On the Wreath Product of Group

$PSL(2,13) \wr PSL(2,11)$ by Some Other Groups

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Abstract

In this paper, we will generate the wreath product $PSL(2,13) \wr PSL(2,11)$ using only two permutations. We will show the structure of some groups containing the wreath product $PSL(2,13) \wr PSL(2,11)$. The structure of the group constructed is determined in terms of wreath product $(PSL(2,13) \wr PSL(2,11)) \wr C_k$. Some related cases are also included. Also, we will show that $S_{143k+1}$ and $A_{143k+1}$ can be generated using the wreath product $(PSL(2,13) \wr PSL(2,11)) \wr C_k$ and a transposition in $S_{143k+1}$ and an element of order 3 in $A_{143k+1}$. We will also show that $S_{143k+1}$ and $A_{143k+1}$ can be generated using the wreath product $PSL(2,13) \wr PSL(2,11)$ and an element of order $k+1$.

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1 Introduction

Hammas and Al-Amri [1], have shown that $A_{2n+1}$ of degree $2n + 1$ can be generated using a copy of $S_n$ and an element of order 3 in $A_{2n+1}$. They also gave the symmetric generating set of Groups $A_{kn+1}$ and $S_{kn+1}$ using $S_n$ [6].
Basmah H. Shafee

Shafee [2] showed that the groups $A_{kn+1}$ and $S_{kn+1}$ can be generated using the wreath product $A_m \wr S_n$ and an element of order $k+1$. Also she showed how to generate $S_{kn+1}$ and $A_{kn+1}$ symmetrically using $n$ elements each of order $k+1$.

In [3], Shafee showed that the groups $A_{252k+1}$ and $S_{252k+1}$ can be generated using the wreath product $L_2(13) \wr L_2(17)$ and an element of order $k+1$. Also [4] she showed that $S_{403k+1}$ and $A_{403k+1}$ can be generated using the wreath product $PSL(3,3) \wr PSL(3,5)$ and an element of order $k+1$.

The Linear groups $PSL(2,13)$ and $PSL(2,11)$ are two groups of the well known simple groups. In [7] as follows

$$PSL(2,13) = < X, Y \mid 13 = Y^2 = (X^4YX^7Y)^2 = (XY)^3 = 1 >. \tag{1}$$
$$PSL(2,11) = < X, Y \mid 11 = Y^2 = (X^4YX^6Y)^2 = (XY)^3 = 1 >. \tag{2}$$

$PSL(2,13)$ can be generated using two permutations, the first is of order 13 and an involution as follows:

$$PSL(2,13) = < (1,2,...,13)(1,5)(3,4)(6,8)(7,14)(9,13)(10,11) >. \tag{3}$$

$PSL(2,11)$ can be generated using two permutations, the first is of order 11 and an involution as follows:

$$PSL(2,11) = < (1,2,...,11)(1,9)(2,6)(4,5)(7,8) >. \tag{4}$$

Here we will generate the wreath product $PSL(2,13) \wr PSL(2,11)$ using only two permutations and we will show the structure of some groups containing the wreath product $PSL(2,13) \wr PSL(2,11)$. The structure of the groups obtained is determined in terms of wreath product $(PSL(2,13) \wr PSL(2,11)) \wr C_k$.

Some related cases are also included. We will show that $S_{143k+1}$ and $A_{143k+1}$ can be generated using the wreath product $(PSL(2,13) \wr PSL(2,11)) \wr C_k$ and a transposition in $S_{143k+1}$ and an element of order in $A_{143k+1}$. We will also show that $S_{143k+1}$ and $A_{143k+1}$ can be generated using the wreath product $PSL(2,13) \wr PSL(2,11)$ and an element of order.

2 PRELIMINARY RESULTS

DEFINITION 2.1. [7] The general linear group $GL_n(q)$ consists of all the $n \times n$ matrices that have non-zero determinant over the field $F_q$ with $q$-elements.
The special linear group $SL_n(q)$ is the subgroup of $GL_n(q)$ which consists of all matrices of determinant one. The projective general linear group $PGL_n(q)$ and projective special general linear group $PSL_n(q)$ are the groups obtained from $GL_n(q)$ and $SL_n(q)$. The projective special general linear group $PSL_n(q)$ is also denoted by $L_n(q)$. The orders of these groups are

$$|GL_n(q)| = (q - 1)N, |SL_n(q)| = |PGL_n(q)| = N, |PSL_n(q)| = |L_n(q)| = \frac{N}{d},$$

where

$$N = q^{\frac{1}{2}n(n-1)}(q^n - 1)(q^{n-1} - 1)...(q^2 - 1)$$

and $d = (q - 1, n)$

**DEFINITION 2.2.** Let $A$ and $B$ be groups of permutations on nonempty sets $\Omega_1$ and $\Omega_2$, respectively, where $\Omega_1 \cap \Omega_2 = \phi$. The wreath product of $A$ and $B$ is denoted by $A \wr B$ and defined as $A \Omega_2 \times_\theta B$, i.e., the direct product of $|\Omega_2|$ copies of $A$ and a mapping $\theta$, where $\theta : B \to \text{Aut}(A \Omega_2)$ is defined by $\theta_{y}(x) = x^y$, for all $x \in A \Omega_2$. It follows that

$$|A \wr B| = (|A|)^{|\Omega_2|}|B|.$$  

(5)

**THEOREM 2.3 (Jorden-Moore)** The group $PSL_n(q)$ is simple if and only if $q \geq 3$.

**THEOREM 2.1** Let $G$ be the group generated by the $n$-cycle $(1, 2, \ldots, n)$ and the $2$-cycle $(n, a)$. If $1 < a < n$, is an integer with $n = am$, then

$$G \cong S_m \wr C_a.$$  

(6)

**THEOREM 2.5** Let $1 \leq a \neq b < n$ be an integers. Let $n$ be an odd integer and let $G$ be the group generated by the $n$-cycle $(1, 2, \ldots, n)$ and the $3$-cycle $(n, a, b)$. If $\text{hcf}(n, a, b) = 1$, then $G \cong A_n$. While if $n$ can be even then

$$G \cong S_n.$$  

(7)

**THEOREM 2.6** Let $1 \leq a \leq n$ be any integer. Let $G = \langle (1, 2, \ldots, n), (n, a) \rangle$. If $\text{hcf}(n, a) = 1$, then $G \cong S_n$.

**THEOREM 2.7** Let $1 \leq a \neq b < n$ be an integers. Let $n$ be an even integer and let $G$ be the group generated by the $n$-cycle $(1, 2, \ldots, n)$ and the $3$-cycle $(n, a, b)$. Then

$$G \cong A_n.$$  

(8)
3 THE RESULTS

THEOREM 3.1 The wreath product $PSL(2,13) \wr PSL(2,11)$ can be generated using two permutations, the first is of order $143k$ and the second is of order 4.


which is the product of 12 cycles each of order 4 and two of transpositions.

Let $\alpha_1 = ((XY)^6[X, Y]^5)^{18}$. Then $\alpha_1 = (11, 22, 33, 44, 55, 66, 77, 88, 99, 110, 121, 132, 143)$, which is a cycle of order 13. Let $\alpha_2 = \alpha_1^{-1}X$.

It is easy to show that

$\alpha_2 = (1, 2, 3, \ldots, 11)(12, 13, 14, \ldots, 22) \ldots (133, 134, 135, \ldots, 143)$, which is the product of 13 cycles each of order 11.

Let: $\beta_1 = (Y^2)^{(XY)^{18}} = (9, 20)(12, 23)(31, 53)(34, 56), \beta_2 = \beta_1^{-1} = (1, 9, 12, 20)(2, 4, 5)(7, 8)(13, 17)(15, 16)(18, 19)(23, 31, 45, 53)(24, 28)(26, 27)(29, 30)(34, 42)(35, 39)(37, 38)(40, 41)(46, 50)(48, 49)(51, 52)(56, 66)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71)(73, 74), \beta_3 = (Y^3\beta_2)^2 = (1, 45)(12, 23), \beta_4 = \beta_3^{-1}(\alpha_2\alpha_1) \beta_3 = (11, 44)(55, 66)$ and $\beta_5 = \beta_4^{-1} = (11, 132)(44, 55)$. Let $\alpha_3 = \beta_5^{(\alpha_2\alpha_1)}$. Hence

$\alpha_3 = (11, 22)(33, 55)$.

Let $\alpha_4 = YX^{-1}\alpha_3^{-1}X$. We can conclude that


which is a product of twenty eight transpositions.

Let $K = \langle \alpha_2, \alpha_4 \rangle$. Let $\theta : K \rightarrow PSL(2,11)$ be the mapping defined by

$\theta(i) = j, \forall 0 \leq i \leq 11 , \forall 0 \leq j \leq 11$.

Since $\theta(\alpha_2) = (1, 2, \ldots, 11)$ and $\theta(\alpha_4) = (1, 9)(2, 6)(4, 5)(7, 8)$, then $K \cong \theta(K) = PSL(2,11)$. Let $H_0 = \langle \alpha_1, \alpha_3 \rangle$. Then $H_0 \cong PSL(2,13)$.

Moreover, $K$ conjugates $H_0$ into $H_1$, $H_1$ into $H_2$ and so it conjugates $H_{11}$ into $H_0$, where

$H_i = \langle (i, i+1, i+2, i+3, i+4, i+5, i+6, i+7, i+8, i+9, i+10), (i, i+1, i+2, i+3, i+4, i+5, i+6, i+7, i+8, i+9, i+10) \rangle \quad \forall 0 \leq i \leq 11$

Hence we get $PSL(2,13) \wr PSL(2,11) \subseteq G$. On the other hand, since $X = \alpha_1\alpha_2$ and $Y = \alpha_4\alpha_3^X$ then $G \subseteq PSL(2,13) \wr PSL(2,11)$. 

Basmah H. Shafee
Hence $G = PSL(2, 13)\wr PSL(2, 11)$.  \\

**Theorem 3.2** The wreath product $(PSL(2, 13)\wr PSL(2, 11))\wr C_K$ can be generated using two permutations, the first is of order $143k$ and an involution, for all integers $K \geq 1$.

**Proof:**

Let $\sigma = (1, 2, ..., 143k)$,

$$\tau = (k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15, 16k)$$

$$(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34, 42k, 56k, 64k)(35k, 39k)$$

$$(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)$$

$$(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k),$$

If $k=1$, then we get the group $PSL(2, 13)\wr PSL(2, 11)$ which can be considered as the trivial wreath product $PSL(2, 13)\wr PSL(2, 11)$ wr $<id>$. Assume that $k > 1$. Let $\alpha = \prod_{i=0}^{1}\sigma^i$, we get an element

$$\delta = \sigma^{45} = (k, 2k, 3k, ..., 143k).$$

Let $G_i = < \delta^{i}, \tau^{a^{i}} >$ be the groups acts on the sets $\Gamma_i = \{ i, k+i, 2k+i, ..., 142k+i \}$, for all $1 \leq i \leq k$. Since $\cap_{i=1}^{k} \Gamma_i = \phi$, then we get the direct product $G_1 \times G_2 \times ... \times G_k$, where, by Theorem 3.1 each $G_i \cong PSL(2, 13)\wr PSL(2, 11)$. Let $\beta = \delta^{-1}\sigma = (1, 2, ..., k)(k+1, k+2, ..., 2k) ... (76k+1, 76k+2, ..., 143k)$. Let $H = < \beta \geq \cong C_k \times H$ conjugates $G_1$ into $G_2$,$G_2$ into $G_3$,... and $G_k$ into $G_1$. Hence we get the wreath product $(PSL(2, 13)\wr PSL(2, 11))\wr C_K \subseteq G$. On the other hand, since $\delta\beta = (1, 2, ..., k, k+1, k+2, ..., 2k, ..., 142k+1, 142k+2, ..., 143k) \equiv \sigma$, then $\sigma \in (PSL(2, 13)\wr PSL(2, 11))\wr C_K$. Hence $G = < \sigma, \tau \geq \cong (PSL(2, 13)\wr PSL(2, 11))\wr C_K$.

**Theorem 3.3** The wreath product $(PSL(2, 13)\wr PSL(2, 11))\wr S_K$ can be generated by using three permutations, the first is of order $143k$, the second and the third are involutions, for all $K \geq 2$.

**Proof:**

Let $\sigma = (1, 2, ..., 143k)$,

$$\tau = (k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15, 16k)$$

$$(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34, 42k, 56k, 64k)(35k, 39k)$$

$$(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)$$

$$(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k), \mu = (k, a)(2k, k+a)(3k, 2k+a)...(143k + 142k + a).$$

Since by theorem 3.2 $< \sigma, \tau \geq \cong (PSL(2, 13)\wr PSL(2, 11))\wr C_k$ and $(1, 2, ..., k)$

$(k+1, 2k) ... (142k, 143k) \in (PSL(2, 13)\wr PSL(2, 11))\wr C_K$ then $< (1, ..., k)(k+1, 2k) ... (142k + 1, 143k, \mu \geq \cong S_k$. Hence $G = < \sigma, \tau, \mu \geq \cong (PSL(2, 13)\wr PSL(2, 11))\wr S_K$.  

$\diamondsuit$
COROLLARY 3.4 The wreath product \((PSL(2,13) \wr PSL(2,11)) \wr A_k\) can be generated by using three permutations, the first is of order 143k, the second is an involution and the third is of order 3, for all odd integers \(k \geq 3\).

**Proof**: The proof is similar to the previous one. \(\diamondsuit\)

THEOREM 3.5 The wreath product \((PSL(2,13) \wr PSL(2,11)) \wr (S_m \wr C_a)\) can be generated by using three permutations, the first is of order 143k, the second and the third are involutions, where \(k = am\) be any integer with \(1 < a < k\).

**Proof**: Let \(\sigma = (1,2,...,143k)\), \(\tau = (k,9k)(2k,6k)(4k,5k)(7k,8k)(12k,20k,23k,31k)(13k,17k)(15,16k)(18k,19k)(24k,28k)(26k,27k)(29k,30k)(34,42k,56k,64k)(35k,39k)(37k,38k)(40k,41k)(45k,53k)(46k,50k)(49k)(51k,52k)(57k,61k)(59k,60k)(62k,63k)(67k,75k)(68k,72k)(70k,71k), \(\mu = (k,a)(2k,k+a)(3k,2k+a)\)\((143k + 142k + a)\).

Since by theorem 3.2 \(<\sigma, \tau \succ \simeq (PSL(2,13) \wr PSL(2,11)(w) \wr C_k\) and \((1,...,k)\)
\((k+1,...,2k)\)\((142k + 1,...,143k)\)\(\in (PSL(2,13) \wr PSL(2,11)) \wr C_K\) then \(<\sigma, \tau \succ \simeq (S_m \wr C_a)\). Hence \(G = <\sigma, \tau, \mu \succ \simeq (PSL(2,13) \wr PSL(2,11)) \wr (S_m \wr C_a)\). \(\square\)

THEOREM 3.6 \(S_{143K+1}\) and \(A_{143K+1}\) can be generated using the wreath product \((PSL(2,13) \wr PSL(2,11)) \wr C_k\) and a transposition in \(S_{143K+1}\) for all integers \(k > 1\) and an element of order 11 in \(A_{143K+1}\) for all odd integers \(k > 1\).

**Proof**: Let \(\sigma = (1,2,...,143k)\), \(\tau = (k,9k)(2k,6k)(4k,5k)(7k,8k)(12k,20k,23k,31k)(13k,17k)(15,16k)(18k,19k)(24k,28k)(26k,27k)(29k,30k)(34,42k,56k,64k)(35k,39k)(37k,38k)(40k,41k)(45k,53k)(46k,50k)(49k)(51k,52k)(57k,61k)(59k,60k)(62k,63k)(67k,75k)(68k,72k)(70k,71k), \(\mu = (143k + 1,1)\) and \(\mu^1 = (1,1,143k + 1)\) be four Permutations, of order 143k,2,2, and 3 respectively.

Let \(H = <\sigma, \tau >\). By theorem 3.2 \(H \simeq (PSL(2,13) \wr PSL(2,11)) \wr C_k\).

**Case 1**: Let \(G = <\sigma, \tau, \mu >\). Let \(\alpha = \sigma \mu\), then \(\alpha = (1,2,...,143k,143k + 1)\) which is a cycle of order 143k+1. By theorem 2.6 \(G = <\sigma, \tau, \mu >\simeq <\alpha, \mu >\simeq S_{143K+1}\).

**Case 2**: Let \(G = <\sigma, \tau, \mu >\). By theorem 2.7 \(<\sigma, \mu >\simeq A_{143K+1}\). Since \(\tau\) is an even Permutation, then \(G \simeq A_{143K+1}\).
THEOREM 3.7 $S_{143k+1}$ and $A_{143k+1}$ can be generated using the wreath product $PSL(2,13) \wr PSL(2,11)$ and an element of order $k+1$ in $S_{143k+1}$ and $A_{143k+1}$ for all integers $k \geq 1$.

Proof: Let $G = \langle \sigma, \tau, \mu \rangle$, Where

\[ \sigma = (1, 2, 3, \ldots, 143)(143(k - (k - 1)) + 1, \ldots, 143(k - (k - 1)) + 143) \]
\[ \ldots (143(k - 1) + 1, \ldots, 143(k - 1) + 143), \]

\[ \tau = (1, 9)(2, 6)(4, 5)(7, 8)(12, 20, 23, 31)(13, 17)(15, 16)(18, 19)(24, 28) \]
\[ (26, 27)(29, 30)(34, 42, 56, 64)(35, 39)(37, 38)(40, 41)(45, 53)(46, 50) \]
\[ (48, 49)(51, 52)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71)(73, 74) \]
\[ (143(k - 1) + 1, 143(k - 1) + 9) \ldots (143(k - 1) + 73, 143(k - 1) + 74), \]

$\mu = (143, 154, \ldots, 143k, 143k + 1)$, where $k - i > 0$, be three permutations of order $143$, $4$ and $k + 1$ respectively.

Let $H = \langle \sigma, \tau \rangle$. Define the mapping $\theta$ as follows

\[ \theta(11(k - i) + j) = j \forall 1 \leq j \leq 11 \]

$H = \langle \sigma, \tau \rangle \cong PSL(2,13) \wr PSL(2,11)$. Let $\alpha = \mu \sigma$ it is easy to show that $\alpha = (1, 2, \ldots, 143k, 143k + 1)$, Which is a cycle of order $143k + 1$.

Let $\mu \mu = \mu^\sigma = (1, 144, \ldots, 143(k - 1) + 1, 143k + 1)$ and $\beta = [\mu, \mu^\lambda] = (1, 143, 143k + 1)$. Since $h.c.f(1, 143, 143k + 1) = 1$, then by theorem 2.5 $G = \langle \sigma, \tau, \mu \rangle \cong S_{143k+1}$ or $A_{143k+1}$ depending on whether $k$ is an odd or an even integer respectively. ♦

References


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