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Characterization of Non-nilpotent Elements of the $\mathbb{Z}$-Module $\mathbb{Z}/(p_1^{k_1} \times \cdots \times p_n^{k_n})\mathbb{Z}$

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Abstract

We characterize non-nilpotent elements of the $\mathbb{Z}$-module $D_{\mathcal{K}} = \mathbb{Z}/(p_1^{k_1} \times \cdots \times p_n^{k_n})\mathbb{Z}$. The projective limit of non-nilpotent elements of the $\mathbb{Z}$-module $D_{\mathcal{K}}$ adjoined with zero is an ideal of the ring $\prod_{i=1}^{n} \mathbb{Z}_{p_i}$, where $\mathbb{Z}_{p_i}$ is the ring of p_i-adic integers. It turns out that the results that were obtained in [5] are just corollaries and Question 2.1 that was posed in [5] is also answered.

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1 Introduction

All modules considered are left unital modules, the rings are unital and associative. The notion of nilpotent elements of a module studied in this paper is quite new - it was first introduced in [1] and later studied in [5]. In [5] a characterization of non-nilpotent elements of the $\mathbb{Z}$-module $\mathbb{Z}/p^k\mathbb{Z}$ $k \in \mathbb{Z}^+$ was done. The motivation of this definition stems from the fact that one gets a nilpotent element in a ring R if R is not completely semiprime, (i.e., not reduced). A module $_RM$ is completely semiprime if for all $a \in R$ and every $m \in M$, $a^k m = 0$ for some $k \in \mathbb{Z}^+$ implies $am = 0$. So, if M is not completely semiprime then it contains at least one nilpotent element. Hence, a nonzero element m of an R-module M is nilpotent of degree k if there exists $a \in R$ and $k \in \mathbb{Z}^+$ such that $a^k m = 0$ and $am \neq 0$. The zero element of a module is taken to be nilpotent.

Let $A_k = \mathbb{Z}/p^k\mathbb{Z}$, the function $f_k : A_k \to A_{k-1}$ given by $a \pmod{p^k} \mapsto a \pmod{p^{k-1}}$ is a ring epimorphism with kernel $p^{k-1}A_k$. The non-nilpotent elements B_k of the $\mathbb{Z}$-module A_k for $k > 1$ are given by $B_k = \{np^{k-1}\}_{n=1}^{p-1}$. If $B_k^0 = B_k \cup \{0\}$, then B_k^0 is a zero ring under the usual addition modulo p^k and multiplication modulo p^k. For each $k > 1$, B_k^0 is an ideal of A_k and

$$\lim_{\leftarrow} B_k^0 \triangleleft \lim_{\leftarrow} A_k = \mathbb{Z}_p$$

where $\mathbb{Z}_p$ is the ring of p-adic integers, $\lim\limits_{\leftarrow}$ denotes the projective (inverse) limit and $\triangleleft$ denotes an ideal of a ring. For more about p-adic integers, the reader may consult [2], [3] and [4] among other sources. The natural question that arises is that, if we can characterize the non-nilpotent elements of the $\mathbb{Z}$-module $\mathbb{Z}/p^k\mathbb{Z}$ in terms of p-adic integers, can we also characterize the non-nilpotent elements of the general $\mathbb{Z}$-module $\mathbb{Z}/(p_1^{k_1} \times \cdots \times p_n^{k_n})\mathbb{Z}$? This question was posed in [5] and we here answer it in affirmative. It then follows that all the results that were obtained in [5] are just corollaries of what we have here.

Let $D_{\mathcal{K}} = \mathbb{Z}/(p_1^{k_1} \times \cdots \times p_n^{k_n})\mathbb{Z}$ with p_i prime and $k_i > 1$ for $1 \leq i \leq n$. We say $\mathcal{K}$ corresponds to the primes $p_1, \cdots p_n$ with powers $k_1, \cdots k_n$. The function $g_k : D_{\mathcal{K}} \to D_{\mathcal{K}-1}$ given by $d \pmod{p_1^{k_1} \cdots p_n^{k_n}} \mapsto d \pmod{p_1^{k_1-1} \cdots p_n^{k_n-1}}$ is a ring epimorphism with kernel $p_1^{k_1-1} \cdots p_n^{k_n-1} D_{\mathcal{K}}$. By the definition of g_k and f_k and the fact that $\mathbb{Z}/(p_1^{k_1} \times \cdots \times p_n^{k_n})\mathbb{Z} \cong \mathbb{Z}/p_1^{k_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_n^{k_n}\mathbb{Z}$, it is easy to see that $g_k = (f_1, f_2, \cdots, f_k)$. The non-nilpotent elements $E_{\mathcal{K}}$ of $D_{\mathcal{K}}$ are $E_{\mathcal{K}} = \{mp_1^{k_1-1} \cdots p_n^{k_n-1}\}_{m=1}^{p_1 p_2 \cdots p_n - 1}$. If $|E_{\mathcal{K}}|$ denotes the number of elements in $E_{\mathcal{K}}$ then for any given $t \in \mathbb{Z}^+$, $|E_{\mathcal{K}}| = |E_{\mathcal{K}+t}| = p_1 p_2 \cdots p_n - 1$, i.e., the number of non-nilpotent elements in $E_{\mathcal{K}}$ is independent of the k_is in $D_{\mathcal{K}}$ but dependent only on the primes p_i involved. Determination of non-nilpotent elements of the $\mathbb{Z}$-module $D_{\mathcal{K}}$ is amenable to computation by sage - thanks to Dr. Tom Denton who pointed out this to me.

Proposition 1.1 *Suppose $E_{\mathcal{K}}^0 = E_{\mathcal{K}} \cup \{0\}$, then $E_{\mathcal{K}}^0$ is a zero ring under the usual addition modulo $p_1^{k_1} \cdots p_n^{k_n}$ and multiplication modulo $p_1^{k_1} \cdots p_n^{k_n}$.*

Theorem 1.1 *For any $\mathcal{K}$ corresponding to the primes $p_1, p_2, \cdots, p_n$ with powers $k_1, k_2, \cdots, k_n$ respectively, we have*

$$E_{\mathcal{K}}^0 \cong B_{k_1}^0 \times B_{k_2}^0 \times \cdots \times B_{k_n}^0.$$

Proof: If $0 \neq x \in E^0_{\mathcal{K}}$, $x = mp_1^{k_1-1} \cdots p_n^{k_n-1}$ with $0 \neq m \in \mathbb{Z}/(p_1 \cdots p_n)\mathbb{Z}$. Since $E_{\mathcal{K}} \subseteq D_{\mathcal{K}}$, through the isomorphism $\phi : \mathbb{Z}/(p_1^{k_1} \times \cdots \times p_n^{k_n})\mathbb{Z} \to \mathbb{Z}/p_1^{k_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_n^{k_n}\mathbb{Z}$, x is mapped onto $(t_1 p_1^{k_1-1}, \cdots, t_n p_n^{k_n-1})$ where $mp_i^{k_i-1} \equiv t_i p_i^{k_i-1} (\text{mod } p_i^{k_i})$ for all $1 \leq i \leq n$. It is evident that $(t_1 p_1^{k_1-1}, \cdots, t_n p_n^{k_n-1}) \in B^0_{k_1} \times \cdots \times B^0_{k_n}$. Now let $0 \neq y \in B^0_{k_1} \times \cdots \times B^0_{k_n}$. $y = (y_1, \cdots, y_n)$ with $y_i = m_i p_i^{k_i-1}$ and $m_i \in \mathbb{Z}/p_i\mathbb{Z}$ for all $1 \leq i \leq n$. If $m = m_1 \times \cdots \times m_n$, then under the isomorphism ϕ above y corresponds to $mp_1^{k_1-1} \times \cdots \times p_n^{k_n-1} \in E^0_{\mathcal{K}}$. ∎

From the proof of Theorem 1.1 above, we have the following.

Corollary 1.1 *The restriction of the isomorphism ϕ on $E_{\mathcal{K}}$ is $B^0_{k_1} \times B^0_{k_2} \times \cdots \times B^0_{k_n}$, i.e., $\phi|_{E_{\mathcal{K}}} = B^0_{k_1} \times B^0_{k_2} \times \cdots \times B^0_{k_n}$.*

Proposition 1.2 *The sum of all elements in $E_{\mathcal{K}}$ is 0 (mod $p_1^{k_1} \cdots p_n^{k_n}$).*

Proof: $\sum_{m=1}^{p_1 p_2 \cdots p_n - 1} m(p_1^{k_1-1} \cdots p_n^{k_n-1}) = (p_1^{k_1-1} \cdots p_n^{k_n-1}) \sum_{m=1}^{p_1 p_2 \cdots p_n - 1} m = \frac{1}{2}(p_1^{k_1-1} \cdots p_n^{k_n-1})(p_1 \cdots p_n - 1)(p_1 \cdots p_n) = 0 \ (\text{mod } p_1^{k_1} \cdots p_n^{k_n})$. ∎

Proposition 1.3 *The rings $E^0_{\mathcal{K}}$ and $E^0_{\mathcal{K}-1}$ are isomorphic under*

$$np_1^{k_1} \cdots p_n^{k_n} \mapsto np_1^{k_1-1} \cdots p_n^{k_n-1}.$$

Theorem 1.2 *For any $\mathcal{K}$ corresponding to the primes $p_1, p_2, \cdots, p_n$ with powers $k_1, k_2, \cdots, k_n$ respectively, we have*

$$\varprojlim E^0_{\mathcal{K}} \lhd \varprojlim D_{\mathcal{K}} = \prod_{i=1}^{n} \mathbb{Z}_{p_i}.$$

Proof: We claim $E^0_{\mathcal{K}} \lhd D_{\mathcal{K}}$. This is easy to see since for each i, $B^0_{k_i} \lhd A_{k_i}$. Since $E^0_{\mathcal{K}} \lhd D_{\mathcal{K}}$, $\varprojlim E^0_{\mathcal{K}} \lhd \varprojlim D_{\mathcal{K}}$. But $\varprojlim D_{\mathcal{K}} = \prod_{i=1}^{n} \mathbb{Z}_{p_i}$, see [2, Example 2.3.11 p.39]. ∎

Corollary 1.2

$$\varprojlim E^0_{\mathcal{K}} = \prod_{i=1}^{n} p^{t_i} \mathbb{Z}_{p_i}$$

Proof: Any nonzero ideal of the p-adic integer $\mathbb{Z}_p$ is of the form $p^t \mathbb{Z}_p$ for some $t \in \mathbb{Z}$, see [3, Sec. 1.5 p.6]. The rest follows from Theorem 1.2. ∎

Proposition 1.4 *As $\mathbb{Z}$-modules, $E^0_{\mathcal{K}}$ and $\mathbb{Z}/(p_1 \cdots p_n)\mathbb{Z}$ are isomorphic under*

$$mp_1^{k_1-1} \cdots p_n^{k_n-1} \mapsto m(\text{mod } p_1 \cdots p_n)$$

but as rings, they are not isomorphic; the former is a zero ring (i.e., a product of any two elements is zero) and non-unital where as the latter is not a zero ring and has unity.

Corollary 1.3 *The $\mathbb{Z}$-module $\mathbb{Z}/(p_1 \cdots p_n)\mathbb{Z}$ is reduced, i.e., has no nonzero nilpotent elements.*

Proof: Follows from Proposition 1.4 and the fact that all the nonzero elements in $E^0_{\mathcal{K}}$ are non-nilpotent. ■

Proposition 1.5 *Every non-nilpotent element of the $\mathbb{Z}$-module D_k is nilpotent (with degree 2) in the ring D_k but not conversely.*

Proof: Suppose m is a non-nilpotent element of the $\mathbb{Z}$-module D_k. Then m is of the form $m = tp_1^{k_1-1} \cdots p_n^{k_n-1}$ for some $t \in \mathbb{Z}/(p_1 \times \cdots \times p_n)\mathbb{Z}$. Looking at m as an element of a ring D_k we get

$$m^2 = t^2 p_1^{2k_1-2} \cdots p_n^{2k_n-2} \cong\ 0 \ (\text{mod}\ p_1^{k_1} \cdots p_n^{k_n}).$$

To see that the converse does not hold, 2 is nilpotent in the ring $\mathbb{Z}/4\mathbb{Z}$ but not nilpotent in the $\mathbb{Z}$-module $\mathbb{Z}/4\mathbb{Z}$. ■

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