The Characterization of Congruences on Additive Inverse Semirings

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Abstract

In this paper, we provide a set of independent axioms characterizing the kernel of a congruence on an additive inverse semiring and show how to reconstruct the congruence from its kernel. Next we show that the mapping \( \rho \rightarrow tr\rho \) is a complete homomorphism of the congruence lattice of \( S \) onto the lattice of normal congruences on \( E^+(S) \), and that the congruence induced by it has all its classes complete modular lattices.

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1 Introduction

A semiring \( S \) is defined as an algebra system \((S, +, \cdot)\), such that \((S, +)\) and \((S, \cdot)\) are semigroups connected by \(a(b + c) = ab + ac\) and \((b + c)a = ba + ca\) for all \(a, b, c \in S\). A semiring \( S \) is a skew ring if its additive reduct \((S, +)\) is a group. A semiring \( S \) is an additive inverse semiring if \((S, +)\) is an inverse semigroup.

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A nonempty subset $A$ of a semiring $S$ is called an ideal of $S$ if $a + b \in A$, $sa, as \in A$ for all $a, b \in A, s \in S$. If $S$ is an inverse semiring, we denote $-a$ the unique additive inverse of $a$, $E^+(S)$ the set of all additive idempotents. Easily, we can prove if $S$ is an additive inverse semiring, then $E^+(S)$ is an ideal of $S$.

Let $S$ be an additive inverse semiring, $\sigma$ a congruence on $E^+(S)$. $\sigma$ is normal, in the sense that

(i) $e\sigma f \Rightarrow (\forall a \in S) (-a + e + a)\sigma(-a + f + a);
(ii) e\sigma f \Rightarrow (\forall c \in S) ecfc$ and $cefc\sigma f$.

Since the intersection of a non-empty family of congruences on a general semiring $S$ is a congruence on $S$, we can induce that for any relation $R$ on $S$ there is an unique smallest congruence $R^\#$ containing $R$. We now give a result analogous to semigroup which will give us a usable description of $R^\#$.

Let $S$ be a semiring. Define a semiring $S^0$ as follows: if $(S, +)$ has a zero element, then let $S^0 = S$; otherwise, define $S^0 = S \cup \{0\}$ to be a new semiring by compensating the operations of $S$ as follows:

$$(\forall x \in S^0)x + 0 = 0 + x = x, x0 = 0x = 0.$$ 

For any relation $R$ on $S$ we denote

$$R^c = \{(u + xay + v, u + xby + v)|u, v, x, y \in S^0, (a, b) \in R\}$$

$$\cup\{(u + a + v, u + b + v)|u, v \in S^0, (a, b) \in R\}$$

$$\cup\{(u + xa + v, u + xb + v)|u, v, x \in S^0, (a, b) \in R\}$$

$$\cup\{(u + ay + v, u + by + v)|u, v, y \in S^0, (a, b) \in R\}.$$ 

If $R$ is a relation on a set $X$, then the minimum equivalence containing $R$ is

$$R^e = [R \cup R^{-1} \cup 1_X]^\infty = \bigcup_{n=1}^\infty [R \cup R^{-1} \cup 1_X]^n.$$ 

If $R$ is a relation on a semiring $S$, then $R^# = (R^e)^e$. So if $\{\rho_i\}$ is a family of congruences on a semiring $S$, then $\forall i\rho_i = (\cup_i\rho_i)^\infty$. In fact, $\forall i\rho_i = ((\cup_i\rho_i)^c \cup ((\cup_i\rho_i)^c)^{-1} \cup 1_S)^\infty = ((\cup_i\rho_i^c) \cup (\cup_i\rho_i)^{-1} \cup 1_S)^\infty = (\cup_i\rho_i)^\infty$.

Semirings with special additive reduct are studied [3], [8], [6]. The congruences and the congruence pairs on an inverse semigroup[5],[2] and orthorings[7] are characterized respectively. Other definitions and results about semigroups and semirings are referred to [2],[4],[5] and [1]. In this paper, we only considered the additive inverse semirings and we will generalize some results about congruences, congruence pairs on semirings and the lattice of congruences and give some characterizations and structure of additive inverse semirings.
2 Characterizations and Structure

If \( \rho \) is a congruence on an additive inverse semiring \( S \), let \( \mathcal{K}(\rho) = \{ e \rho | e \in E^+(S) \} \). Such collections of subsets of \( S \) can be characterized abstractly, and we thus have the following concept.

**Definition 2.1.** Let \( \mathcal{K} \) be a family of pairwise disjoint additive inverse subsemirings of an additive inverse semiring \( S \). Then \( \mathcal{K} \) is a kernel normal system for \( S \), if \( \mathcal{K} \) satisfies

1. \( E^+(S) \subseteq \bigcup_{K \in \mathcal{K}} K \).
2. For each \( a \in S \) and \( K \in \mathcal{K} \), there exists \( L \in \mathcal{K} \) such that \( (-a) + K + a \subseteq L \).
3. If \( a, a + b, b + (-b) \in K \), then \( b \in K \).
4. If \( a \in L \), \( b \in T \), \( L, T \in \mathcal{K} \), there exists \( U \in \mathcal{K} \) such that \( ab \in U \).
5. For \( e, f \in E^+(S) \), if \( e, f \in L, L \in \mathcal{K} \), there exist \( K_1, K_2 \in \mathcal{K} \) such that \( ec, fc \in K_1, ce, cf \in K_2 \) for each \( c \in S \).

For such a family \( \mathcal{K} \), we define a relation \( \xi_{\mathcal{K}} \) on \( S \) by

\[
a \xi_{\mathcal{K}} b \iff a + (-a), b + (-b), a + (-b) \in K \text{ for some } K \in \mathcal{K}.
\]

With this notation and definition, we have one new characterization of congruences on additive inverse semiring as follows.

**Theorem 2.2.** Let \( S \) be an additive inverse semiring. If \( \mathcal{K} \) is a kernel normal system for \( S \), then \( \xi_{\mathcal{K}} \) is the unique congruence \( \xi \) on \( S \) for which \( \mathcal{K}(\xi) = \mathcal{K} \). Conversely, if \( \xi \) is a congruence on \( S \), then \( \mathcal{K}(\xi) \) is a kernel normal system for \( S \) and \( \xi_{\mathcal{K}}(\xi) = \xi \).

**Proof.** First, we shall prove the direct part of the theorem. Let \( \mathcal{K} \) be a kernel normal system for \( S \), let \( K = \bigcup_{L \in \mathcal{K}} L \) and define \( \tau \) on \( E^+(S) \) by

\[
e \tau f \iff e, f \in L \text{ for some } L \text{ in } \mathcal{K}.
\]

We now verify that \( (K, \tau) \) is a congruence pair for \( S \) and that \( \xi_{\mathcal{K}} = \rho_{(K, \tau)} \). Since the members of \( \mathcal{K} \) are pairwise disjoint and 2.1(i) holds, we obtain that \( \tau \) is an equivalent relation. Condition 2.1(ii) and (v) implies that \( \tau \) is a normal congruence. We now show that \( K \) is closed under multiplication. Hence let \( a \in L \) and \( b \in T \) for \( L, T \in \mathcal{K} \). Let \( e = (-a) + a \) and \( f = b + (-b) \). Since \( \tau \) is a normal congruence, there exists \( U \in \mathcal{K} \) such that \( E^+_L + E^+_L \subseteq U_{\tau} \). But then \( e + E^+_T + e \subseteq E^+_U \), which by 2.1(ii) implies that \( e + T + e \subseteq U \). Now \( e + b + e \in e + T + e \subseteq U \), which implies that \( (e + b + e) + (e + (-b) + e) \in U \). Also,

\[
(e + b + e) + (e + (-b) + e) + (e + b) = e + b + e + (-b) + e + b
\]

\[
= e + (b + e + (-b)) + e + b = e + b + e + (-b) + b
\]

\[
= e + b + ((-b) + b) + e = e + b + e \in U,
\]
and \((e + b) + ((-b) + e) = e + b + (-b) \in E^+_L + E^+_T \subseteq U\). Now applying 2.1(iii), we get \(e + b \in U\). Since \(E^+_T + E^+_L = E^+_L + E^+_T \subseteq E^+_U\) and \(-a \in L\), for \(L\) is an additive inverse subsemiring of \(S\), we can similarly prove that \(f + (-a) \in U\). Thus \(a + f = -(f + (-a)) \in U\) and we obtain

\[
a + b = (a + e) + (f + b) = (a + f) + (e + b) \in U + U \subseteq U.
\]

and applying 2.1(iv), we get \(ab \in V\) for some \(V \in K\). So \(K\) is a subsemiring of \(S\). Since each \(L\) in \(K\) is closed under taking of additive inverses, so is \(K\), and is thus an additive subsemiring of \(S\). Condition 2.1(i) and (ii) ensure that \(K\) is full and self-conjugate. Hence \(K\) is a normal subsemiring of \(S\). We now verify \((K, \tau)\) is a congruence pair next. Hence let \(a \in S, e \in E^+(S), a + e \in K\) and \(e \tau(-a) + a\). Then \(a + e \in L\) and \(e, (-a) + a \in T\) for some \(L, T \in K\). It follows that \(e + (-a) = -(a + e) \in L\) and thus \(e + (-a) + a + e = e + ((-a) + a) + e \in L \cap T\), so that \(L = T\). Hence \(e, e + (-a), (-a) + a \in L\), which by 2.1(iii) gives \(-a \in L\), so that also \(a \in L\), then \(a \in K\), as required. If \(a \in K\), then \(a \in L\) for some \(L \in K\), and thus \(a + (-a), (-a) + a \in L\) and hence \(a + (-a) \tau(-a) + a\). Consequently, \((K, \tau)\) is a congruence pair.

We now check that \(\rho_{(K, \tau)} = \xi_K\). First let \(a \rho_{(K, \tau)} b\), so that \((-a) + a \tau(-b) + b\) and \(a + (-b) \in K\). Hence \((-a) + a, (-b) + b \in L\) and \(a + (-b) \in T\) for some \(L, T \in K\). By 2.1(ii), we have \(a + L + (-a) \subseteq L_1\) and \(b + L + (-b) \subseteq L_2\) for some \(L_1, L_2 \in K\). Now

\[
a + (-b) + b + (-a) = a + ((-b) + b) + (-a) \in T \cap L_1
\]

so that \(T = L_1\); also

\[
b + (-a) + a + (-b) = b + ((-a) + a) + (-b) \in T \cap L_2
\]

and hence \(T = L_2\). Since also \(a + (-a) = a + ((-a) + a) + (-a) \in L_1\) and similarly \(b + (-b) \in L_2\), we deduce that \(a + (-a), b + (-b), a + (-b) \in T\) and thus \(a \xi_K b\). Conversely, let \(a \xi_K b\). Then \(a + (-a), b + (-b), a + (-b) \in L\) for some \(L \in K\). We have \((-a) + L + a \subseteq L_1\) and \((-b) + L + b \subseteq L_2\) for some \(L_1, L_2 \in K\). Now

\[
(-a) + a + (-b) + b = (-a) + [a + (-b) + b + (-a)] + a = (-b) + [b + (-a) + (a + (-b))] + b \in L_1 \cap L_2
\]

since \(a + (-b), b + (-a) \in L\). It follows that \(L_1 = L_2\). Since \((-a) + a = (-a) + (a + (-a)) + a \in L_1\) and similarly \((-b) + b \in L_2\), we deduce that \((-a) + a, (-b) + b \in L_1 = L_2\). We thus have \((-a) + a \tau(-b) + b\) and \(a + (-b) \in K\) which gives \(a \rho_{(K, \tau)} b\).

We verify next that \(K(\xi_K) = K\). First let \(L \in K\). Then \(L\) is an additive inverse subsemiring of \(S\). In order to show that \(L \in K(\xi_K)\), it suffices to show that \(L\) is a \(\xi_K\)-class. If \(a, b \in L\), then \(a + (-a), b + (-b), a + (-b) \in L\) and thus \(a \xi_K b\). Let \(a \in L\) and \(a \xi_K b\). Then \(a + (-a), b + (-b), a + (-b) \in T\) for some \(T \in K\). But \(a + (-a) \in L \cap T\), so that \(L = T\). Since \(-a \xi_K - b\), we also have \((-a) + a, (-b) + b, (-a) + b \in U\) for
some $U \in \mathcal{K}$, and as above, $U = L$. Thus, in particular, $(-a), (-a) + b, b + (-b) \in L$ and 2.1(iii) implies that $b \in L$. Consequently, $L$ is a $\xi_K$-class and thus $L \in \mathcal{K}(\xi_K)$. Conversely, let $L \in \mathcal{K}(\xi_K)$. Then $L$ contains an idempotent $e$, and thus $e \in T$ for some $T \in \mathcal{K}$ by 2.1(i). By the above, $T$ also is a $\xi_K$-class, and we must have $L = T$. Consequently, $L \in \mathcal{K}$, which completes the verification that $\mathcal{K}(\xi_K) = \mathcal{K}$.

If now $\rho$ is any congruence on $S$ for which $\mathcal{K}(\rho) = \mathcal{K}$, then we have ker $\rho = \ker \xi_K$ and $\text{tr} \rho = \text{tr} \xi_K$, and we get $\rho = \xi_K$. This establishes the uniqueness of $\xi_K$ and complete the proof of the direct part of the theorem.

Second, we shall prove the converse part. Indeed, let $\xi$ be a congruence on $S$. It is obvious that $\mathcal{K}(\xi)$ consists of a family of pairwise disjoint additive inverse subsemiring of $S$ whose union contains $E^+(S)$. Let $a \in S$ and let $K$ be a $\xi$-class containing an idempotent $e$. Then for any $K \in \mathcal{K}, (-a) + K + a\xi(-a) + e + a$ so that $(-a) + K + a \subseteq L$ where $L$ is the $\xi$-class containing the idempotent $(-a) + e + a$. This verifies 2.1(ii). With the same notation, assume $a, a + b, b + (-b) \in K$. Then

$$b = (b + (-b)) + b\xi e + b\xi a + b\xi e$$

so that $b \in K$. This verifies 2.1(iii). Since $\xi$ is a congruence, condition 2.1(ii) and (v) can be easily proved. Hence we complete the proof that $\mathcal{K}(\xi)$ is a kernel normal system. By part of the proof above, we have $\mathcal{K}(\xi_K(\xi)) = \mathcal{K}(\xi)$. Thus $\xi_K(\xi)$ and $\xi$ have the same kernel normal system, so by the uniqueness proved in the theorem, we obtain $\xi_K(\xi) = \xi$.

By the introduction about the description of $R^\#$, we can get:

**Proposition 2.3.** Let $\rho, \sigma$ be congruences on a semiring $S$. Then

$$\rho \lor \sigma = (\rho \lor \sigma)^\infty.$$  

For any additive inverse semiring $S$, let $\mathcal{C}(S)$ be the lattice of all congruences on $S$ and let $\mathfrak{N}(E^+(S))$ be the lattice of all normal congruences on $E^+(S)$.

**Corollary 2.4.** Let $S$ be an additive inverse semiring. Then the join in $\mathcal{C}(E^+(S))$ of two normal congruences on $E^+(S)$ is a normal congruence on $E^+(S)$.

**Proof.** Let $\rho, \sigma \in \mathcal{C}(E^+(S))$. Applying proposition 2.3, we get $\rho \lor \sigma = (\rho \lor \sigma)^\infty$. Let $e, f \in E^+(S)$, then for any $a \in S$ and $c \in S$,

$$e(\rho \lor \sigma)f \iff e(\rho \lor \sigma)a_1, a_1(\rho \lor \sigma)a_2, ..., a_n - 1(\rho \lor \sigma)f.$$  

$$\iff -a + e + a(\rho \lor \sigma) - a + a_1 + a, ..., -a + a_{n-1} + a(\rho \lor \sigma) - a + f + a$$

and $ec(\rho \lor \sigma)c_1, ..., a_{n-1}(\rho \lor \sigma)f, ce(\rho \lor \sigma)c_1, ..., ca_{n-1}(\rho \lor \sigma)f$.

$$\iff (-a) + e + a(\rho \lor \sigma)(-a) + f + a$$

and $ec(\rho \lor \sigma)f, ce(\rho \lor \sigma)f$.

Thus $\rho \lor \sigma$ is a normal congruence on $E^+(S)$. \hfill \square

**Remark 2.5.** Let $S$ be an additive inverse semiring. For any congruence $\rho$ on $S$, define a relation $\rho_{\min}$ on $S$ by

$$a\rho_{\min}b \iff a + e = b + e \text{ for some } e \in E^+(S), e\rho(-a) + a\rho(-b) + b.$$
Similarly to the proof in semigroup, we now give a result in semiring:

**Proposition 2.6.** Let \( \mathcal{K} \) be a sublattice of the lattice \((\mathcal{C}(S), \subseteq, \wedge, \vee)\) of congruences of a semiring \( S \), and suppose that \( \rho \circ \sigma = \sigma \circ \rho \) for all \( \rho, \sigma \) in \( \mathcal{K} \). Then \( \mathcal{K} \) is a modular lattice.

We are now ready for the result concerning the lattice of congruences in relation to their traces.

**Theorem 2.7.** Let \( S \) be an additive inverse semiring. Define a mapping \( tr \) by

\[
tr: \rho \rightarrow tr\rho \quad (\rho \in \mathcal{C}(S)).
\]

Then \( tr \) is a complete homomorphism of \( \mathcal{C}(S) \) onto \( \mathfrak{N}(E^+(S)) \). Let \( \theta \) be the congruence on \( \mathcal{C}(S) \) induced by \( tr \). Then for any \( \rho \in \mathcal{C}(S) \), \( \rho_{\min} \) is the least element of \( \rho\theta \) and \( \rho\theta \) is a complete modular sublattice of \( \mathcal{C}(S) \).

**Proof.** Let \( F \) be a nonempty family of congruences on \( S \). Then for any \( e, f \in E^+(S) \), we obtain

\[
etr(\bigwedge_{\rho \in F} \rho)f \Leftrightarrow (\bigwedge_{\rho \in F} \rho)f \Leftrightarrow ef \text{ for all } \rho \in F
\]

\[
\Leftrightarrow e tr\rho f \text{ for all } \rho \in F
\]

which proves \( tr(\bigwedge_{\rho \in F} \rho) = \bigwedge_{\rho \in F} tr\rho \). Using proposition 2.4, we get

\[
etr(\bigvee_{\rho \in F} \rho)f \Leftrightarrow (\bigvee_{\rho \in F} \rho)f
\]

\[
\Leftrightarrow e_1 x_1, x_1 x_2, ..., x_{n-1} x_n f \text{ for some } x_i \in S, \rho_i \in F
\]

\[
\Rightarrow e_1 x_1 + (-x_1), x_1 + (-x_1) x_2 + (-x_2), ..., x_{n-1} + (-x_{n-1}) x_n f
\]

\[
\Rightarrow e(\bigvee_{\rho \in F} tr\rho)f,
\]

which shows that \( tr(\bigvee_{\rho \in F} \rho) \subseteq \bigvee_{\rho \in F} tr\rho \). The converse inclusion follows easily:

\[
e\bigvee_{\rho \in F} tr\rho f \Leftrightarrow e tr\rho_1 e_1, e_1 tr\rho_2 e_2, ..., e_{n-1} tr\rho_n f
\]

\[
\Rightarrow e_1 e_1, e_1 e_2, ..., e_{n-1} e_n f
\]

\[
\Rightarrow e(\bigvee_{\rho \in F} \rho)f
\]

\[
\Rightarrow tr(\bigvee_{\rho \in F} \rho)f
\]

This proves \( tr(\bigvee_{\rho \in F} \rho) = \bigvee_{\rho \in F} tr\rho \). Consequently, \( tr \) is a complete homomorphism of \( \mathcal{C}(S) \) into \( \mathfrak{N}(E^+(S)) \).

Let \( \tau \in \mathfrak{N}(E^+(S)) \) and define \( \rho \) on \( S \) by

\[
apb \Leftrightarrow a + e = b + e \text{ for some } e \in E^+(S), e\tau(-a) + a\tau(-b) + b.
\]

It is obvious that \( \rho \) is an equivalence relation on \( S \). Let \( apb \) and \( c \in S \). Then

\[
-(c+a)+c+a = (-a)+((c)+c)+a = ((a)+a)+((-a)+((-a)+((c)+c)+a),
\]

\[
-(c+b)+c+b = (-b)+((c)+c)+b
\]
\[
-e + (\neg a) + ((\neg c) + c) + a + e = e + (\neg a) + ((\neg c) + c) + a,
\]
which shows that
\[
e + (\neg a) + ((\neg c) + c) + a + e = e + (\neg a) + ((\neg c) + c) + a \in E^+(S).
\]
Also
\[
c + a + e + (\neg a) + ((\neg c) + c) + a = c + b + e + (\neg a) + ((\neg c) + c) + a.
\]
So we obtain \( c + a \rho b + c. \) Further,
\[
-(a + c) + a + c = (\neg c) + ((\neg a) + a) + c\tau(-c) + e + c,
\]
\[
-(b + c) + b + c = (\neg c) + ((\neg b) + b) + c\tau(-c) + e + c,
\]
which shows that
\[
(\neg c) + e + c\tau - (a + c) + a + c\tau - (b + c) + b + c for (\neg c) + e + c \in E^+(S).
\]
Also
\[
a + c + (\neg c) + e + c = b + c + (\neg c) + e + c,
\]
and thus \( a + c\rho b + c. \) Also we have
\[
ac + ec = (a + e)c = (b + e)c = bc + ec for ec \in E^+(S), e\tau((\neg a) + a)\tau((-b) + b)c,
\]
\[
ca + ce = c(a + e) = c(b + e) = cb + ce for ce \in E^+(S), c\tau a\tau a\tau((-b) + b),
\]
where we have used normality of \( \tau. \) So that \( c\rho a\tau c, c\rho b. \) It follows that \( \rho \) is a congruence on \( S. \) Further, for any \( e, f \in E^+(S),
\]
\[
e\rho f \iff e + g\tau e + f for all \ g \in E^+(S) \iff e\tau f.
\]
so that \( tr\rho = \tau \) which also shows that \( tr \) maps \( C(S) \) onto \( \mathfrak{M}(E^+(S)). \)

It is easy to verify that \( \rho_{min} \) is an equivalent relation. Let \( a + e = b + e \) for some \( e \in E^+(S), e\rho(-a) + a\rho(-b) + b \) and \( c \in S. \) Then
\[
-(c + a) + c + a = (\neg a) + ((\neg c) + c) + a = ((\neg a) + a) + ((\neg c) + c) + a + e = e + (\neg a) + ((\neg c) + c) + a,
\]
\[
-(c + b) + c + b = (\neg b) + ((\neg c) + c) + b = ((\neg b) + b) + ((\neg c) + c) + b + ((\neg b) + b)
\]
\[
\rho e + (\neg b) + ((\neg c) + c) + b + e = e + (\neg a) + ((\neg c) + c) + a + e = e + (\neg a) + ((\neg c) + c) + a,
\]
which shows that
\[
e + (\neg a) + ((\neg c) + c) + a + e = e + (\neg a) + ((\neg c) + c) + a \in E^+(S).
\]
Also
\[ c + a + e + (-a) + ((-c) + c) + a = c + b + e + (-a) + ((-c) + c) + a. \]
So we obtain \( c + a\rho_{\text{min}} e + b. \) Further,
\[ -(a + c) + a + c = (-c) + ((-a) + a) + c\rho(-c) + e + c, \]
\[ -(b + c) + b + c = (-c) + ((-b) + b) + c\rho(-c) + e + c, \]
which shows that
\[ (-c) + e + c\rho - (a + c) + a + c\rho - (b + c) + b + c\rho e r f(-c) + e + c \in E^+(S). \]
Also
\[ a + c + (-c) + e + c = b + c + (-c) + e + c, \]
and thus \( a + c\rho_{\text{min}} b + c. \) Also we have
\[ ac + ec = (a + e)c = (b + e)c = bc + ec \text{ for } ec \in E^+(S), ec\rho((-a) + a)\rho((-b) + b)c, \]
\[ ca + ce = c(a + e) = c(b + e) = cb + ce \text{ for } ce \in E^+(S), ce\rho c((-a) + a)\rho c((-b) + b). \]
So that \( ca\rho c\rho cb, cb\rho c\rho ca. \) Consequently, \( \rho_{\text{min}} \) is a congruence on \( S. \) If \( a + c = b + e \) and \( e\rho(-a) + a\rho(-b) + b, \) then \( a\rho a + e = b + e\rho b. \) This proves that \( \rho_{\text{min}} \subseteq \rho. \)
If \( e, f \in E^+(S) \) and \( e\rho f, \) then \( e + (e + f) = f + (e + f) \) and \( e + f\rho e\rho f \) so that \( e\rho_{\text{min}} f. \) This shows that \( tr\rho = tr\rho_{\text{min}}, \) and since the definition of \( \rho_{\text{min}} \) depends only on idempotents, it follows that \( \rho_{\text{min}} \) is the least element of \( \rho \theta. \)

Next we show that \( \rho \theta \) is the complete sublattice of \( S. \) For any \( \rho_i \in \rho \theta, \) we have
\[ tr\rho_i = tr\rho. \] Then \( tr(\vee_{\rho_i \in \rho \theta})\rho_i = \vee_{\rho_i \in \rho \theta} tr\rho = tr\rho = tr\rho. \) Similarly, \( tr(\wedge_{\rho_i \in \rho \theta})\rho_i = tr\rho, \) hence \( \vee \rho_i, \wedge \rho_i \in \rho \theta. \) So that \( \rho \theta \) is the complete sublattice of \( S. \)

Now let \( \rho, \lambda \in \mathcal{C}(S) \) be such that \( \rho \theta = \lambda \theta, \) and let \( a\rho \lambda b. \) Then \( a\rho c \) and \( c\lambda b \) for some \( c \in S. \) Hence \( a + (-a)c + (c) \) and thus \( a + (-a)c + (c) \) since \( tr\rho = tr\lambda. \) Also \( c + (c)\lambda b + (c) \) and thus \( a + (-a)\lambda b + (c) \) which yields \( a\lambda b + (c) \) and \( a\lambda b. \) A similar argument shows that \( b\rho b + (c) \) and \( a\lambda b + (c) \) and \( a\lambda b \) and by symmetry, we deduce that \( \rho \lambda = \lambda \rho. \) An application of proposition 2.8 gives that \( \rho \theta \)

is a modular lattice. \( \square \)

References


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