Notes on a Problem on Weakly Exponential Δ-Semigroups

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Abstract

Abstract. A semigroup $S$ is called a weakly exponential semigroup if, for every couple $(a, b) \in S \times S$ and every positive integer $n$, there is a non-negative integer $m$ such that $(ab)^{n+m} = a^n b^m = (ab)^m a^n b^n$. A semigroup $S$ is called a Δ-semigroup if the lattice of all congruences of $S$ is a chain with respect to inclusion. The weakly exponential Δ-semigroups were described in [5]. Although the existence of two types of them (T2R and T2L semigroups) is an open question, Theorem 3.11 of [5] gives necessary and sufficient conditions for a semigroup to be a T2R [T2L] semigroup. In our present paper we give a little correction of condition (v) of Theorem 3.11 of [5], and prove some new results which are addendum to the problem: Does there exist a T2R [T2L] semigroup?

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Introduction

A semigroup $S$ is called a $\Delta$-semigroup if the lattice of all congruences of $S$ is a chain with respect to inclusion. In the literature of the semigroups, there are lots of papers and a book which deal with $\Delta$-semigroups in special subclasses of the class of semigroups (see [1], [3] and [5] - [18]).

A semigroup $S$ is called a weakly exponential semigroup if, for every couple $(a, b) \in S \times S$ and every positive integer $n$, there is a non-negative integer $m$ such that $(ab)^{n+m} = a^n b^n (ab)^m = (ab)^m a^n b^n$ ([4]). The weakly exponential $\Delta$-semigroups were examined in [5]. A $\Delta$-semigroup $S$ is called a $T_1 [T_{2R}, T_{2L}]$ semigroup if $S$ is a semilattice of a non-trivial nil ideal $S_0$ and a subsemigroup $S_1$ which is a one-element semigroup [two-element right zero semigroup, two-element left zero semigroup]. It is clear that the $T_1 [T_{2R}, T_{2L}]$ semigroups are weakly exponential. In [5], it is proved that a semigroup is a weakly exponential $\Delta$-semigroup if and only if it is isomorphic to one of the following semigroups: (1) $G$ or $G^0$, where $G$ is a subgroup of a quasicyclic $p$-group, $p$ is a prime; (2) $B$ or $B^0$ or $B^1$, where $B$ is a two-element rectangular band; (3) a nil semigroup whose principal ideals are chain ordered by inclusion; (4) a $T_1$ semigroup or a $T_{2R}$ semigroup or a $T_{2L}$ semigroup.

Although the existence of $T_{2R} [T_{2L}]$ semigroups was not proved in [5], we characterized them in Theorem 3.11 of [5]. For an element $a$ of a semigroup $S$, let $J(a) = S^1 a S^1$ and $I(a) = J(a) - J_a$, where $J_a = \{ s \in S : J(s) = J(a) \}$. In Theorem 3.11 of [5] it was asserted that if $S$ is a $T_{2R}$ semigroup then $S$ satisfies the following condition (condition $(v)$ of Theorem 3.11 of [5]): for each $b \in S$, if $|J_b| = 2$ and $a \in I(b)$ then there are elements $x, y \in S^1$ such that $xJ_by \cap J_a \neq \emptyset$ and $xJ_by \nsubseteq J_a$. This condition needs a little correction, because in the proof of Theorem 3.11 of [5] we proved actually that if $S$ is a $T_{2R}$ semigroup then $S$ satisfies the following condition: for each $b \in S$, if $|J_b| = 2$, $I(b) \neq \{0\}$ and $a \in I(b)$ then there are elements $x, y \in S^1$ such that $xJ_by \cap J_a \neq \emptyset$ and $xJ_by \nsubseteq J_a$. When we proved in [5] that a semigroup $S$ satisfying conditions (i)-(v) of Theorem 3.11 of [5] is a $T_{2R}$ semigroup, condition $(v)$ of Theorem 3.11 was used for only such element $b$ of $S$, which satisfies both of $|J_b| = 2$ and $I(b) \neq \{0\}$. Thus the proof of Theorem 3.11 of [5] is basically the proof of the following theorem.

**Theorem 1** $S$ is a $T_{2R}$ semigroup if and only if it satisfies all of the following conditions.

1. $S$ is a semilattice of a non-trivial nil semigroup $S_0$ and a two-element right zero semigroup $S_1$ such that $S_0 S_1 \subseteq S_0$.

2. The ideals of $S$ form a chain with respect to inclusion.

3. For each $b \in S_0$, either $b \in bS_1$ or $bS_1 \subseteq S^1 b S_0$. 

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(4) For each \( b \in S_0 \), either \( \{0\} = S_1b \) or \( S_1b \cap (S_0bS^1 \cup S^1bS_0) \neq \emptyset \).

(5) For each \( b \in S \), if \( |J_b| = 2 \), \( I(b) \neq \{0\} \) and \( a \in I(b) \) then there are elements \( x, y \in S^1 \) such that \( xJ_by \cap J_a \neq \emptyset \) and \( xJ_by \nsubseteq J_a \).

In this paper, if \( S \) denotes a T2R semigroup then \( S_0 \) and \( S_1 \) will denote the subsemigroups of \( S \) appearing in condition (1) of Theorem 1. The elements of \( S_1 \) will be denoted by \( u \) and \( v \).

**Proposition 2** If \( b \) is an element of a T2R semigroup \( S \) such that \( |J_b| = 2 \) and \( I(b) = \{0\} \) then, for every \( x, y \in S^1 \), either \( 0 \notin xJ_by \) or \( xJ_by = \{0\} \). Moreover, \( J_bS_0 = S_0J_b = \{0\} \) and either \( S_1J_b = \{0\} \) or \( S_1J_b = J_b \).

**Proof.** Let \( b \) be an element of a T2R semigroup \( S \) such that \( |J_b| = 2 \) and \( I(b) = \{0\} \). Then \( b \in S_0 \). By Lemma 3.9 of [5], \( J_b = bS_1 = \{bu, bv\} \). By Lemma 2.7 of [1], \( J_b \) is a normal complex, that is, \( xJ_by \cap J_b \neq \emptyset \) implies \( xJ_by \subseteq J_b \) for every \( x, y \in S^1 \). As \( xJ_by \subseteq J(b) = J_b \cup \{0\} \), we get either \( 0 \notin xJ_by \) or \( xJ_by = \{0\} \) for every \( x, y \in S^1 \).

Next we show that \( J_bS_0 = S_0J_b = \{0\} \). If \( J_by \neq \{0\} \) for some \( y \in S_0 \) then, \( 0 \notin J_by \) and so \( buy \in J_b \). Thus \( buyu = bu \) from which we get \( bu(yu)^n = bu \) for every positive integer \( n \). As \( S_0 \) is a nil semigroup and \( yu \in S_0 \), we have \( bu = 0 \). This is a contradiction. Hence \( J_bS_0 = \{0\} \). If \( xJ_b \neq \{0\} \) for some \( x \in S_0 \) then, \( 0 \notin xJ_b \) and so \( xbu \in J_b \). Then \( xbu = bu \). From this we get \( x^nbu = bu \) for every positive integer \( n \). As \( x \in S_0 \) and \( S_0 \) is a nil semigroup, we get \( bu = 0 \). This is a contradiction. Hence \( S_0J_b = \{0\} \).

Next we show that \( uJ_b = \{0\} \) if and only if \( vJ_b = \{0\} \). Assume \( uJ_b = \{0\} \) and \( vJ_b \neq \{0\} \). Then \( 0 \notin vJ_b \) and so \( vbu \in J_b \). Then \( vbu = bu \) from this we get \( bu = vbu = uvbu = ubu = 0 \). This is a contradiction. Thus \( uJ_b = \{0\} \) implies \( vJ_b = \{0\} \). Similarly, \( vJ_b = \{0\} \) implies \( uJ_b = \{0\} \). Hence \( uJ_b = \{0\} \) iff \( vJ_b = \{0\} \).

Next we show that either \( S_1J_b = \{0\} \) or \( S_1J_b = J_b \). First of all, we note that \( S_1J_b = J_b \) is satisfied if and only if \( ef = f \) is satisfied for every \( e \in S_1 \) and \( f \in J_b \). Assume \( S_1J_b \neq \{0\} \). As \( uJ_b = \{0\} \) iff \( vJ_b = \{0\} \), \( uJ_b \neq \{0\} \) and \( vJ_b \neq \{0\} \). Thus \( 0 \notin uJ_b \) and \( 0 \notin vJ_b \) from which we get that, for every \( x \in S_1 \), there are elements \( y, z \in S_1 \) such that \( ubx = by \) and \( vbx = bz \). Then \( uw = w \) and \( vw = w \) for every \( w \in J_b \), that is, \( S_1J_b = J_b \).

**Corollary 3** If \( S \) is a T2R semigroup and \( b \in S_0 \) is arbitrary with \( |J_b| = 2 \) then \( S_0J_b \subseteq I(b) \), \( J_bS_0 \subseteq I(b) \) and either \( S_1J_b \subseteq I(b) \) or \( S_1J_b = J_b \).

**Proof.** Let \( b \in S_0 \) be an arbitrary element of a T2R semigroup \( S \) such that \( |J_b| = 2 \). By Lemma 2 of [17], the Rees factor semigroup of \( S \) by the ideal \( I(b) \) is a T2R semigroup, in which \( J(b) = J_b \cup \{0\} \). Thus our assertion follows from Proposition 2.
Proposition 4 If $S$ is a T2R semigroup then there is an element $b \in S_0$ such that $|J_b| = 2$.

Proof. Assume, in an indirect way, that $S$ is a T2R semigroup in which $|J_b| \neq 2$ for every $b \in S_0$. Then, by Lemma 3.9 of [5], $J_b = \{b\}$ for every $b \in S_0$.

First we show that $u$ and $v$ are left identity elements of $S$. Let $a \in S_0$ be an arbitrary element. Then $a \in I(u) = S_0 \neq \{0\}$. By (5) of Theorem 1, there are elements $x, y \in S^1$ such that $xJ_ay \cap J_a \neq \emptyset$ and $xJ_ay \not\subseteq J_a$. As $J_a = \{a\}$, we have $xuy = a$ and $xvy \neq a$ or $xuv \neq a$.

By the symmetry, we can consider only one of the above two cases. Assume, for example, $xuv = a$, $xvy \neq a$. If $x \in S_0$ then $xu \in SS_1$ and so (by Lemma 3.9 of [5]) $J_{xu} = xuS_1 = \{xu, xv\}$. As $xu \in S_0$, we have $|J_{xu}| = 1$ and so $xu = xv$. From this it follows that $xuv = xvy$ which is a contradiction. Thus $x \in S_1$ and so $xu = u$. From $uy = xuv = a$ we get $ua = a$ and so we also have $va = a$. Thus $u$ and $v$ are left identity elements of $S$.

By the previous part of the proof, if $a$ is an arbitrary element of $S_0$ then there is an element $y \in S_0$ such that $uy = a$ and $vy \neq a$ or $vy = a$ and $uy \neq a$. Both cases are impossible, because $uy = a$ is satisfied if and only if $y = a$ if and only if $vy = a$, because $u$ and $v$ are left identity elements of $S$. \qed

Proposition 5 If there exists a T2R semigroup then there exists a T2R semigroup $S$ which contains an element $b \in S_0$ with $|J_b| = 2$ and $I(b) = \{0\}$.

Proof. Suppose that there exist a T2R semigroup $H$ which is a semilattice of a non-trivial nil semigroup $H_0$ and a two-element right zero semigroup $H_1$. By Proposition 4, there is an element $b \in H_0$ such that $|J_b| = 2$. Denote $S$ the Rees factor semigroup $H/I(b)$ defined by the ideal $I(b)$. By Lemma 2 of [17], $S$ is a $\Delta$-semigroup. It is clear that $S$ is a T2R-semigroup in which $S_1 = H_1$ and $S_0 = H_0/I(b)$. Identifying the elements of $S - \{0\}$ and $H - I(b)$, for $b \in S_0$, we have (in $S$) $|J_b| = 2$ and $I(b) = \{0\}$.

\[ \square \]

Proposition 6 In every T2R semigroup $S$ there is an element $b \in S_0$ such that $ub \neq b$ and $vb \neq b$.

Proof. Assume, in an indirect way, that there is a T2R semigroup $S$ in which $ub = vb = b$ is satisfied for every $b \in S_0$. Let $b \in S_0$ be an arbitrary element with $|J_b| = 2$. By Proposition 4, such element exists. By (5) of Theorem 1, there are elements $x, y \in S^1$ such that $xJ_ay \cap J_b \neq \emptyset$ and $xJ_ay \not\subseteq J_b$. Let $b^* \in J_b$ denote the element for which $b^* \in xJ_ay$ is satisfied. Then $xuy = b^*$ or $xvy = b^*$. Consider the case $xuy = b^*$ (the proof is similar in the case $xvy = b^*$). By $xJ_ay \not\subseteq J_b$, we have $xvy \notin J_b$. Then $xuy = b^*$ and $xvy \neq b^*$ and so $uy \neq vy$ from which we get $y \notin S$, that is, $y = 1$. Then $xvy = xv = xuv = b^*v \in J_b$ which contradicts $xvy \notin J_b$. \[ \square \]
Proposition 7  In every T2R semigroup $S$, $S_0^2 = S_0$.

Proof. It is sufficient to show that, in every T2R semigroup $S$, $S_0^2 \neq \{0\}$. This implies our assertion, because if $S_0^2 \neq S_0$ was in a T2R semigroup $S$, then we would have $H_0^2 = \{0\}$ in the Rees factor semigroup $H = S/S_0^2$ of $S$ defined by the ideal $S_0^2$ of $S$ (which is a T2R semigroup in which $H_0 = S_0/S_0^2$).

Assume, in an indirect way, that there is a T2R semigroup $S$ in which $S_0^2 = \{0\}$. By Proposition 6, $uS_0 \neq S_0$ (and $vS_0 \neq S_0$). Let $a \in S_0 - uS_0$ be an arbitrary element. By (5) of Theorem 1, there are elements $x, y \in S^1$ such that $xJ_ay \cap J_a \neq \emptyset$ and $xJ_ay \subseteq J_a$. Let $a^* \in J_a$ denote the element for which $a^* \in xJ_ay$ is satisfied. Then $xuv = a^*$ or $xvy = a^*$. Consider the case $xuv = a^*$ (the proof is similar in case $xvy = a^*$). Then $xuv \neq a^*$. If $|J_a| = 1$ then $a = a^*$ and so $ua^* \neq a^*$. If $|J_a| = 2$ then $a \in J_a = J_{a^*} = \{a^*u, a^*v\}$ and so there is an element $x \in \{u, v\}$ such that $a = a^*x$. Then $ua^* \neq a^*$, because the opposite case implies $a = a^*x = (ua^*)x = u(a^*x) = ua$ which is a contradiction. Consequently (in both cases) $a^* \notin uS_0$. Thus, from the above equation $xuv = a^*$, it follows that $x \in S_0$. If $y = 1$ then $a^* = xv \in SS_1$ and so, by Lemma 3.9 of [5], $J_a = J_{a^*} = \{a^*u, a^*v\}$. Then $xvy = xv = xuv = a^*v \in J_{a^*} = J_a$ which is a contradiction. If $y \in S_1$ then $uy = vy$ and so $xuv = xvy = a^*$ which is also a contradiction. If $y \in S_0$ then, using also $x \in S_0$, we have $a^* = xuv \in S_0^2 = \{0\}$ from which we get $a^* = ua^* \in uS_0$. This is a contradiction. As in all cases we get a contradiction, the indirect assumption is not true. $\square$

Remarks (1): From Theorem 3.3 of [1] it follows that there is no a finite T2R semigroup. This result also follows from Proposition 7 of this paper, because every finite nil semigroup is nilpotent.

(2): A semigroup $S$ is called an $\mathcal{R}$-commutative semigroup if, for every $s, t \in S$, there is an element $r \in S^1$ such that $st = tsr$. If $b \in S_0$ is an arbitrary element of an $\mathcal{R}$-commutative T2R semigroup $S$ then, by $S\mu S = S$, there are elements $x, y \in S$ and $z \in S^1$ such that $b = xuv = uxzy$. Then $ub = b$. This contradicts Proposition 6. Consequently there is no an $\mathcal{R}$-commutative T2R semigroup.

(3): A semigroup $S$ is called a permutative semigroup if it satisfies a non-identity permutational identity. A semigroup $S$ is called a medial [left commutative] semigroup if it satisfies the identity $axyb = ayxb \ [xya = yxa]$ ($a, b, x, y \in S$). By Theorem 1 of [12], there is no a permutative T2R semigroup. This result also follows from Proposition 6 and Proposition 7 of this paper. By Theorem 4 of [12], every permutative $\Delta$-semigroup is medial. Thus it is sufficient to show that there is no a medial T2R semigroup. First we show that there is no a left commutative T2R semigroup. Assume, in an indirect way, that there is a left commutative T2R semigroup $S$. Let $x \in S_0$ be an arbitrary element. As $S\mu S = S$, there are elements $a, b \in S$ such that $x = aub = uab$ and
so \( ux = x \). By Proposition 6, it is impossible. In the next we prove that there is no a medial T2R semigroup. Assume, in an indirect way, that there is a medial T2R semigroup \( S \). It is clear that \( \varrho = \{ (a, b) \in S \times S : (\forall s \in S) sa = sb \} \) is a congruence of \( S \). Let \([x]\varrho\) denotes the \( \varrho \)-class of \( S \) containing the element \( x \) of \( S \). Then \([u]\varrho = \{ u \}\) and \([v]\varrho = \{ v \}\). If \([0]\varrho = S_0\) then \((a, 0) \in \varrho\) for every \( a \in S_0\). Thus, for every \( a, b \in S_0\), \( ba = b0 = 0 \) which means that \((S_0)^2 = \{0\}\). This result contradicts Proposition 7. Thus \([0]\varrho \neq S_0\) and so the factor semigroup \( S/\varrho\) of \( S \) is a T2R semigroup (see also Lemma 2 of [17]). As \( sxyb = syxb \) is satisfied for every \( s, x, y, b \in S \), we have \((xyb, yxb) \in \varrho\) for every \( x, y, b \in S \). Thus the T2R semigroup \( S/\varrho\) is left commutative. This is a contradiction.

References


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