A Brief Statement on the Absolute-valued
Algebras with One-sided Unit

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Abstract. We note that every absolute-valued algebra having both left-unit and a non-zero central element is finite-dimensional, next we specify.

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1. Introduction

Absolute-valued algebras (AVA) with left-unit have been extensively studied. Classified in finite dimension [A 47], [Ram 99], [Roc 03], [Rod 04], [CDD 10] and their existence happen in arbitrary infinite dimension [Cu 92], [Rod 92], [EP 97]. A constructive method for obtaining all infinite-dimensional AVA with left-unit is also given [Rod 92, Theorem 2], [Rod 04, Theorem 3.6].
Other studies on AVA with left-unit have emerged recently. The finiteness of the dimension happen if some additional condition is assumed, as minor identities [CR 08], [DRR 11] or existence of a non-zero central element [BM 11].

The objective of the present paper is to provide a classification of all AVA $A$ with left-unit whenever the existence of non-zero central element $a$ is assumed. We briefly establish the finiteness of the dimension in two distinct ways. First (Proposition 3) by using a classical theorem by Albert-Urbanik-Wright [A 49, Theorem 2], [UW 60, Theorem 1] asserting that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ are the only AVA with two-sided unit, and [Rod 92, Remark 4. i]), then (Proposition 4) noting that the element $a$ satisfies to $aA = Aa = A$ and using [Rod 92, Proposition 4], [Rod 04, Theorem 2.2]. Next, using [Roc 03, Th. 4.3], [Rod 04, Proposition 1.8], we give a classification. The list obtained is reduced to $\mathbb{R}$, $\mathbb{C}$, and the algebras $A_a$ where $A$ stands for either $\mathbb{H}$ or $\mathbb{O}$, $a$ being norm-one in $A$, with real part positive, and $A_a$ denotes the AVA obtained by endowing the normed space of $A$ with the product $x \circ y := (ax\bar{x})y$. Moreover, given norm-one $a, b \in A$ with real parts positive, the algebras $A_a$ and $A_b$ are isomorphic if and only if $a$ and $b$ have the same real parts.

Our work is an extension of Albert-Urbanik-Wright’s theorem. It improves [BM 11] and simplifies the content by avoiding tedious calculations.

2. Notations

An algebra is a vector space $A$ over the field $\mathbb{R}$ of real numbers endowed with a product, that is a bilinear mapping $(x, y) \mapsto xy$ from $A \times A$ to $A$. An algebra is said to be alternative if it satisfies the identities $x^2y = x(xy)$ and $(yx)x = yx^2$. Absolute-valued algebras are defined as those algebras $A$ satisfying $||xy|| = ||x|| \cdot ||y||$ for a given norm $||.||$ on $A$, and all $x, y \in A$.

Let $A$ be one of main AVA $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (quaternions), $\mathbb{O}$ (octonions), with two-sided unit 1 and standard involution $\sigma_h : x \mapsto \bar{x}$. Its absolute value and the corresponding unit sphere are given, respectively, by $||x|| = \sqrt{xx}$ and $S(A) = \{x \in A : xx = 1\}$. The space $A$ is a direct sum of its sub-spaces of scalars $\mathbb{R}$.1 and purely imaginary elements $\text{Im}(A) = \{x \in A : \bar{x} = -x\}$ which is also the orthogonal complement of $\mathbb{R}$.1 [HKR 91]. We denote by $\text{Aut}(A)$ the group of automorphisms of algebra $A$. Any non-zero sub-algebra of $A$ contains 1 [S 54] and is invariant under $\sigma_h$. A well known theorem of Artin [ZSSS 82, Theorem 2.3.2] shows that for every $x, y$ in the alternative algebra $A$, the set $\{x, y, \bar{x}, \bar{y}\}$ is contained in an associative subalgebra of $A$. This fact will be used in the sequel without further reference.
3. Absolute-valued algebras containing one sided-unit

Rodriguez [Rod 92, Remark 4. i], [Rod 04, Theorem 3.5] proved the following famous result:

**Theorem 1.** Let $A$ be an absolute-valued algebra with left-unit $e$. Then the norm $|| \cdot ||$ of $A$ comes from an inner product $\langle \cdot, \cdot \rangle$, and, for $x^* = 2\langle e, x \rangle e - x$, we have $\langle xy, z \rangle = \langle y, x^* z \rangle$ and $x^*(xy) = ||x||^2 y$ for all $x, y, z \in A$. ■

We also have $||x^*|| = ||x||$, $x \in A$, and it is easy to check the following:

(3.1) \[(x^*)^* = x, \ x \in A.\]

(3.2) \[x(x^*y) = ||x||^2 y, \ x, y \in A.\]

We have the well known result [Roc 03, Theorem 4.3], [Rod 04, Proposition 1.8]:

**Theorem 2.** The finite-dimensional absolute-valued algebras with a left unit are precisely those of the form $A_\varphi$, where $A$ stands for either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, $\varphi : A \to A$ is a linear isometry fixing 1, and $A_\varphi$ denotes the absolute-valued algebra obtained by endowing the normed space of $A$ with the product $x \odot y := \varphi(x)y$. Moreover, given linear isometries $\varphi, \phi : A \to A$ fixing 1, the algebras $A_\varphi$ and $A_\phi$ are isomorphic if and only if there exists an algebra automorphism $\psi$ of $A$ satisfying $\phi = \psi \circ \varphi \circ \psi^{-1}$. ■

Among above algebras we specify those having a non-zero central element:

**Proposition 1.** The following two statements are equivalent:

1. The algebra $A_\varphi = (A, \odot)$ contains a central element $a \in S(A)$,
2. The linear isometry $\varphi : A \to A$ is given by $\varphi(x) = axa$ for all $x \in A$.

**Proof.** (2) $\Rightarrow$ (1). For every $x \in A$, we have:

\[x \odot a = \varphi(x)a = axa = ax = \varphi(a)x = a \odot x.\]

(1) $\Rightarrow$ (2). For all $x \in A$, we have

\[a \odot x = x \odot a \Leftrightarrow \varphi(a)x = \varphi(x)a \Leftrightarrow \varphi(x) = \left(\varphi(a)x\right)\overline{a}.\]

For $x = 1$ this gives $\varphi(a) = a$. So $\varphi(x) = axa$ for all $x \in A$. ■
4. The result

In this last section $A$ will be assumed to be an absolute-valued algebra containing both a left-unit $e$ and a norm-one central element $a$. If $A$ has dimension $\leq 2$ then $A$ possesses a two-sided unit, and is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ according to Albert-Urbanik-Wright’s theorem. Subsequently we will focus on the case where $A$ has dimension $\geq 4$.

We start with:

**Proposition 2.** The sub-algebra $A(e, a)$ of $A$ generated by $\{e, a\}$ has finite dimension $\leq 2$ and is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.

**Proof.** By Theorem 1, we have $a^*a = a^*(ea) = a^*(ae) = e$. In other hand $a^*a = 2(e, a)a - a^2$, so $a^2$ belongs to the lineal space spanned by $\{e, a\}$. ■

We can now state the key result:

**Proposition 3.** Let $A$ be an absolute-valued algebra containing both a left-unit $e$ and a norm-one central element $a$. Then $A$ is finite-dimensional.

**Proof.** The normed space of $A$ endowed with the product $x \odot y = (a^*x)y$ is an absolute-valued algebra $(A, \odot)$. In addition $a$ is a two-sided unit for algebra $(A, \odot)$. Indeed, for every $x$ in $A$, we have:

$$a \odot x = (a^*a)x = x \quad \text{and} \quad x \odot a = (a^*x)a = a(a^*x)^{3.2} = x.$$  

The result is then concluded by Albert-Urbanik-Wright’s theorem. ■

The previous result can be both refined and its proof shortened:

**Proposition 4.** Let $A$ be an absolute-valued algebra containing elements $e, a$ such that $eA = A$, $a \neq 0$, and $[a, A] = 0$. Then $A$ is finite-dimensional.

**Proof.** Since $eA = A$ and $a \neq 0$, [Rod 92, Proposition 4] applies, so that $aA = A$. But, since $[a, A] = 0$, we have also $Aa = A$ and [Rod 04, Theorem 2.2] concludes. ■

The main result:

**Theorem 3.** Every absolute-valued algebra with both a left unit and a non-zero central element is finite-dimensional. Such an algebras are precisely $\mathbb{R}$, $\mathbb{C}$, or those of the form $A_a$, where $A$ stands for either $\mathbb{H}$ or $\mathbb{O}$, $a$ being norm-one in $A$, with positive real part, and $A_a$ denotes the absolute-valued algebra obtained by endowing the normed space of $A$ with the product $x \odot y := (axa)y$. Moreover, given norm-one $a, b \in A$ with positive real parts, the algebras $A_a$ and $A_b$ are isomorphic if and only $a$ and $b$ have the same real part. The element $a$ can be choosen in arbitrary copy of $\mathbb{C}$ fixed in $A$. 


Proof. The first assertion follows by Proposition 3 (or Proposition 4), algebras of dimension \( \leq 2 \) being either \( \mathbb{R} \) or \( \mathbb{C} \). The first statement of Theorem 2 and Proposition 1 assert that those algebras of dimensions 4 and 8 are of the form \( A_{c} \), \( A \) stands for either \( \mathbb{H} \) or \( \mathbb{O} \), \( \varphi_{c} : A \to A \) is the linear isometry given by \( \varphi_{c}(x) = cx \) for fixed \( c \in S(A) \) and all \( x \in A \), and \( A_{c} \) denotes the absolute-valued algebra obtained by endowing the normed space of \( A \) with the product \( x \odot y := \varphi_{c}(x)y \). We set \( A_{c} := A_{c} \), and we have \( A_{-c} = A_{c} \) so we can take positive the real part \( Re(c) \) of \( c \).

Let now \( a, b \) be in \( S(A) \) with \( Re(a), Re(b) \geq 0 \), using the second statement of Theorem 2, and that any automorphism of \( A \) fixed 1 and leaves invariant \( Im(A) \), we have:

\[
A_{a} \simeq A_{b} \iff \exists \Phi \in \text{Aut}(A) : \varphi_{a} = \Phi \circ \varphi_{a} \circ \Phi^{-1} \\
\iff \exists \Phi \in \text{Aut}(A) : bx = \Phi(a)x\Phi(\overline{a}) \text{ for all } x \in A \\
\iff \exists \Phi \in \text{Aut}(A) : bx = \Phi(a)x\Phi(a) \text{ for all } x \in A \\
\iff \exists \Phi \in \text{Aut}(A) : \Phi(a) \in S(\mathbb{R}) = \{1, -1\} \\
\iff \exists \Phi \in \text{Aut}(A) : b = \Phi(a) \text{ because } Re(a), Re(b) \geq 0 \\
\iff Re(b) = Re(a) \text{ by [Po 85, Lemme 1, pp. 269-270] .}
\]

Now, fix a norm-one \( u \) in \( Im(A) \) and denote by \( C \subset A \) the (unique) copy of \( \mathbb{C} \) containing \( u \). Let \( a \) be norm-one in \( A \) then \( a \) has an orthonormal decomposition \( Re(a) + \lambda v \) in \( A = \mathbb{R}.1 \oplus Im(A) \), so \( A_{a} \) is isomorphic to \( A_{b} \) where \( b = Re(a) + \lambda u \in C \). □

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