Division Algebras Satisfying

\[(x^p x^q)x^r = x^p(x^q x^r)\]

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Abstract. Let \( A \) be an algebra over a field of characteristic zero and let \((x, y, z)\) be the associator \((xy)z - x(yz)\) of \(x, y, z \in A\). Assume that \( A \) has no divisors of zero satisfying \((x^p, x^q, x^r) = 0\) for fixed \(p, q, r \in \{1, 2\}\). A slightly generalized unit in \( A \), if it exists, becomes an unit element and the algebra \( A \) is third power-associative. If, in addition, \( A \) is finite-dimensional of degree \(\leq 4\) then \( A \) is power-commutative. We deduce that any 4-dimensional real division algebra, with a slightly generalized unit, satisfying \((x^p, x^q, x^r) = 0\) is quadratic. This persists for identity \((x, x^q, x^r) = 0\) if we replace the expression ”slightly generalized unit” by ”slightly generalized left-unit”.

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1. Introduction

The study of finite-dimensional (FD) real division algebras is a fascinating topic that arose after the discovery of quaternions $\mathbb{H}$ and octonions $\mathbb{O}$. Classical results show that $\{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ classifies the FD real associative division algebras [Fr 1878], [HKR 91] and that $\{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ classifies the FD real alternative\(^1\) division algebras [Zo 31], [HKR 91].

The (1, 2, 4, 8)-theorem [Ho 40], [BM 58], [Ke 58], [HKR 91], stating that 1, 2, 4, 8 are the only possibilities for the dimension of every FD real division algebra, revolutionized the theory and has become an indispensable tool in almost all subsequent work. On the other hand the classification of such an algebras is trivial in dimension 1 [Sc 66, p. 2], [Rod 04, Proposition 1.2]. It was completed and refined quite recently in dimension 2 [HP 04], [Di 05] after a few attempts [BBO 81, 82], [AK 83], [Bu 85], [G 98]. Thus work focused onto algebras of dimension 4 and 8.

Studies in [BBO 82], where the pseudo-octonions algebra $\mathbb{P}$ [Ok 78] played an important role, led to a classification of the FD real flexible\(^2\) division algebras [CDKR 99], [Da 06] generalizing [Zo 31].

Another generalizing result [Os 62], [Di 00] classifies the four-dimensional real quadratic\(^3\) division algebras, and despite a series of additional works in dimension 8 [DL 03], [Li 04], [DFL 06], [Di-Ru 10] the classification of eight-dimensional real quadratic division algebras remains an open problem [Da 10].

The class of power-commutative algebras contains the flexible [Raf 50\_2], and quadratic algebras. Note here that among the finite-dimensional real division algebras, power-associative algebras are the same as the quadratic algebras [Roe 94, Corollaire 2.27]. The problem of determining all real division algebras is more difficult in dimension 8 than on dimension 4 where the works are progressing relatively more quickly, as occurred for the power-commutative algebras [Da-Ro 11].

Third power-associative algebras, that are a natural generalization of power-commutative algebras, was already studied [Elm 83, 87], [EP 94] but apparently never in the context of division algebras. In the present work we begin our study with a third power-associative algebra over an arbitrary field of characteristic zero. We show that the additional assumption of the existence of a slightly generalized left-unit and that the algebra is FD of degree $\leq 4$,

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\(^1\)An algebra $A$ over an arbitrary field is said to be alternative if it satisfies $(x, x, y) = (y, x, x) = 0$ for all $x, y \in A$.

\(^2\)Algebra $A$ is said to be flexible if it satisfies $(x, y, x) = 0$ for all $x, y \in A$.

\(^3\)Algebra $A$ is said to be quadratic if it contains a unit element $e$ and for all $x \in A$ the elements $e, x, x^2$ are linearly dependent.
ensures the commutativity of powers (Lemma 3). We deduce that every third power-associative FD real division algebra, with a slightly generalized left-unit, having degree \(\leq 4\) is quadratic (Theorem 3).

Algebras satisfying the identity \((x^p, x^q, x^r) = 0\) for fixed \(p, q, r \in \{1, 2\}\) were also studied outside the general context of division algebras [CR 08], [CRR 11], [Elm 01], [EE 04]. Here we show first that every FD real division algebra, with left-unit, satisfying \((x, x, x^2) = 0\) is quadratic (Theorem 1). This remains true for every FD real division algebra, with slightly generalized unit, of degree \(\leq 4\) satisfying \((x^p, x^q, x^r) = 0\) (Theorem 2). Above result persists for every identity \((x, x^q, x^r) = 0\) if we replace the expression “slightly generalized unit” by “slightly generalized left-unit” (Theorem 3). A consequence of either Theorem 1 or Theorem 2 is that any third power-associative real division algebra with left-unit contains no elements of degree 4 (Corollary 2). In the other hand there are examples of FD real division algebras of degree 2 with a left-unit, satisfying \((x^2, x^q, x^r) = 0\) for all \(q, r \in \{1, 2\}\), and without right-unit (Remark 1 (2)). Hence the need, here, for the assumption of the existence of a two-sided unit.

2. Notations

Algebras \(A\) will be considered over a field \(\mathbb{K}\) of characteristic zero. They are assumed to have an arbitrary dimension in sections 2, 3 and 4. We denote by \((x, y, z)\) the associator \((xy)z - x(yz)\) of \(x, y, z \in A\). The subalgebra of \(A\) generated by every element \(x \in A\) is denoted by \(A(x)\).

1. We give here an extension of a unit (left-unit). Let \(e\) be a non-zero element in \(A\).

(a) The element \(e\) is called a generalized left-unit if it satisfies to \(e(xy) = x(ey)\) for all \(x, y \in A\). It is called a generalized unit if it satisfies to both \(e(xy) = x(ey)\) and \((xy)e = (xe)y\) for all \(x, y \in A\).

(b) The element \(e\) is called a square root of a left-unit if it satisfies to \(e(ex) = x\) for all \(x \in A\). It is called a square root of a unit if it satisfies to \(e(ex) = (xe)e = x\) for all \(x \in A\).

(c) Assume now that \(e\) is a non-zero idempotent. It is called a slightly generalized left-unit if it is both generalized left-unit and square root of a left-unit. It is called a slightly generalized unit if it is both a generalized unit and square root of a unit. ■
2. Algebra $A$ is called

(a) **third power-associative** if $(x, x, x) = 0$ for all $x \in A$,

(b) **power-associative** if $A(x)$ is associative for all $x \in A$,

(c) **power-commutative** if $A(x)$ is commutative for all $x \in A$.

3. $A$ is called a division algebra if it is FD and the operators $L_x$ and $R_x$ of left and right multiplication by $x$ are bijective for all $x \in A - \{0\}$.

4. An element $a$ in $A$ is said to be of degree $n \in \mathbb{N}$ if the sub-algebra $A(a)$ has dimension $n$. If $A$ is FD then the set $\{\dim(A(y)) : y \in A\}$ admit a bigger finite element $d$ called the degree of $A$ [Rod 94]. The $(1, 2, 4, 8)$-theorem shows that the degree of every real division algebra is 1, 2, 4 or 8. ■

Let now $A$ be one of classical real division algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions) or $\mathbb{O}$ (octonions), and $^*A$, $\hat{A}$, the standard isotopes of $A$ having $A$ as vectorial space and products $x \star y$ given respectively by $\overline{x}y$, $\overline{x} \overline{y}$ where $x \mapsto \overline{x}$ is the standard conjugation of $A$.

3. The identities $(x^p, x^q, x^r) = 0$

In this section $x \bullet y$ denotes $xy + yx$ for all $x, y \in A$.

Let now $p, q, r$ be natural numbers in $\{1, 2\}$, there are maps

$$f^{(m)}_{p,q,r} : A \times A \to A \quad \text{for} \quad m = 0, \ldots, p + q + r$$

such that the equality

$$\left((x + \lambda y)^p, (x + \lambda y)^q, (x + \lambda y)^r\right) = \sum_{k=0}^{p+q+r} \lambda^k f^{(k)}_{p,q,r}(x, y)$$

holds for all non-zero scalar $\lambda$ and $x, y$ in $A$. Moreover

$$f^{(m)}_{p,q,r}(y, x) = f^{(p+q+r-m)}_{p,q,r}(x, y) \quad \text{for all} \quad m \in \{0, \ldots, p + q + r\}.$$ 

Concretely $f^{(0)}_{p,q,r}(x, y) = (x^p, x^q, x^r) = f^{(p+q+r)}_{p,q,r}(y, x)$. For $p = q = r = 2$ we have
\[ f_{2,2,2}^{(1)}(x, y) = (x^2, x^2, x \cdot y) + (x^2, x \cdot y, x^2) + (x \cdot y, x^2, x^2) = f_{2,2,2}^{(5)}(y, x) \]

\[ f_{2,2,2}^{(2)}(x, y) = (x^2, x^2, y^2) + (x^2, x \cdot y, x \cdot y) + (x^2, y^2, x^2) + (x \cdot y, x^2, x \cdot y) + (x \cdot y, x \cdot y, x^2) + (y^2, y^2, x^2) = f_{2,2,2}^{(4)}(y, x) \]

\[ f_{2,2,2}^{(3)}(x, y) = (x^2, x \cdot y, y^2) + (x^2, y^2, x \cdot y) + (x \cdot y, x \cdot y, x \cdot y) + (x \cdot y, y^2, x^2) + (y^2, x^2, x \cdot y) + (y^2, x \cdot y, x^2) = f_{2,2,2}^{(3)}(y, x) \]

So

\[ f_{2,2,2}^{(m)}(y, x) = f_{2,2,2}^{(6-m)}(x, y) \quad \text{for all } m \in \{1, \ldots, 5\}. \]

The identity \((x^p, x^q, x^r) = 0\) in \(A\) is equivalent to \(f_{p,q,r}^{(m)}(y, x) \equiv 0\) for all \(m\). We denote \(f_{p,q,r}^{(m)} \equiv 0\) by \((p,q,r,m)\) and call it the \(m^{th}\) identity obtained by linearization of \((x^p, x^q, x^r) = 0\). The following three tables specify the cases \((p,q,r,1)\), \((p,q,r,2)\) and \((2,2,2,3)\) respectively:

<table>
<thead>
<tr>
<th>((x^p, x^q, x^r) = 0)</th>
<th>The first corresponding identity ((p,q,r,1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x, x, x) = 0)</td>
<td>((x, x, y) + (x, y, x) + (y, x, x) = 0)</td>
</tr>
<tr>
<td>((x, x, x^2) = 0)</td>
<td>((x, x, y) + (x, y, x^2) + (y, x, x^2) = 0)</td>
</tr>
<tr>
<td>((x, x^2, x) = 0)</td>
<td>((x, x^2, y) + (x, x \cdot y, x) + (y, x^2, x) = 0)</td>
</tr>
<tr>
<td>((x, x^2, x^2) = 0)</td>
<td>((x, x^2, x \cdot y) + (x, x \cdot y, x^2) + (y, x^2, x^2) = 0)</td>
</tr>
<tr>
<td>((x^2, x, x) = 0)</td>
<td>((x^2, x, y) + (x^2, y, x) + (x \cdot y, x, x) = 0)</td>
</tr>
<tr>
<td>((x^2, x, x^2) = 0)</td>
<td>((x^2, x, y) + (x^2, y, x^2) + (x \cdot y, x, x^2) = 0)</td>
</tr>
<tr>
<td>((x^2, x^2, x) = 0)</td>
<td>((x^2, x^2, y) + (x^2, x \cdot y, x) + (x \cdot y, x^2, x) = 0)</td>
</tr>
<tr>
<td>((x^2, x^2, x^2) = 0)</td>
<td>((x^2, x^2, x \cdot y) + (x^2, x \cdot y, x^2) + (x \cdot y, x^2, x^2) = 0)</td>
</tr>
</tbody>
</table>
The second corresponding identity (p.q.r.2)

\[
(x, x, x) = 0 \quad \Rightarrow \quad (x, x, y) + (x, y, x) + (y, x, x) = 0
\]

\[
(x, x^2) = 0 \quad \Rightarrow \quad (x, x, y) + (x, y, x) + (y, x, x) = 0
\]

\[
(x, x^2, x) = 0 \quad \Rightarrow \quad (x, x, y) + (x, y, x) + (y, x, x) = 0
\]

\[
(x, x^2, x^2) = 0 \quad \Rightarrow \quad (x, x, y) + (x, y, x) + (y, x, x) = 0
\]

\[
(x^2, x, x) = 0 \quad \Rightarrow \quad (x^2, y, y) + (x^2, x, y) + (x, y, x, x) = 0
\]

\[
(x^2, x^2) = 0 \quad \Rightarrow \quad (x^2, y, y) + (x^2, x, y) + (x, y, x, x) = 0
\]

\[
(x^2, x^2, x) = 0 \quad \Rightarrow \quad (x^2, x, y) + (x^2, y, x) + (x, y, x, x) = 0
\]

\[
(x^2, x^2, x^2) = 0 \quad \Rightarrow \quad (x^2, x, y) + (x^2, y, x) + (x, y, x, x) = 0
\]

The third corresponding identity (2.2.2.3)

\[
(x^2, x^2, x^2) = 0 \quad \Rightarrow \quad (x^2, x, y) + (x^2, y, x) + (x, y, x, x) = 0
\]

4. Preliminary results

We begin with the following useful preliminary result involving a square root of a left-unit:

**Lemma 1.** Let \( A \) be an algebra containing a square root \( e \) of a left-unit. Then the following two properties are equivalent:

1. The element \( ex - x \) is fixed by the operator \( L_e \) for all \( x \in A \).
2. \( e \) is a left-unit of algebra \( A \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( x \) be arbitrary in \( A \), we have:

\[
2(ex - x) = (ex - x) + L_e(ex - x)
\]

\[
= (ex - x) + e(ex - x)
\]

\[
= (ex - x) + (x - ex)
\]

\[
= 0.
\]
The implication \((2) \Rightarrow (1)\) is clear. ■

**Remark 1.** Let \(A\) be an algebra containing a generalized left-unit \(e\) which is an idempotent. Then every element of the form \(xe\) is fixed by the operator \(L_e\). Indeed, \(L_e(xe) = e(xe) = xe^2 = xe\). 

We have some relationships between the identities \((x^p, x^q, x^r) = 0:\)

**Proposition 1.** Every third power associative algebra satisfies the identity \((x, x^2, x) = 0\).

**Proof.** The result is well known [A 48, Lemma 2] and follows here from (1.1.1.1) by putting \(y = x^2\) and taking into account the characteristic of \(K\). ■

**Remark 2.** Third power-associativity is stronger than identity \((x, x^2, x) = 0\) [Ch 09, Remarque 1.17, p. 18]. However, we know no examples of division algebras satisfying the identity \((x, x^2, x) = 0\) which are not third power associative. ■

**Proposition 2.** Let \(A\) be an algebra, with unit \(e\), satisfying the identity \((x^p, x^q, x^r) = 0\) for fixed \(p, q, r\) in \(\{1, 2\}\). Then \(A\) is third power associative.

**Proof.** We can obviously assume that \((p, q, r) \neq (1, 1, 1)\). Then the result follows immediately for each one of the seven other cases of \((p, q, r)\) by putting \(y = e\) in the equality \((p.q.r.m)\), where \(m = p + q + r - 3\), and taking into account the characteristic of \(K\). ■

**Proposition 3.** Let \(A\) be an algebra with a left-unit \(e\), having no non-zero divisors of zero. If \(A\) satisfies the identity \((x, x^q, x^r) = 0\) for fixed \(q, r\) in \(\{1, 2\}\), then \(A\) has unit \(e\) and is third power associative.

**Proof.** We can assume that \((q, r) \neq (1, 1)\) and putting \(x = e\) in the equality \((1.q.r.1)\), we have: \(0 = (y, e, e) = (ye - y)e\). As \(A\) has no non-zero divisors of zero, \(e\) is an unit element and Proposition 2 concludes. ■

The identities \((x, x, x^2) = 0, (x^2, x, x) = 0\) have a particular importance:

**Proposition 4.** Let \(A\) be an algebra with left-unit \(e\), having no non-zero divisors of zero. If \(A\) satisfy the identity \((x, x, x^2) = 0\) then \(A\) has unit \(e\) and is power-associative.
Proof. Consequence of Proposition 3 and [A 48, Theorem 2]. ■

As consequence:

Theorem 1. Let $A$ be a finite-dimensional real division algebra, with left-unit $e$, satisfying the identity $(x, x, x^2) = 0$. Then $A$ has unit element $e$ and is a quadratic algebra.

Proof. $A$ is power-associative and [Roc 94, Corollaire 2.27] concludes. ■

We will improve the Propositions 2, 3, 4. To do this, we need the following preliminary results:

Lemma 2. Let $A$ be an algebra containing a slightly generalized left-unit $e$. Then the following equalities holds for all $y \in A$:

\[
\begin{align*}
(e, e, ey + ye) &= y - ey. \\
(e, ey + ye, e) &= R_e(y - ey). \\
(ey + ye, e, e) &= R_e((ey + ye)e - (ey + ye)). \\
(e, e, y) &= ey - y. \\
(e, y, e) &= R_e(ey - y). \\
(y, e, e) &= R_e(ye - y). \quad \blacksquare
\end{align*}
\]

We can now state the following curious result involving the identities $(x^p, x^q, x^r) = 0$:

Proposition 5. Let $A$ be an algebra containing a slightly generalized left-unit $e$ and let $p, q, r$ be fixed integers in $\{1, 2\}$. Assume that $A$ satisfies to the identity $(x^p, x^q, x^r) = 0$ then

1. $e$ is a left-unit.
2. If, moreover, $A$ has no non-zero divisors of zero, then $e$ is also a square root of a unit. In particular, when $A$ satisfies to the identity $(x, x^q, x^r) = 0$, then $e$ is a unit element.

Proof. We will use the equalities (pqr1).

1. By putting $x = e$ in the equality (2221) and taking into account the Lemma 2, we get:
\[0 = (e, e, ey + ye) + (e, ey + ye, e) + (ey + ye, e, e) \text{ for all } y \in A\]

\[= (y - ey) + R_e\left(y - ey + (ey + ye)e - (ey + ye)e\right) \text{ for all } y \in A.\]

The Remark 1 shows that the element

\[ey - y = R_e\left(y - ey + (ey + ye)e - (ey + ye)e\right)\]

is fixed by \(L_e\). So \(e\) is a left-unit by Lemma 1.

It should be noted that the reasoning is similar for the other identities (pqr1) where it is seen easily that any quantity \(ey - y\) is the image by \(R_e\) of an element belonging in \(A\).

2. We now assume that \(A\) has no zero divisors, and we can distinguish the following two cases

(a) If \(A\) satisfies to the identity \((x^2, x^q, x^r) = 0\), we put \(x = e\) in the equality (2qr1) and for all \(y \in A\) we get:

\[0 = (y + ye, e, e) = (ye)e - y e = 0.\]

The absence of non-zero divisors of zero shows that \((y)e = y\).

(b) If \(A\) satisfies to the identity \((x, x^q, x^r) = 0\), we put \(x = e\) in the equality (1qr1) and for all \(y \in A\) we get:

\[0 = (y, e, e) = (ye - y)e.\]

A simplification on the right by \(e\) shows that \(e\) is a (two-sided) unit. ■

As consequence:

**Corollary 1.** Let \(A\) be an algebra containing a slightly generalized unit \(e\) and let \(p, q, r\) be fixed integers in \(\{1, 2\}\). Assume that \(A\) satisfies to the identity \((x^p, x^q, x^r) = 0\) then \(e\) is a unit. ■
In this last section algebras $A$ are assumed to be finite-dimensional. We have the following preliminary result:

**Lemma 3.** Let $A$ be an algebra of degree $\le 4$ and having a slightly generalized left-unit $e$. Then $A$ is power-commutative in any one of the following cases:

1. $e$ is a slightly generalized unit and $A$ satisfies the identity $(x^p, x^q, x^r) = 0$ for fixed $p, q, r$ in $\{1, 2\}$.
2. $A$ has no non-zero divisor of zero and satisfies the identity $(x, x^q, x^r) = 0$ for fixed $q, r$ in $\{1, 2\}$.

**Proof.** According to the Corollary 1 the element $e$ is a (two-sided) unit. So the algebra $A$ is third power-associative by Proposition 2. Thus the result follows in case (1) by [Raf 50, Théorème 3, p. 578]. Also the result follows in case (2) from the one in case (1) and Proposition 5. ■

We can now state:

**Theorem 2.** Let $A$ be a real division algebra of degree $\le 4$, with slightly generalized unit $e$. Then the following assertions are equivalent:

1. $A$ satisfy the identity $(x^p, x^q, x^r) = 0$ for fixed $p, q, r$ in $\{1, 2\}$.
2. $A$ is power-associative.
3. $A$ is quadratic.

**Proof.** (1) $\Rightarrow$ (2). $A$ is power-commutative by Lemma 3, having a unit $e$ by Corollary 1. The result is then a consequence of Hopf’s commutative theorem [H 40, p. 238-239], [HKR 91, p. 235-236] and Yang-Petro’s theorem [Y 81, Theorem 1], [Pet 87].

(2) $\Rightarrow$ (3). See [Roc 94, Corollaire 2.27].

(3) $\Rightarrow$ (1) is obvious. ■

Theorem 2 remain true if one replaces the assumption ”$A$ has a slightly generalized unit” by ”$A$ has a slightly generalized left-unit” and identity ”$(x^p, x^q, x^r) = 0$” by ”$(x, x^q, x^r) = 0$”, thanks to Proposition 5:

**Theorem 3.** Let $A$ be a real division algebra of degree $\le 4$ having a slightly generalized left-unit. Then the following assertions are equivalent:

1. $A$ satisfy the identity $(x, x^q, x^r) = 0$ for fixed $q, r$ in $\{1, 2\}$.
2. $A$ is quadratic.
Corollary 2. Let $B$ be a third power-associative real division algebra with left-unit of arbitrary dimension. Then $B$ contains no elements of degree 4.

Proof. If $B$ contains an element $a$ of degree 4 then $B(a)$ has both dimension 4 and degree 4. In the other hand the left-unit being the only non-zero idempotent is contained in $B(a)$ [Se 54, Teorema 1 p. 162]. By Theorem 3 $B(a)$ is a quadratic algebra and must have dimension 2. Absurd. ■

Open problem 1. According to Corollary 2 the degree of every finite-dimensional third power-associative real division algebra with left-unit cannot be equal to 4. It is then natural to be known if there are examples of eight-dimensional third power-associative real division algebra with left-unit having degree 8.

We conclude this paper by the following

Remarks 1. Let $A$ any one of real division algebras $\mathbb{C}, \mathbb{H}, \mathbb{O}$.

1. The isotope standard $A^*$ of $A$ and the pseudo-octonion algebra $\mathbb{P}$ are flexible but not power-associative algebras. Thus the hypothesis of the existence of a slightly generalized left-unit in Theorem 3 is necessary.

2. The division algebra $A^*$, having left-unit and degree 2, satisfies all the identities $(x^2, x^a, x^r) = 0$ but is not third power-associative. There is also recent examples of real division algebras with left-unit and degree 4 satisfying $(x^2, x^2, x^2) = 0$, but not third power-associative [DRR 11]. So Proposition 3 and Theorem 3 does not have a similar for the identities: $(x^2, x^a, x^r) = 0$.

3. Taking into account Theorem 2 and the $(1, 2, 4, 8)$-theorem it is natural to wonder if there are third power-associative real division algebras of degree 8 having unit element. ■

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Division algebras


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