb{E}_6$, we have $Y_1 = X_{32}, Y_2 = X_{16}, Y_3 = X_{20}, Y_4 = X_5, Y_5 = X_7, Y_6 = X_2, Y_7 = X_1, Y_8 = X_{11}, Y_9 = X_{36}, Y_{10} = X_{51}, Y_{11} = X_{24}, Y_{12} = X_{22}, Y_{13} = X_6, Y_{14} = X_{19}, Y_{15} = X_{23}, Y_{16} = X_{29}, Y_{17} = X_{60}, Y_{18} = X_{50}, Y_{19} = X_{47}, Y_{20} = X_{18}, Y_{21} = X_{40}, Y_{22} = X_{38}, Y_{23} = X_{52}, Y_{24} = X_{62}, Y_{25} = X_{56}, Y_{26} = X_{45}, Y_{27} = X_{35}, Y_{28} = X_{28}, Y_{29} = X_{25}, Y_{30} = X_{54}, Y_{31} = X_{63}, Y_{32} = X_{59}, Y_{33} = X_{30}, Y_{34} = X_{15}, Y_{35} = X_{49}, Y_{36} = X_{39}, Y_{37} = X_{55}, Y_{38} = X_{61}, Y_{39} = X_{58}, Y_{40} = X_{48}, Y_{41} = X_{21}, Y_{42} = X_{44}, Y_{43} = X_{13}, Y_{44} = X_{46}, Y_{45} = X_{57}, Y_{46} = X_{53}, Y_{47} = X_{41}, Y_{48} = X_{34}, Y_{49} = X_{27}, Y_{50} = X_{17}, Y_{51} = X_{26}, Y_{52} = X_{42}, Y_{53} = X_{43}, Y_{54} = X_{14}, Y_{55} = X_{10}, Y_{56} = X_{37}, Y_{57} = X_{31}, Y_{58} = X_{33}, Y_{59} = X_8, Y_{60} = X_{12}, Y_{61} = X_3, Y_{62} = X_4, Y_{63} = X_9$.

In the same way as §5, we can calculate the table of $\dim_K \operatorname{Ext}^1_\Lambda(Y_i, Y_j)$ as follows.

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}^tA_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}^tA_2 & {}^tA_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 & 0 & 0 & 0 \\
{}^tA_5 & {}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 & 0 & 0 \\
{}^tA_6 & {}^tA_5 & {}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 & 0 \\
{}^tA_7 & {}^tA_6 & {}^tA_5 & {}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 \\
{}^tA_8 & {}^tA_7 & {}^tA_6 & {}^tA_5 & {}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0
\end{pmatrix} \tag{6.1}$$

and each 7×7 matrices $A_1, \ldots, A_8$ is given by

$$A_1 = \begin{pmatrix} 1&0&0&0&0&0&0\\ 1&1&1&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&1&1&1&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&1&1&0\\ 0&0&1&0&0&0&1 \end{pmatrix}, A_2 = \begin{pmatrix} 0&1&1&0&0&0&0\\ 0&1&2&1&1&0&1\\ 1&1&2&1&1&0&1\\ 1&1&2&1&1&1&1\\ 0&0&1&1&1&1&0\\ 0&0&1&1&0&0&0\\ 1&1&1&1&1&0&0 \end{pmatrix}, A_3 = \begin{pmatrix} 0&0&1&1&1&0&1\\ 1&1&2&2&2&1&1\\ 1&2&3&2&2&1&1\\ 1&2&3&2&1&0&1\\ 1&1&2&1&0&0&1\\ 1&1&1&0&0&0&1\\ 0&1&2&1&1&1&1 \end{pmatrix}, A_4 = \begin{pmatrix} 1&1&1&1&1&1&0\\ 1&2&3&2&1&1&1\\ 1&2&4&3&2&1&2\\ 1&2&3&2&2&1&2\\ 1&2&2&1&1&0&1\\ 0&1&1&1&1&0&0\\ 1&1&2&2&1&0&1 \end{pmatrix},$$

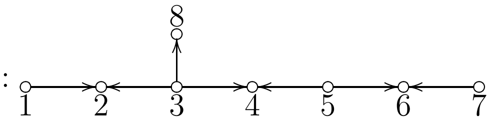
$$A_5 = \begin{pmatrix} 0&1&2&1&0&0&1\\ 1&2&3&2&1&0&2\\ 2&3&4&3&2&1&2\\ 1&2&3&3&2&1&1\\ 0&1&2&2&2&1&1\\ 0&0&1&1&1&1&1\\ 1&2&2&1&1&1&1 \end{pmatrix}, A_6 = \begin{pmatrix} 1&1&1&1&1&0&1\\ 1&2&2&2&2&1&1\\ 1&3&4&3&2&1&2\\ 1&2&3&2&1&1&2\\ 1&1&2&2&1&1&1\\ 1&1&1&1&0&0&0\\ 0&1&2&2&1&0&1 \end{pmatrix}, A_7 = \begin{pmatrix} 0&1&1&1&1&1&0\\ 0&1&2&2&1&1&1\\ 1&2&3&3&2&1&2\\ 1&2&2&2&1&0&1\\ 1&2&2&1&0&0&1\\ 0&1&1&0&0&0&1\\ 1&1&1&1&1&1&1 \end{pmatrix}, A_8 = \begin{pmatrix} 0&0&1&1&0&0&1\\ 1&1&1&1&0&0&1\\ 1&2&2&2&1&1&1\\ 0&1&1&1&1&1&1\\ 0&1&1&1&1&0&1\\ 0&0&0&1&1&0&0\\ 0&1&1&1&0&0&0 \end{pmatrix}.$$

Theorem 6.1. (The number of the isomorphism classes of the basic partial tilting $\Lambda(\mathfrak{M}, \Omega)$-modules of type $\mathbb{E}_7$)

There exist 777 (resp. 3927, 9933, 13299, 9009, 2431) isomorphism classes of basic parial tilting modules which are direct sums of 2 (resp. 3,4,5,6,7) indecomposable modules.

7 The Case for exceptional type $\mathbb{E}_8$

In this section, we consider the valued graph $(\Gamma, \boldsymbol{v})$ with $\Gamma = \{1,2,3,4,5,6,7\}$

and $\boldsymbol{v} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$ with an orientation Ω :  ,

i.e., of type $\mathbb{E}_8$. There exist 120 positive roots $\alpha_1, \ldots, \alpha_{120}$ given by
$\alpha_1 = (0,0,0,0,0,0,0,1)$, $\alpha_2 = (0,0,0,0,0,0,1,0)$, $\alpha_3 = (0,0,0,0,0,1,0,0)$,
$\alpha_4 = (0,0,0,0,0,1,1,0)$, $\alpha_5 = (0,0,0,0,1,0,0,0)$, $\alpha_6 = (0,0,0,0,1,1,0,0)$,
$\alpha_7 = (0,0,0,0,1,1,1,0)$, $\alpha_8 = (0,0,0,1,0,0,0,0)$, $\alpha_9 = (0,0,0,1,1,0,0,0)$,
$\alpha_{10} = (0,0,0,1,1,1,0,0)$, $\alpha_{11} = (0,0,0,1,1,1,1,0)$, $\alpha_{12} = (0,0,1,0,0,0,0,0)$,
$\alpha_{13} = (0,0,1,0,0,0,0,1)$, $\alpha_{14} = (0,0,1,1,0,0,0,0)$, $\alpha_{15} = (0,0,1,1,0,0,0,1)$,
$\alpha_{16} = (0,0,1,1,1,0,0,0)$, $\alpha_{17} = (0,0,1,1,1,0,0,1)$, $\alpha_{18} = (0,0,1,1,1,1,0,0)$,
$\alpha_{19} = (0,0,1,1,1,1,0,1)$, $\alpha_{20} = (0,0,1,1,1,1,1,0)$, $\alpha_{21} = (0,0,1,1,1,1,1,1)$,
$\alpha_{22} = (0,1,0,0,0,0,0,0)$, $\alpha_{23} = (0,1,1,0,0,0,0,0)$, $\alpha_{24} = (0,1,1,0,0,0,0,1)$,
$\alpha_{25} = (0,1,1,1,0,0,0,0)$, $\alpha_{26} = (0,1,1,1,0,0,0,1)$, $\alpha_{27} = (0,1,1,1,1,0,0,0)$,
$\alpha_{28} = (0,1,1,1,1,0,0,1)$, $\alpha_{29} = (0,1,1,1,1,1,0,0)$, $\alpha_{30} = (0,1,1,1,1,1,0,1)$,
$\alpha_{31} = (0,1,1,1,1,1,1,0)$, $\alpha_{32} = (0,1,1,1,1,1,1,1)$, $\alpha_{33} = (0,1,2,1,0,0,0,1)$,
$\alpha_{34} = (0,1,2,1,1,0,0,1)$, $\alpha_{35} = (0,1,2,1,1,1,0,1)$, $\alpha_{36} = (0,1,2,1,1,1,1,1)$,
$\alpha_{37} = (0,1,2,2,1,0,0,1)$, $\alpha_{38} = (0,1,2,2,1,1,0,1)$, $\alpha_{39} = (0,1,2,2,1,1,1,1)$,
$\alpha_{40} = (0,1,2,2,2,1,0,1)$, $\alpha_{41} = (0,1,2,2,2,1,1,1)$, $\alpha_{42} = (0,1,2,2,2,2,1,1)$,
$\alpha_{43} = (1,0,0,0,0,0,0,0)$, $\alpha_{44} = (1,1,0,0,0,0,0,0)$, $\alpha_{45} = (1,1,1,0,0,0,0,0)$,
$\alpha_{46} = (1,1,1,0,0,0,0,1)$, $\alpha_{47} = (1,1,1,1,0,0,0,0)$, $\alpha_{48} = (1,1,1,1,0,0,0,1)$,
$\alpha_{49} = (1,1,1,1,1,0,0,0)$, $\alpha_{50} = (1,1,1,1,1,0,0,1)$, $\alpha_{51} = (1,1,1,1,1,1,0,0)$,
$\alpha_{52} = (1,1,1,1,1,1,0,1)$, $\alpha_{53} = (1,1,1,1,1,1,1,0)$, $\alpha_{54} = (1,1,1,1,1,1,1,1)$,
$\alpha_{55} = (1,1,2,1,0,0,0,1)$, $\alpha_{56} = (1,1,2,1,1,0,0,1)$, $\alpha_{57} = (1,1,2,1,1,1,0,1)$,
$\alpha_{58} = (1,1,2,1,1,1,1,1)$, $\alpha_{59} = (1,1,2,2,1,0,0,1)$, $\alpha_{60} = (1,1,2,2,1,1,0,1)$,
$\alpha_{61} = (1,1,2,2,1,1,1,1)$, $\alpha_{62} = (1,1,2,2,2,1,0,1)$, $\alpha_{63} = (1,1,2,2,2,1,1,1)$,
$\alpha_{64} = (1,1,2,2,2,2,1,1)$, $\alpha_{65} = (1,2,2,1,0,0,0,1)$, $\alpha_{66} = (1,2,2,1,1,0,0,1)$,
$\alpha_{67} = (1,2,2,1,1,1,0,1)$, $\alpha_{68} = (1,2,2,1,1,1,1,1)$, $\alpha_{69} = (1,2,2,2,1,0,0,1)$,
$\alpha_{70} = (1,2,2,2,1,1,0,1)$, $\alpha_{71} = (1,2,2,2,1,1,1,1)$, $\alpha_{72} = (1,2,2,2,2,1,0,1)$,
$\alpha_{73} = (1,2,2,2,2,1,1,1)$, $\alpha_{74} = (1,2,2,2,2,2,1,1)$, $\alpha_{75} = (1,2,3,2,1,0,0,1)$,
$\alpha_{76} = (1,2,3,2,1,0,0,2)$, $\alpha_{77} = (1,2,3,2,1,1,0,1)$, $\alpha_{78} = (1,2,3,2,1,1,0,2)$,
$\alpha_{79} = (1,2,3,2,1,1,1,1)$, $\alpha_{80} = (1,2,3,2,1,1,1,2)$, $\alpha_{81} = (1,2,3,2,2,1,0,1)$,
$\alpha_{82} = (1,2,3,2,2,1,0,2)$, $\alpha_{83} = (1,2,3,2,2,1,1,1)$, $\alpha_{84} = (1,2,3,2,2,1,1,2)$,
$\alpha_{85} = (1,2,3,2,2,2,1,1)$, $\alpha_{86} = (1,2,3,2,2,2,1,2)$, $\alpha_{87} = (1,2,3,3,2,1,0,1)$,
$\alpha_{88} = (1,2,3,3,2,1,0,2)$, $\alpha_{89} = (1,2,3,3,2,1,1,1)$, $\alpha_{90} = (1,2,3,3,2,1,1,2)$
$\alpha_{91} = (1,2,3,3,2,2,1,1)$, $\alpha_{92} = (1,2,3,3,2,2,1,2)$, $\alpha_{93} = (1,2,3,3,3,2,1,1)$,

$\alpha_{94} = (1,2,3,3,3,2,1,2)$, $\alpha_{95} = (1,2,4,3,2,1,0,2)$, $\alpha_{96} = (1,2,4,3,2,1,1,2)$, $\alpha_{97} = (1,2,4,3,2,2,1,2)$, $\alpha_{98} = (1,2,4,3,3,2,1,2)$, $\alpha_{99} = (1,2,4,4,3,2,1,2)$, $\alpha_{100} = (1,3,4,3,2,1,0,2)$, $\alpha_{101} = (1,3,4,3,2,1,1,2)$, $\alpha_{102} = (1,3,4,3,2,2,1,2)$, $\alpha_{103} = (1,3,4,3,3,2,1,2)$, $\alpha_{104} = (1,3,4,4,3,2,1,2)$, $\alpha_{105} = (1,3,5,4,3,2,1,2)$, $\alpha_{106} = (1,3,5,4,3,2,1,3)$, $\alpha_{107} = (2,3,4,3,2,1,0,2)$, $\alpha_{108} = (2,3,4,3,2,1,1,2)$, $\alpha_{109} = (2,3,4,3,2,2,1,2)$, $\alpha_{110} = (2,3,4,3,3,2,1,2)$, $\alpha_{111} = (2,3,4,4,3,2,1,2)$, $\alpha_{112} = (2,3,5,4,3,2,1,2)$, $\alpha_{113} = (2,3,5,4,3,2,1,3)$, $\alpha_{114} = (2,4,5,4,3,2,1,2)$, $\alpha_{115} = (2,4,5,4,3,2,1,3)$, $\alpha_{116} = (2,4,6,4,3,2,1,3)$, $\alpha_{117} = (2,4,6,5,3,2,1,3)$, $\alpha_{118} = (2,4,6,5,4,2,1,3)$, $\alpha_{119} = (2,4,6,5,4,3,1,3)$, $\alpha_{120} = (2,4,6,5,4,3,2,3)$.

Let X_k be the Λ-module corresponding to the positive root α_k ($1 \le k \le 120$). If we define Y_i similarly to the case of $\mathbb{E}_6$, we have $Y_1 = X_{44}, Y_2 = X_{22}, Y_3 = X_{26}, Y_4 = X_8, Y_5 = X_{10}, Y_6 = X_3, Y_7 = X_4, Y_8 = X_1, Y_9 = X_{15}, Y_{10} = X_{48}, Y_{11} = X_{70}, Y_{12} = X_{30}, Y_{13} = X_{32}, Y_{14} = X_{11}, Y_{15} = X_9, Y_{16} = X_{25}, Y_{17} = X_{29}, Y_{18} = X_{38}, Y_{19} = X_{92}, Y_{20} = X_{71}, Y_{21} = X_{69}, Y_{22} = X_{28}, Y_{23} = X_{24}, Y_{24} = X_{52}, Y_{25} = X_{54}, Y_{26} = X_{74}, Y_{27} = X_{104}, Y_{28} = X_{88}, Y_{29} = X_{78}, Y_{30} = X_{65}, Y_{31} = X_{47}, Y_{32} = X_{39}, Y_{33} = X_{37}, Y_{34} = X_{90}, Y_{35} = X_{115}, Y_{36} = X_{102}, Y_{37} = X_{91}, Y_{38} = X_{60}, Y_{39} = X_{19}, Y_{40} = X_{72}, Y_{41} = X_{67}, Y_{42} = X_{100}, Y_{43} = X_{117}, Y_{44} = X_{111}, Y_{45} = X_{94}, Y_{46} = X_{42}, Y_{47} = X_{31}, Y_{48} = X_{80}, Y_{49} = X_{61}, Y_{50} = X_{109}, Y_{51} = X_{119}, Y_{52} = X_{106}, Y_{53} = X_{101}, Y_{54} = X_{73}, Y_{55} = X_{50}, Y_{56} = X_{87}, Y_{57} = X_{40}, Y_{58} = X_{99}, Y_{59} = X_{120}, Y_{60} = X_{114}, Y_{61} = X_{107}, Y_{62} = X_{76}, Y_{63} = X_{33}, Y_{64} = X_{86}, Y_{65} = X_{68}, Y_{66} = X_{103}, Y_{67} = X_{118}, Y_{68} = X_{113}, Y_{69} = X_{97}, Y_{70} = X_{77}, Y_{71} = X_{51}, Y_{72} = X_{89}, Y_{73} = X_{59}, Y_{74} = X_{108}, Y_{75} = X_{116}, Y_{76} = X_{105}, Y_{77} = X_{93}, Y_{78} = X_{64}, Y_{79} = X_{21}, Y_{80} = X_{82}, Y_{81} = X_{35}, Y_{82} = X_{95}, Y_{83} = X_{112}, Y_{84} = X_{110}, Y_{85} = X_{84}, Y_{86} = X_{41}, Y_{87} = X_{27}, Y_{88} = X_{79}, Y_{89} = X_{53}, Y_{90} = X_{85}, Y_{91} = X_{98}, Y_{92} = X_{96}, Y_{93} = X_{75}, Y_{94} = X_{66}, Y_{95} = X_{46}, Y_{96} = X_{62}, Y_{97} = X_{17}, Y_{98} = X_{63}, Y_{99} = X_{83}, Y_{100} = X_{81}, Y_{101} = X_{57}, Y_{102} = X_{55}, Y_{103} = X_{14}, Y_{104} = X_{36}, Y_{105} = X_{23}, Y_{106} = X_{34}, Y_{107} = X_{56}, Y_{108} = X_{58}, Y_{109} = X_{20}, Y_{110} = X_{18}, Y_{111} = X_6, Y_{112} = X_{49}, Y_{113} = X_{43}, Y_{114} = X_{45}, Y_{115} = X_{12}, Y_{116} = X_{16}, Y_{117} = X_5, Y_{118} = X_7, Y_{119} = X_2, Y_{120} = X_{13}$.

In the same way as §5, we can calculate the matrix of $\dim_K \operatorname{Ext}^1_\Lambda(Y_i, Y_j)$ as follows.

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}^tA_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}^tA_2 & {}^tA_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}^tA_5 & {}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}^tA_6 & {}^tA_5 & {}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}^tA_7 & {}^tA_6 & {}^tA_5 & {}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}^tA_8 & {}^tA_7 & {}^tA_6 & {}^tA_5 & {}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}^tA_9 & {}^tA_8 & {}^tA_7 & {}^tA_6 & {}^tA_5 & {}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}^tA_{10} & {}^tA_9 & {}^tA_8 & {}^tA_7 & {}^tA_6 & {}^tA_5 & {}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 & 0 & 0 & 0 \\
{}^tA_{11} & {}^tA_{10} & {}^tA_9 & {}^tA_8 & {}^tA_7 & {}^tA_6 & {}^tA_5 & {}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 & 0 & 0 \\
{}^tA_{12} & {}^tA_{11} & {}^tA_{10} & {}^tA_9 & {}^tA_8 & {}^tA_7 & {}^tA_6 & {}^tA_5 & {}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 & 0 \\
{}^tA_{13} & {}^tA_{12} & {}^tA_{11} & {}^tA_{10} & {}^tA_9 & {}^tA_8 & {}^tA_7 & {}^tA_6 & {}^tA_5 & {}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0 & 0 \\
{}^tA_{14} & {}^tA_{13} & {}^tA_{12} & {}^tA_{11} & {}^tA_{10} & {}^tA_9 & {}^tA_8 & {}^tA_7 & {}^tA_6 & {}^tA_5 & {}^tA_4 & {}^tA_3 & {}^tA_2 & {}^tA_1 & 0
\end{pmatrix}
\tag{7.1}
$$

and each 8×8 matrices $A_1, \ldots, A_{14}$ is given by

$$
A_1 = \begin{pmatrix}
1&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&1&1&1&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&1&1&1&0\\
0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&1
\end{pmatrix},\
A_2 = \begin{pmatrix}
0&1&1&0&0&0&0&0\\
0&1&2&1&1&0&0&1\\
1&1&2&1&1&0&0&1\\
1&1&2&1&1&1&1&1\\
0&0&1&1&1&1&1&0\\
0&0&1&1&1&1&0&0\\
0&0&0&0&1&1&0&0\\
1&1&1&1&1&0&0&0
\end{pmatrix},\
A_3 = \begin{pmatrix}
0&0&1&1&1&0&0&1\\
1&1&2&2&2&1&1&1\\
1&2&3&2&2&1&1&1\\
1&2&3&2&2&1&0&1\\
1&1&2&1&1&1&0&1\\
1&1&2&1&0&0&0&1\\
0&0&1&1&0&0&0&0\\
0&1&2&1&1&1&1&1
\end{pmatrix},
$$

$$
A_4 = \begin{pmatrix}
1&1&1&1&1&1&1&0\\
1&2&3&2&2&2&1&1\\
1&2&4&3&3&2&1&2\\
1&2&4&3&2&1&1&2\\
1&2&3&2&1&0&0&1\\
1&2&2&1&1&0&0&1\\
1&1&1&0&0&0&0&1\\
1&1&2&2&2&1&0&1
\end{pmatrix},\
A_5 = \begin{pmatrix}
0&1&2&1&1&1&0&1\\
1&2&4&3&2&1&0&2\\
2&3&5&4&3&2&1&2\\
2&3&4&3&3&2&1&2\\
1&2&3&2&2&1&1&2\\
0&1&2&2&2&1&1&1\\
0&1&1&1&1&0&0&0\\
1&2&3&2&1&1&1&1
\end{pmatrix},\
A_6 = \begin{pmatrix}
1&1&2&2&1&0&0&1\\
2&3&4&3&2&1&1&2\\
2&4&6&4&3&2&1&3\\
1&3&5&4&3&2&1&2\\
1&2&3&3&3&2&1&1\\
1&1&2&2&2&2&1&1\\
0&0&1&1&1&1&1&1\\
1&2&3&2&2&1&0&2
\end{pmatrix},
$$

$$
A_7 = \begin{pmatrix}
1&2&2&1&1&1&1&1\\
1&3&4&3&3&2&1&2\\
2&4&6&5&4&2&1&3\\
2&3&5&4&3&2&1&3\\
1&2&4&3&2&2&1&2\\
1&2&3&2&1&1&0&1\\
1&1&1&1&1&1&0&0\\
1&2&3&3&2&1&1&1
\end{pmatrix},\
A_8 = \begin{pmatrix}
0&1&2&2&2&1&0&1\\
1&2&4&4&3&2&1&2\\
2&4&6&5&4&3&2&3\\
2&4&5&4&3&2&1&2\\
2&3&4&3&2&1&0&2\\
1&2&3&2&1&0&0&2\\
0&1&2&1&0&0&0&1\\
1&2&3&2&2&2&1&2
\end{pmatrix},\
A_9 = \begin{pmatrix}
1&1&2&2&1&1&1&1\\
2&3&4&3&2&2&1&2\\
2&4&6&5&4&3&1&3\\
1&3&5&4&3&2&1&3\\
1&3&4&3&2&1&1&2\\
1&2&2&2&2&1&1&1\\
1&1&1&1&1&0&0&1\\
1&2&3&3&2&1&0&1
\end{pmatrix},
$$

$$
A_{10} = \begin{pmatrix}
1&2&2&1&1&1&0&1\\
1&3&4&3&2&1&0&2\\
2&4&6&5&3&2&1&3\\
2&3&4&4&3&2&1&2\\
1&2&3&3&3&2&1&2\\
0&1&2&2&2&2&1&1\\
0&1&1&1&1&1&1&0\\
1&2&3&2&1&1&1&2
\end{pmatrix},\
A_{11} = \begin{pmatrix}
0&1&2&2&1&0&0&1\\
1&2&3&3&2&1&1&2\\
2&4&5&4&3&2&1&3\\
1&3&4&3&2&2&1&2\\
1&2&3&3&2&2&1&1\\
1&1&2&2&1&1&0&1\\
0&0&1&1&1&1&0&1\\
1&2&2&2&2&1&0&1
\end{pmatrix},\
A_{12} = \begin{pmatrix}
1&1&1&1&1&1&1&1\\
1&2&2&2&2&2&1&1\\
1&3&4&4&3&2&1&2\\
1&2&3&3&2&1&0&2\\
1&2&3&2&1&1&0&2\\
1&2&2&1&0&0&0&1\\
1&1&1&1&0&0&0&0\\
0&1&2&2&1&1&1&1
\end{pmatrix},
$$

$$
A_{13} = \begin{pmatrix}
0&1&1&1&1&1&0&0\\
0&1&2&2&1&1&0&1\\
1&2&3&3&2&2&1&2\\
1&2&2&2&1&1&1&1\\
1&2&2&2&1&0&0&1\\
0&1&1&1&1&0&0&1\\
0&1&1&0&0&0&0&1\\
1&1&1&1&1&1&0&1
\end{pmatrix},\
A_{14} = \begin{pmatrix}
0&0&1&1&0&0&0&1\\
1&1&1&1&0&0&0&1\\
1&2&2&2&1&1&0&1\\
0&1&1&1&1&1&0&1\\
0&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&0\\
0&0&0&1&1&0&0&0\\
0&1&1&1&0&0&0&0
\end{pmatrix}.
$$

Theorem 7.1. (The number of the isomorphism classes of the basic partial tilting $\Lambda(\mathfrak{M}, \Omega)$-modules of type $\mathbb{E}_8$)

There exist 2135 (resp. 15120, 54327, 108360, 121555, 71760, 17342) isomorphism classes of basic parial tilting modules which are direct sums of 2 (resp. 3,4,5,6,7,8) indecomposable modules.

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