On Number of Subgroups of

Finite Abelian Group \( \mathbb{Z}_m \otimes \mathbb{Z}_n \)

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Abstract

In this paper, we determine the number of subgroups of group \( \mathbb{Z}_{p^m} \otimes \mathbb{Z}_{p^n} \) which may be cyclic or non-cyclic by using simple number-theoretic formulae.

Keywords: Subgroup;

1. Introduction

Consider a finite abelian group \( \mathbb{Z}_m \otimes \mathbb{Z}_n \) of order \( mn \). If \( m \) and \( n \) are relatively prime then \( \mathbb{Z}_m \otimes \mathbb{Z}_n \) is cyclic, otherwise non-cyclic. In [1], if group \( \mathbb{Z}_m \otimes \mathbb{Z}_n \) is cyclic then number of subgroup is equal number of divisor of \( mn \). In [2] (Murali and Makamba, see Lemma 3.5) for a prime \( p \), the number of nontrivial subgroups of
\[ G = Z_p \otimes Z_p \] is \( p+3 \). If \( p \) is prime then \( p+3 \) never equal to number of divisor of \( p^2 \). In [3] (CĂLUGĂREANU, GR.G) prove that number of subgroup of \( Z_4 \otimes Z_4 \) are 15. In [4] (MARIUS TĂRNĂUCEANU), the total number of subgroups of \( Z_p^{\alpha_1} \otimes Z_p^{\alpha_2} \) is \[ \frac{[(\alpha_2-\alpha_1+1)p^{\alpha_1+2}-p^{\alpha_1+2}-(\alpha_2-\alpha_1-1)p^{\alpha_1+2}-(\alpha_2+\alpha_1+3)p+(\alpha_2+\alpha_1+1)]}{(p-1)^2} \] where \( \alpha_1 \leq \alpha_2 \). In this paper we derive a formula which works in both the case either group \( Z_m \otimes Z_n \) cyclic or non-cyclic.

2. Preliminaries

**Theorem 2.1:** (The Fundamental Theorem of Arithmetic) Every positive integer greater than one can be written uniquely as a product of primes, with prime factors in the product written in order of non decreasing size.

**Theorem 2.2:** (The Fundamental Theorem of Finite Abelian Groups) Every finite Abelian group is a direct product of cyclic groups of prime-power order. Moreover, the factorization is unique except for rearrangement of factors.

**Theorem 2.3:** Prove that \( G \otimes H \approx H \otimes G \) where \( H \) and \( G \) both are groups.

**Theorem 2.4:** If \( a \) and \( b \) are elements abelian groups \( G_1 \) and \( G_2 \) respectively and their orders are finite as well as co-prime, then \( \langle a \rangle \otimes \langle b \rangle = \langle (a, b) \rangle \)

**Proof:** We have \( o((a,b))=\text{lcm}\{o(a), o(b)\}=o(a)o(b) \)

Hence subgroup \( \langle (a, b) \rangle \) has order \( o(a)o(b) \) which same order of \( \langle a \rangle \otimes \langle b \rangle \)

Now want to prove that \( \langle (a, b) \rangle = \langle a \rangle \otimes \langle b \rangle \)

Let \( x = (a^k, b^k) \) be any arbitrary element of \( \langle (a, b) \rangle \)

\[ x = (a^k, b^k) \Rightarrow \quad a^k \in \langle a \rangle \quad \text{and} \quad b^k \in \langle b \rangle \Rightarrow (a^k, b^k) \in \langle a \rangle \otimes \langle b \rangle \]

\[ x \in \langle a \rangle \otimes \langle b \rangle \]

Hence \( \langle (a, b) \rangle \subseteq \langle a \rangle \otimes \langle b \rangle \)

Let \( x = (a^k, b^l) \) be any arbitrary element of \( \langle a \rangle \otimes \langle b \rangle \) and without loss of generality assume that \( k \leq l \)

Case 1:-
If \( k=l \) then \( x = (a^k, b^k) \) then \( x \in \langle (a, b) \rangle \)

Case 2:- If \( k<l \) then \( x = (a^k, b^l) = (a^k, b^k)(e, b^{l-k}) \)

We know that \( o(a) \) and \( o(b) \) are co-prime, then there exists integers \( \alpha \) and \( \beta \) such that \( 1 = \alpha o(a) + \beta o(b) \)
Then \((e, b) = (e, b^{a \cdot o(a) + \beta \cdot o(b)}) = (e, (b^{o(a)})^{\alpha} (b^{o(b)})^{\beta}) = (e, (b^{o(a)})^{\alpha}) = (e, b^{o(a)})^{\alpha}\)

Here \((a, b) \in \langle(a, b)\rangle \implies (a, b)^{o(a)} \in \langle(a, b)\rangle \implies (e, b^{o(a)}) \in \langle(a, b)\rangle \implies (e, b^{t-k}) \in \langle(a, b)\rangle \implies (a^k, b^k) \in \langle(a, b)\rangle \implies (*)

Also \((a^k, b^k) \in \langle(a, b)\rangle \implies (**)\)

From (*) and (**), we get \((a^k, b^l) \in \langle(a, b)\rangle \implies x \in \langle(a, b)\rangle \)

On basis of case 1 and case 2, we conclude that \(\langle a \rangle \otimes \langle b \rangle \subseteq \langle(a, b)\rangle\)

Therefore \(\langle a \rangle \otimes \langle b \rangle = \langle(a, b)\rangle\)

**Theorem 2.5**: If order of abelian groups \(G_1\) and \(G_2\) are finite as well as co-prime, then every subgroup of \(G_1 \otimes G_2\) can be written as be written as external product of subgroup of \(G_1\) and subgroup of \(G_2\).

**Proof**: Assume \(H\) is any subgroup of \(G_1 \otimes G_2\), we have to prove that \(H\) can be written as external product of subgroup of \(G_1\) and subgroup of \(G_2\).

Firstly we prove that \(H\) must have a subgroup which can be written as external product of subgroup of \(G_1\) and subgroup of \(G_2\).

Here \(\{e_1, e_2\}\) is a subgroup of \(H\) and \(\{e_1, e_2\} \approx \{e_1\} \otimes \{e_2\}\)

Therefore, our first claim is proved.

Assume that \(H_1 \otimes H_2\) is the largest subgroup of \(H\) which can be written as external product of subgroup of \(G_1\) and subgroup of \(G_2\). There are two cases arises here.

(i) \(H_1 \otimes H_2 = H\)

(ii) \(H_1 \otimes H_2 \not\subseteq H\)

Case 1 :- If \(H_1 \otimes H_2 = H\), then nothing to prove.

Case 2: If \(H_1 \otimes H_2 \not\subseteq H\), then there exists \((a, b) \in H\) such that \((a, b) \not\in H_1 \otimes H_2\)

It is given that \(G_1\) is an abelian group; Therefore \(H_1\) and \(\langle a \rangle\) are also abelian subgroup of \(G_1\); Hence \(H_1 \langle a \rangle = \langle a \rangle H_1 \implies H_1 \langle a \rangle\) is a subgroup of \(G_1\)

Similarly we can prove that \(H_2 (b)\) is a subgroup of \(G_2\)

Then \(H_1 \langle a \rangle \otimes H_2 (b)\) is also subgroup of \(G_1 \otimes G_2\)

Now we have to show that \(H_1 \otimes H_2 \not\subseteq H_1 \langle a \rangle \otimes H_2 (b) \subseteq H\)

Let \((x, y) \in H_1 \otimes H_2 \implies x \in H_1\) and \(y \in H_2 \implies x \in H_1 \langle a \rangle\) and \(y \in H_2 \langle b \rangle\)

\(\implies (x, y) \in H_1 \langle a \rangle \otimes H_2 \langle b \rangle\)
Hence $H_1 \otimes H_2 \subseteq H_1 \langle a \rangle \otimes H_2 \langle b \rangle$  
.... (1)

It is given in this case that $(a, b) \notin H_1 \otimes H_2$

But $a \in \langle a \rangle \Rightarrow a \in H_1 \langle a \rangle$

Similarly, $b \in H_2 \langle b \rangle$

Hence $(a, b) \in H_1 \langle a \rangle \otimes H_2 \langle b \rangle$  
.... (2)

From (*) and (**), we get $H_1 \otimes H_2 \not\subseteq H_1 \langle a \rangle \otimes H_2 \langle b \rangle$  
.... (3)

Let $(x, y) \in H_1 \langle a \rangle \otimes H_2 \langle b \rangle$

$\Rightarrow x \in H_1 \langle a \rangle$ and $y \in H_2 \langle b \rangle$

$\Rightarrow x = h_1 a^k$ and $y = h_2 b^l$ where $h_1 \in H_1$, $h_2 \in H_2$ and $k, l \in Z$

$(x, y) = (h_1 a^k, h_2 b^l) = (h_1, h_2)(a^k, b^l)$  
.... (4)

It is also given that order of groups $G_1$ and $G_2$ are finite as well as co-prime, hence $o(a)$ and $o(b)$ are also co-prime and finite

By use of theorem 2.4, we get $\langle a \rangle \otimes \langle b \rangle \subseteq H$

Hence $(a^k, b^l) \in H$  
.... (5)

Also $(h_1, h_2) \in H_1 \otimes H_2 \subseteq H$  
.... (6)

By use of (4), (5) and (6) with use of concept that H is subgroup, we get

$(x, y) = (h_1 a^k, h_2 b^l) = (h_1, h_2)(a^k, b^l) \in H$

Hence $H_1 \langle a \rangle \otimes H_2 \langle b \rangle \subseteq H$  
.... (7)

From (3) and (7), we get $H_1 \otimes H_2 \not\subseteq H_1 \langle a \rangle \otimes H_2 \langle b \rangle \subseteq H$

Above result is a contraction with fact that $H_1 \otimes H_2$ is largest subgroup of H
which can be external product of two subgroups of $G_1$ and $G_2$. Hence H is itself
external product of two subgroups of $G_1$ and $G_2$.

But H is any arbitrary subgroup of $G_1 \otimes G_2$, hence every subgroup of $G_1 \otimes G_2$
can be written as be written as external product of subgroup of $G_1$ and subgroup of $G_2$.

3. New Number-theoretic Formula for number of subgroup of $Z_{p_1^\alpha_1} \otimes Z_{p_2^\alpha_2}$

**Theorem 3.1:**- Prove that the total number of subgroups of $Z_{p_1^\alpha_1} \otimes Z_{p_2^\alpha_2}$ are

$$\sum_{q|\langle p_1^\alpha_1, p_2^\alpha_2 \rangle} \tau \left( \frac{p_1^\alpha_1}{q} \right) \tau \left( \frac{p_2^\alpha_2}{q} \right) \Phi(q)$$

**Proof:**- Without loss of generality, assume that $\alpha_1 \leq \alpha_2$

Let $S = \sum_{q|\langle p_1^\alpha_1, p_2^\alpha_2 \rangle} \tau \left( \frac{p_1^\alpha_1}{q} \right) \tau \left( \frac{p_2^\alpha_2}{q} \right) \Phi(q)$ where $\alpha_1 \leq \alpha_2$
Subgroups of finite Abelian group

\[ S = \sum_{q \mid p^a} \tau \left( \frac{p^a}{q} \right) \tau \left( \frac{p^a}{q} \right) \Phi(q) \]

\[ S = \tau(p^{a_1}) \tau(p^{a_2}) \Phi(1) + \tau(p^{a_1-1}) \tau(p^{a_2-1}) \Phi(p) + \tau(p^{a_1-2}) \tau(p^{a_2-2}) \Phi(p^2) + \ldots + \tau(p^{a_1-a_1}) \tau(p^{a_2-a_2}) \Phi(p^{a_1}) \]

\[ S = (\alpha_1 + 1)(\alpha_2 + 1) + (\alpha_1)(\alpha_2)(p - 1) + (\alpha_1 - 1)(\alpha_2 - 1)p(p - 1) + (\alpha_1 - 2)(\alpha_2 - 2)p^2(p - 1) + \ldots + (\alpha_1 - \alpha_1 + 1)(\alpha_2 - \alpha_1 + 1)p^{a_1 - 1}(p - 1) \quad \ldots(1) \]

Multiply (1) by \( p \), we get

\[ Sp = (\alpha_1 + 1)(\alpha_2 + 1)p + (\alpha_1)(\alpha_2)(p - 1)p + (\alpha_1 - 1)(\alpha_2 - 1)p^2(p - 1) + (\alpha_1 - 2)(\alpha_2 - 2)p^3(p - 1) + \ldots + (\alpha_1 - \alpha_1 + 1)(\alpha_2 - \alpha_1 + 1)p^{a_1}(p - 1) \quad \ldots(2) \]

Then (2)-(1), we get

\[ S(p-1) = (\alpha_1 + 1)(\alpha_2 + 1)(p - 1) - (\alpha_1)(\alpha_2)(p - 1) + (\alpha_1 + \alpha_2 - 1 + 2(1))p(p - 1) + (\alpha_1 + \alpha_2 - 1 + 2(2))p^2(p - 1) + (\alpha_1 + \alpha_2 - 1 + 2(3))p^3(p - 1) + \ldots + (\alpha_1 + \alpha_2 - 1 + 2(\alpha_1 - 1))p^{a_1 - 1}(p - 1) + (\alpha_2 - \alpha_1 + 1)p^{a_1} \]

\[ S = (\alpha_1 + 1)(\alpha_2 + 1) - (\alpha_1)(\alpha_2) + (\alpha_1 + \alpha_2 + 1 - 2(1))p + (\alpha_1 + \alpha_2 + 1 - 2(2))p^2 + (\alpha_1 + \alpha_2 + 1 - 2(3))p^3 + \ldots + (\alpha_1 + \alpha_2 + 1 - 2(\alpha_1 - 1))p^{a_1 - 1} + (\alpha_2 - \alpha_1 + 1)p^{a_1} \]

\[ S = (\alpha_1 + \alpha_2 + 1)(1 + p + p^2 + \ldots + p^{a_1 - 1}) - 2(p + 2p^2 + 3p^3 + \ldots + (\alpha_1 - 1)p^{a_1 - 1}) + (\alpha_2 - \alpha_1 + 1)p^{a_1} \]

\[ S = (\alpha_1 + \alpha_2 + 1) \frac{p^{a_1 - 1}}{p - 1} - 2 \left( \frac{\alpha_1 p^{a_1}}{p - 1} - \frac{p(p^{a_1 - 1})}{(p - 1)^2} \right) + (\alpha_2 - \alpha_1 + 1)p^{a_1} \]

\[ S = \frac{(\alpha_1 + \alpha_2 + 1)(p^{a_1 - 1})(p - 1) - 2(\alpha_1 p^{a_1}(p - 1) - p(p^{a_1 - 1}) + (\alpha_2 - \alpha_1 + 1)p^{a_1}(p - 1)\right)^2}{(p - 1)^2} \]

\[ S = \frac{[(\alpha_2 - \alpha_1 + 1)p^{a_1 + 2} - (\alpha_2 - \alpha_1 - 1)p^{a_1 + 1} - (\alpha_2 + \alpha_1 + 3)p + (\alpha_2 + \alpha_1 + 1)]}{(p - 1)^2} \]
Which is same as in [4] (MARIUS TĂRNĂUCEANU), the total number of subgroups of $Z_{p^{\alpha_1}} \otimes Z_{p^{\alpha_2}}$ are

$$[(a_2-a_1+1)p^{\alpha_1+2}-(a_2-a_1-1)p^{\alpha_1+1}-(a_2+a_1+3)p+(a_2+a_1+1)] \ \text{where} \ \alpha_1 \leq \alpha_2$$

Hence, total number of subgroups of $Z_{p^{\alpha_1}} \otimes Z_{p^{\alpha_2}}$ are

$$\sum_{q|\langle p^{\alpha_1}, p^{\alpha_2} \rangle} \tau\left(\frac{p^{\alpha_1}}{q}\right) \tau\left(\frac{p^{\alpha_2}}{q}\right) \phi(q)$$

4. New Number-theoretic Formula for number of subgroup of $Z_m \otimes Z_n$

Theorem 4.1.- If order of abelian groups $G_1$ and $G_2$ are finite as well as co-prime, then number of subgroups of $G_1 \otimes G_2$ is product of number of subgroups of $G_1$ with number of subgroups of $G_2$.

Proof: - Total number of subgroups of $G_1 \otimes G_2$ which can be written as be written as external product of subgroup of $G_1$ and subgroup of $G_2$ is product of number of subgroups of $G_1$ with number of subgroups of $G_2$.

By use of theorem 2.5, every subgroup of $G_1 \otimes G_2$ can be written as be written as external product of subgroup of $G_1$ and subgroup of $G_2$. Hence there is no subgroup of $G_1 \otimes G_2$ which cannot be written as be written as external product of subgroup of $G_1$ and subgroup of $G_2$.

Hence total number of subgroups of $G_1 \otimes G_2$ is product of number of subgroups of $G_1$ with number of subgroups of $G_2$.

Corollary 1:- If $p$ and $q$ are different primes, then number of subgroup of group $Z_{p^a} \otimes Z_{q^b}$ are $\tau(p^a q^b)$

Proof:- Number of subgroup of $Z_{p^a}$ is $\tau(p^a)$ and Number of subgroup of $Z_{q^b}$ is $\tau(q^b)$

Here order of abelian groups $Z_{p^a}$ and $Z_{q^b}$ are finite as well as co-prime, then number of subgroups of $G_1 \otimes G_2$ is $\tau(p^a) \tau(q^b) = \tau(p^a q^b)$

Corollary 2:- If $p_1$ and $p_2$ are different primes, then number of subgroup of group $Z_{p_1^{a_1}} \otimes Z_{p_2^{a_2}} \otimes Z_{p_2^{\beta_1}} \otimes Z_{p_2^{\beta_2}}$ are

$$\sum_{q|\langle p_1^{a_1}p_2^{\beta_1}, p_1^{a_2}p_2^{\beta_2} \rangle} \tau\left(\frac{p_1^{a_1}p_2^{\beta_1}}{q}\right) \tau\left(\frac{p_1^{a_2}p_2^{\beta_2}}{q}\right) \phi(q)$$
Proof:- Here $G_1 = Z_{p_1^{a_1}} \otimes Z_{p_1^{a_2}}$ and $G_2 = Z_{p_2^{\beta_1}} \otimes Z_{p_2^{\beta_2}}$ which also satisfies condition given in the theorem, hence number of subgroup are

$$\sum q | (p_1^{a_1}, p_1^{a_2}) \sum r | (p_2^{\beta_1}, p_2^{\beta_2}) \tau \left( \frac{p_1^{a_1} p_2^{\beta_1}}{qr} \right) \tau \left( \frac{p_1^{a_2} p_2^{\beta_2}}{qr} \right) \Phi(q) \times \sum r | (p_2^{\beta_1}, p_2^{\beta_2}) \tau \left( \frac{p_2^{\beta_1}}{r} \right) \tau \left( \frac{p_2^{\beta_2}}{r} \right) \Phi(r)$$

Also $q$ and $r$ are always relative primes and $\Phi$ is a multiplicative function.

$$\sum q | (p_1^{a_1}, p_1^{a_2}) \sum r | (p_2^{\beta_1}, p_2^{\beta_2}) \tau \left( \frac{p_1^{a_1} p_2^{\beta_1}}{qr} \right) \tau \left( \frac{p_1^{a_2} p_2^{\beta_2}}{qr} \right) \Phi(q) \Phi(r)$$

We get the desired result.

**Theorem 4.2**- If finite abelian group $G_i$ for $i=1,2,\ldots,k$ and $(|G_i|, |G_j|) = 1 \forall i \neq j$, then number of subgroups of $G_1 \otimes G_2 \otimes G_3 \otimes \ldots \otimes G_k$ is

$$\prod_{i=1}^{k-1} (\text{Number of subgroup of } G_i)$$

Proof:- We use the principal of Mathematical Induction to prove this result.

Take $k=2$, then by use of theorem 4.1 we have

Number of subgroups of $G_1 \otimes G_2$ is $\prod_{i=1}^{2} (\text{Number of subgroup of } G_i)$

Let us assume that the given result is true for $k-1$, we have

Number of subgroups of $G_1 \otimes G_2 \otimes G_3 \otimes \ldots \otimes G_{k-1}$ is $\prod_{i=1}^{k-1} (\text{Number of subgroup of } G_i)$

We have to prove that the given result is true for $k$

Say $H=G_1 \otimes G_2 \otimes G_3 \otimes \ldots \otimes G_{k-1}$ and $K=G_k$

It is given that order of each $G_i$ for $i=1,2,\ldots,k-1$ is finite, therefore order $H$ is finite. It is also given that each $G_i$ for $i=1,2,\ldots,k-1$ is abelian group, therefore $H$ is also abelian group.

Here $(|G_1|, |G_k|) = 1$, $(|G_2|, |G_k|) = 1$ \ldots, $(|G_{k-1}|, |G_k|) = 1$ \Rightarrow $(|G_1| |G_2| \ldots |G_{k-1}| |G_k|) = 1$

\Rightarrow $(|G_1 \otimes G_2 \otimes G_3 \otimes \ldots \otimes G_{k-1}|, |G_k|) = 1$ \Rightarrow $(|H|, |K|) = 1$

Hence number of subgroups of $H \otimes K$ is (number of subgroup of $H$) $\times$ (number of subgroup with $K$).

$$= \prod_{i=1}^{k-1} (\text{Number of subgroup of } G_i) \times \text{Number of subgroup of } G_k$$

$$= \prod_{i=1}^{k} (\text{Number of subgroup of } G_i)$$
Corollary: - Number of subgroups of $Z_m \otimes Z_n$ are $\sum_{d|m,n} \tau \left( \frac{m}{d} \right) \tau \left( \frac{n}{d} \right) \Phi(d)$

Proof: - By use of The Fundamental Theorem of Arithmetic $m$ and $n$ can be written as $m = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3} \ldots p_r^{\alpha_r}$ and $n = p_1^{\beta_1}p_2^{\beta_2}p_3^{\beta_3} \ldots p_r^{\beta_r}$ throughout the paper. Here it is not necessary that all $\alpha_i$ and $\beta_i$ are not zero.

Hence, by use of The Fundamental Theorem of Finite Abelian Groups, we have $Z_m \approx Z_{p_1^{\alpha_1}} \otimes Z_{p_2^{\alpha_2}} \otimes Z_{p_3^{\alpha_3}} \otimes \ldots \otimes Z_{p_r^{\alpha_r}}$ and $Z_n \approx Z_{p_1^{\beta_1}} \otimes Z_{p_2^{\beta_2}} \otimes Z_{p_3^{\beta_3}} \otimes \ldots \otimes Z_{p_r^{\beta_r}}$

Then, we can write $Z_m \otimes Z_n \approx Z_{p_1^{\alpha_1}} \otimes Z_{p_2^{\alpha_2}} \otimes \ldots \otimes Z_{p_r^{\alpha_r}} \otimes Z_{p_1^{\beta_1}} \otimes Z_{p_2^{\beta_2}} \otimes \ldots \otimes Z_{p_r^{\beta_r}}$

Now apply theorem 2.3, we get

$Z_m \otimes Z_n \approx Z_{p_1^{\alpha_1}} \otimes Z_{p_2^{\alpha_2}} \otimes \ldots \otimes Z_{p_r^{\alpha_r}} \otimes Z_{p_1^{\beta_1}} \otimes Z_{p_2^{\beta_2}} \otimes \ldots \otimes Z_{p_r^{\beta_r}}$

Assume $G_1 = Z_{p_1^{\alpha_1}} \otimes Z_{p_1^{\beta_1}}$, $G_2 = Z_{p_2^{\alpha_2}} \otimes Z_{p_2^{\beta_2}}$, $\ldots$, $G_r = Z_{p_r^{\alpha_r}} \otimes Z_{p_r^{\beta_r}}$

Here each $G_i$ is abelian finite group and $|G_i| = 1 \forall i \neq j$

By use of theorem, number of subgroups of $Z_m \otimes Z_n$ is

$\prod_{i=1}^{k}(\text{Number of subgroup of } G_i) = \prod_{i=1}^{k}(\text{Number of subgroup of } Z_{p_i^{\alpha_i}} \otimes Z_{p_i^{\beta_i}})$

$= \prod_{i=1}^{k} \left( \sum_{q_i | p_i^{\alpha_i} \otimes p_i^{\beta_i}} \tau \left( \frac{p_i^{\alpha_i}}{q_i} \right) \tau \left( \frac{p_i^{\beta_i}}{q_i} \right) \Phi(q_i) \right) = \prod_{i=1}^{k} \left( \sum_{q_i | p_i^{\alpha_i} \otimes p_i^{\beta_i}} \tau \left( \frac{p_i^{\alpha_i}}{q_i} \right) \Phi(q) \right)$

$= \sum_{q | m,n} \tau \left( \frac{m}{q} \right) \tau \left( \frac{n}{q} \right) \Phi(q)$

[Here $q_1 q_2 \ldots q_k = q$ (say)]

We get the desired result.

References


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