Prime and Semiprime Bi-ideals in
Ordered Semigroups

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Abstract

In this paper we shall introduce the concept of prime and semiprime bi-ideals of ordered semigroups and we give characterizations of prime bi-ideal and regular of ordered semigroups.

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1 Introduction

The concept of prime and semiprime bi-ideals of associative rings with unity was introduced by A. P. J. van der Walt [7]. In [6] H J le Roux have constructed a number of results by using prime and semiprime bi-ideals of associative rings without unity. In this paper, we define a prime and semiprime bi-ideals of ordered semigroups. Further we shall extend the results of H J le Roux [6] to an ordered semigroups. It is shown that a bi-ideal $B$ of a po-semigroup $S$ is prime if and only if $RL \subseteq B$, with $R$ a right ideal of $S$, $L$ a left ideal of $S$, implies $R \subseteq B$ or $L \subseteq B$. Moreover, let $B$ be a prime bi-ideal of a po-semigroup $S$, then $H(B)$ is a prime ideal of $S$. 
2 Preliminary Notes

In this section, we recall some basic definitions and results that are relevant for this paper.

**Definition 2.1** A po-semigroup (:ordered semigroup) is an ordered set \((S, \leq)\) at the same time a semigroup such that:
\[
a \leq b \Rightarrow ca \leq cb \text{ and } ac \leq bc \quad \forall \ c \in S \ [1].
\]

The following definitions and results were due to N. Kehayopulu.

**Definition 2.2** Let \(S\) be a po-semigroup and \(\phi \neq A \subseteq S\). \(A\) is called a right (resp. left) ideal of \(S\) \([2, 3, 4, 5]\) if
\[
1) \ AS \subseteq A \ (\text{resp. } SA \subseteq A).
\]
\[
2) \ a \in A, \ S \ni b \leq a \Rightarrow b \in A.
\]
\(A\) is called an ideal of \(S\) if it both a right and a left ideal of \(S\).

**Definition 2.3** Let \(S\) be a po-semigroup and \(T \subseteq S\). \(T\) is called prime if \(A, B \subseteq S, AB \subseteq T \Rightarrow A \subseteq T \) or \(B \subseteq T \) \([5]\).
\(T\) is equivalently, \(a, b \in S, \ ab \in T \Rightarrow a \in T \) or \(b \in T\).

**Definition 2.4** Let \(S\) be a po-semigroup and \(T \subseteq S\). \(T\) is called weakly prime if
For all ideals \(A, B\) of \(S\) such that \(AB \subseteq T\), we have \(A \subseteq T\) or \(B \subseteq T\) \([2, 5]\).

**Definition 2.5** Let \(S\) be a po-semigroup and \(T \subseteq S\). \(T\) is called semiprime if
\(A \subseteq S, \ A^2 \subseteq T \Rightarrow A \subseteq T \) \([5]\).
\(T\) is equivalently, \(a \in S, \ a^2 \in T \Rightarrow a \in T\).

**Definition 2.6** Let \(S\) be a po-semigroup and \(\phi \neq Q \subseteq S\). \(Q\) is called a quasi ideal of \(S\) \([4]\) if
\[
1) \ QS \cap SQ \subseteq Q.
\]
\[
2) \ a \in Q, \ S \ni b \leq a \Rightarrow b \in Q.
\]

**Definition 2.7** Let \(S\) be a po-semigroup and \(\phi \neq B \subseteq S\). \(B\) is called a bi-ideal of \(S\) \([4]\) if
\[
1) \ BSB \subseteq B.
\]
\[
2) \ a \in B, \ S \ni b \leq a \Rightarrow b \in B.
\]

**Notation**
For \(H \subseteq S, \ (H) = \{t \in S/t \leq h \text{ for some } h \in H\}\).
We denote by \(I(a) \) (resp. \(L(a), \ R(a)\)) the ideal (resp. left ideal, right ideal) of \(S\) generated by \(a\). One can easily prove that:
\[
I(a) = (a \cup Sa \cup aS \cup SaS), \ L(a) = (a \cup Sa), \ R(a) = (a \cup aS).
\]
Definition 2.8 Let $S$ be a po-semigroup. $S$ is called left (resp. right ideal) regular if
\[ \forall a \in S \exists x \in S : a \leq xa^2 \text{ (resp. } a \leq a^2 x) \] [4, 5].

Equivalently,
1) $a \in (Sa^2)$ (resp. $a \in (a^2 S)$) $\forall a \in S$.
2) $A \subseteq (SA^2)$ (resp. $A \subseteq (A^2 S)$) $\forall A \subseteq S$.

Definition 2.9 Let $S$ be a po-semigroup. $S$ is called regular if
\[ \forall a \in S \exists x \in S : a \leq axa \] [5].

Equivalently,
1) $a \in (aSa)$ $\forall a \in S$.
2) $A \subseteq (ASA)$ $\forall A \subseteq S$.

We note the following Lemma.

Lemma 2.10 For an ordered semigroup $S$, we have
1) $A \subseteq (A) \forall A \subseteq S$.
2) If $A \subseteq B \subseteq S$, then $(A) \subseteq (B)$.
3) $(A)(B) \subseteq (AB)$ $\forall A, B \subseteq S$.
4) $((A)) = (A) \forall A \subseteq S$.
5) For every left (resp. right) ideal or bi-ideal $T$ of $S$, we have $(T) = T$.
6) $((A)(B)) = (AB) \forall A, B \subseteq S$ (cf.[2, the Lemma]).

The following results were due to N. Kehayopulu.

Result 2.11 Let $S$ be a po-semigroup and $T$ be an ideal of $S$. The following are equivalent:
1) $T$ is weakly prime.
2) If $a, b \in S$ such that $(aSb) \subseteq T$, then $a \in T$ or $b \in T$.
3) If $a, b \in S$ such that $I(a)I(b) \subseteq T$, then $a \in T$ or $b \in T$.
4) If $A, B$ are right ideals of $S$ such that $AB \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.
5) If $A, B$ are left ideals of $S$ such that $AB \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.
6) If $A$ a right ideal, $B$ a left ideal of $S$ such that $AB \subseteq T$, then $A \subseteq T$ or $B \subseteq T$ [2].

Result 2.12 An ideal $T$ of a po-semigroup $S$ is weakly semiprime if and only if one of the following four equivalent conditions holds in $S$:
1) For every $a \in S$ such that $(aSa) \subseteq T$, we have $a \in T$.
2) For every $a \in S$ such that $(I(a))^2 \subseteq T$, we have $a \in T$.
3) For every right ideal $A$ of $S$ such that $A^2 \subseteq T$, we have $A \subseteq T$.
4) For every left ideal $B$ of $S$ such that $B^2 \subseteq T$, we have $B \subseteq T$ [2].

Result 2.13 An ideal of a po-semigroup is prime if and only if it is both semiprime and weakly prime. In commutative po-semigroups the prime and weakly prime ideals coincide.
3 Main Results

In this section, we introduce prime and semiprime bi-ideals of ordered semigroups and obtain some properties of it.

**Definition 3.1** Let $S$ be a po-semigroup. A bi-ideal $B$ of $S$ is called prime if

$$xSy \subseteq B \Rightarrow x \in B \text{ or } y \in B.$$  

Equivalently,

the subsets $C, D \subseteq S, CSD \subseteq B \Rightarrow C \subseteq B$ or $D \subseteq B$.

**Definition 3.2** Let $S$ be a po-semigroup. A bi-ideal $B$ of $S$ is called semiprime if

$$xSx \subseteq B \Rightarrow x \in B.$$  

Equivalently,

a subset $C \subseteq S, CSC \subseteq B \Rightarrow C \subseteq B$.

Now we shall generalize the results on associative ring without unity found in [6] for an ordered semigroups.

**Proposition 3.3** A bi-ideal $B$ of a po-semigroup $S$ is prime if and only if $RL \subseteq B$, with $R$ a right ideal of $S, L$ a left ideal of $S$, implies $R \subseteq B$ or $L \subseteq B$.

**Proof.** Let $B$ be a prime bi-ideal of a po-semigroup $S$ and $RL \subseteq B$. Suppose $R \notin B$. For all $x \in L$ and $r \in R \setminus B$, we have $rSx \subseteq RL \subseteq B$.

Since $B$ is prime and $r \notin B$, we have $x \in B$ for all $x \in L$, so $L \subseteq B$.

Conversely, suppose $RL \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$ for any right ideal $R$ of $S$ and any left ideal $L$ of $S$. Let $x, y \in S$ such that $xSy \subseteq B$. Then $(xS)(Sy) \subseteq (xSy) \subseteq (B) = B$.

Since $(xS)$ is a right ideal of $S$ and $(Sy)$ a left ideal of $S$, we have $(xS) \subseteq B$ or $(Sy) \subseteq B$. Suppose $(xS) \subseteq B$. Then $x^2 \in B$.

Consider $R(x)$ and $L(x)$, the right ideal of $S$ and left ideal of $S$ generated by $x$ in $S$, respectively. Now $R(x)L(x) = (x \cup xS)(x \cup xSx) \subseteq ((x \cup xS)(x \cup xSx)) = (x^2 \cup xSx \cup xSx \cup xSSx) \subseteq (x^2 \cup xSx)$. Let $z$ be any element of the product $R(x)L(x)$. Then $z \in (x^2 \cup xSx)$. So $z \leq t$ for some $t \in x^2 \cup xSx$. If $t = x^2$, then $z \leq x^2 \in B$ i.e. $z \in B$, since $B$ is a bi-ideal of $S$. If $t = xyx$ forsome $y \in S$, then $z \leq xyx \in xSx \subseteq xS \subseteq (xS) \subseteq B$ i.e. $z \in B$, since $B$ is a bi-ideal of $S$. Hence $R(x)L(x) \subseteq B$.

From our assumption it follows that $R(x) \subseteq B$ or $L(x) \subseteq B$ and hence $x \in B$. Similarly, if $(Sy) \subseteq B$, then $y \in B$. Hence $B$ is a prime ideal of $S$.

**Proposition 3.4** A prime bi-ideal of a po-semigroup $S$ is prime one-sided ideal of $S$. 

Proof. Let $B$ be a prime bi-ideal of a po-semigroup $S$. It is only necessary to show that $B$ is a one-sided ideal of $S$.

Clearly, $(BS)(SB) \subseteq (BS^2B) \subseteq (BSB) \subseteq (B) = B$. Since $(BS)$ is a right and $(SB)$ a left ideal of $S$, we have, from proposition 3.3, that $(BS) \subseteq B$ or $(SB) \subseteq B$ i.e. $BS \subseteq B$ or $SB \subseteq B$, since $BS \subseteq (BS)$, $SB \subseteq (SB)$. Assume $x \in B$, $S \ni y \leq x$. Then $y \in B$, since $B$ is a bi-ideal of $S$. Hence $B$ is a one-sided ideal of $S$.

Remark

Let $B$ be any bi-ideal of a po-semigroup $S$ and let $L(B) = \{x \in B/Sx \subseteq B\}$ and $H(B) = \{y \in L(B)/yS \subseteq L(B)\}$.

Lemma 3.5 For any bi-ideal $B$ of a po-semigroup $S$ the set $L(B) = \{x \in B/Sx \subseteq B\}$ is a left ideal of $S$.

Proof. If $x \in L(B)$ and $z \in S$, then $zx \in Sx \subseteq B$ and $Szx \subseteq SSx \subseteq Sx \subseteq B$.

Choose $x \in L(B)$ such that $S \ni y \leq x$. Then $y \in B$, since $L(B) \subseteq B$ and $B$ is a bi-ideal of $S$. Since $y \leq x$ and $S$ is a po-semigroup, we have $zy \leq zx \forall z \in S$. So $zy \leq zx \in Sx \subseteq B$ i.e. $zy \in B \forall z \in S$, since $B$ is a bi-ideal of $S$. Thus $Sy \subseteq B$ implies $y \in L(B)$. Hence $L(B)$ is a left ideal of $S$.

Proposition 3.6 If $B$ is any bi-ideal of a po-semigroup $S$, then $H(B)$ is the (unique) largest two-sided ideal of $S$ contained in $B$.

Proof. As in the lines of [6], we prove $xy, yx \in H(B)$.

Since $L(B) \subseteq B$ and $H(B) \subseteq L(B)$, we have that $H(B) \subseteq B$. We now show that $H(B)$ is a two-sided ideal of $S$.

Let $x \in H(B)$ and $y \in S$. Then $x \in B$ and $x$ is also an element of $L(B)$, we have that $Sx \subseteq B$ and $xS \subseteq L(B)$.

Then $yx \in Sx \subseteq B$. So $yx \in B$. Furthermore $Syx \subseteq Sx \subseteq B$. So $yx \in L(B)$. Also $xy \in xS \subseteq L(B)$. Hence $xy \in L(B)$.

Now we shall show that $xy$ and $yx \in H(B)$. $xyS \subseteq xS \subseteq L(B)$. Hence $xy \in H(B)$. $yxS \subseteq SxS \subseteq SL(B) \subseteq L(B)$, since $L(B)$ is a left ideal of $S$. Hence $yx \in H(B)$.

Let $x \in H(B)$, $S \ni y \leq x$. Then $y \in L(B)$, since $H(B) \subseteq L(B)$ and $L(B)$ is a left ideal of $S$. Since $y \leq x$ and $S$ is a po-semigroup, we have $yz \leq xz \forall z \in S$. So $yz \leq xz \in xS \subseteq L(B)$ i.e. $yz \in L(B) \forall z \in S$, since $L(B)$ is a left ideal of $S$. Thus $yS \subseteq L(B)$ implies $y \in H(B)$.

Hence $H(B)$ is a two-sided ideal of $S$.

Let $I$ be any ideal of $S$ and $I \subseteq B$, and let $u$ be an arbitrary element of $I$. Then $u \in B$ and $Su \subseteq I \subseteq B$. Hence $I \subseteq L(B)$.

Furthermore $u \in L(B)$ and $uS \subseteq I \subseteq L(B)$. This implies that $u \in H(B)$ and hence $I \subseteq H(B)$.
**Proposition 3.7** Let \( B \) be a prime bi-ideal of a po-semigroup \( S \). Then \( H(B) \) is a weakly prime ideal of \( S \).

*Proof.* Let \( B \) be a prime bi-ideal of a po-semigroup \( S \). Since \( B \) is a bi-ideal of \( S \), we have, from proposition 3.6, that \( H(B) \) is a two-sided ideal of \( S \). We now show that an ideal \( H(B) \) of \( S \) is weakly prime.

Let \( a, b \in S \) such that \( I(a)I(b) \subseteq H(B) \) for any two-sided ideals \( I(a) \) and \( I(b) \) of \( S \) generated by \( a \) and \( b \) in \( S \), respectively. From proposition 3.3 it follows that \( I(a) \subseteq B \) or \( I(b) \subseteq B \), since \( I(a)I(b) \subseteq B \). From proposition 3.6, we have, that \( H(B) \) is the largest ideal in \( B \). Hence \( I(a) \subseteq H(B) \) or \( I(b) \subseteq H(B) \). This implies that \( a \in H(B) \) or \( b \in H(B) \) and hence by theorem 2.11 that \( H(B) \) is weakly prime.

**Proposition 3.8** Let \( B \) be a semiprime bi-ideal of a po-semigroup \( S \). Then \( L^2 \subseteq B \) (or \( R^2 \subseteq B \)) implies \( L \subseteq B \) (or \( R \subseteq B \)) for any left ideal \( L \) (or right ideal \( R \)) of \( S \).

*Proof.* The proof is same as in the proposition 10 of [6].

**Proposition 3.9** Let \( B \) be a semiprime bi-ideal of a po-semigroup \( S \). Then \( H(B) \) is a weakly semiprime ideal of \( S \).

*Proof.* Let \( B \) be a semiprime bi-ideal of a po-semigroup \( S \). Since \( B \) is a bi-ideal of \( S \), we have, from proposition 3.6, that \( H(B) \) is a two-sided ideal of \( S \). We now show that an ideal \( H(B) \) of \( S \) is weakly semiprime.

Let \( a \in S \) such that \( (I(a))^2 \subseteq H(B) \) for any two-sided ideal \( I(a) \) of \( S \) generated by \( a \) in \( S \). From proposition 3.8 it follows that \( I(a) \subseteq B \), since \( (I(a))^2 \subseteq B \). From proposition 3.6, we have, that \( H(B) \) is the largest ideal in \( B \). Hence \( I(a) \subseteq H(B) \). This implies that \( a \in H(B) \) and hence by theorem 2.12 that \( H(B) \) is weakly semiprime.

**Proposition 3.10** Let \( B \) be a semiprime bi-ideal of a po-semigroup \( S \). Then \( B \) is a quasi ideal of \( S \).

*Proof.* Assume \( y \in BS \cap SB \). Then \( y \in BS \) and \( y \in SB \). \( ySy \subseteq (BS)S(SB) \subseteq BSB \subseteq B \). Since \( B \) is a semiprime bi-ideal of \( S \), we have \( y \in B \). Hence \( BS \cap SB \subseteq B \).

Next, let \( x \in B \), \( S \triangleright y \leq x \). Then \( y \in B \), since \( B \) is a bi-ideal of \( S \). Hence \( B \) is a quasi-ideal of \( S \).

**Proposition 3.11** A po-semigroup \( S \) regular if and only if every bi-ideal in \( S \) is semiprime.
Proof. Let $S$ be a regular po-semigroup and $B$ be any bi-ideal of $S$. Suppose $aSa \subseteq B$ for $a \in S$. Then there exists $x \in S$ such that $a \leq axa$, since $S$ is regular. But $axa \in aSa \subseteq B$ i.e. $axa \in B$. Since $axa \in B$, $S \ni a \leq axa$ and $B$ is a bi-ideal of $S$, we have $a \in B$ and so $B$ is semiprime.

Conversely, suppose that every bi-ideal of $S$ is semiprime. Let $a \in S$. It is clear that $(aSa)$ is a bi-ideal of $S$. Hence $(aSa)$ is semiprime for any $a \in S$. Since $aSa \subseteq (aSa)$ and $(aSa)$ is semiprime, we have $a \in (aSa)$. This implies that $a \leq axa$ for some $x \in S$ and hence $S$ is regular.

Proposition 3.12 A commutative po-semigroup $S$ regular if and only if every ideal of $S$ is semiprime.

Proof. Let $S$ be a regular commutative po-semigroup and $I$ be an ideal of $S$. Suppose $a^2 \in I$ for $a \in S$. Then there exists $x \in S$ such that $a \leq axa$, since $S$ is regular. But $a \leq axa = a(xa) = a(ax) = a^2x \in IS \subseteq I$. This implies that $a \in I$, since $I$ is an ideal of $S$. Hence $I$ is semiprime.

Conversely, suppose that every ideal of $S$ is semiprime. Let $a \in S$. It is clear that $(a^2S)$ is an ideal of $S$. Hence $(a^2S)$ is semiprime for any $a \in S$. Since $a^4 \in (a^2S)$, $(a^2S)$ is semiprime, we have $a^2 \in (a^2S)$ implies $a \in (a^2S)$ i.e. $a \leq a^2x$ for some $x \in S$. This implies that $a \leq aax = axa$ for some $x \in S$. Hence $S$ is regular.

References


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