Decomposition and Order of the Maximal Abelian Quotients of Certain Cyclically Presented Groups

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Abstract

Let $b = \ell k \in \mathbb{Z}$. We consider the Maximal Abelian quotient, $A_n$, of the family of cyclically presented groups $G_n(b) = \langle x_i : (x_i^{-\ell}x_{i+2}^{-\ell})^k x_{i+1}(x_{i+1}^{-\ell}x_{i}^{-\ell})^k \rangle_n$.

We show that the order of the Maximal Abelian quotient is effectively computable, depending only on $n$ and $b$, but independent of the factorization of $b$. Results relating to the decomposition and order enumeration of $A_n$ are also presented. In particular, we show that $A_n$ is 2-generated and its order satisfy the recurrence relation $a_1 = 1$, $a_2 = 4b - 1$, $a_3 = (3b-1)^2$, $a_n = (3b-1)a_{n-1} - b(3b-1)a_{n-2} + b^3a_{n-3}$ for all $n \geq 4$.

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1 Introduction

Let $F_n$ denote the free group on $n$ generators $x_1, x_2, \ldots, x_n$ and let $\eta : F_n \to F_n$ be the automorphism given by $\eta(x_i) = x_{i+1 \mod n}$. Let $\omega \in F_n$ be a reduced word and define $G_n = F_n/N$, where $N$ is the normal closure of $\{\omega, \eta(\omega), \ldots, \eta^{n-1}(\omega)\}$.
Definition 1.1 A group $G$ is said to be cyclically presented if for some $n$ and cyclically reduced word $\omega$, $G \cong G_n(\omega)$ where $G_n(\omega)$ has the presentation

$$G_n(\omega) = \langle x_1, x_2, \ldots, x_n | \omega, \eta(\omega), \ldots, \eta^{n-1}(\omega) \rangle.$$ 

Let $a_i$ be the exponent sum of $x_i$ in $\omega$. The polynomial $f(x) = \sum_{i=0}^{n-1} a_ix^i$ is associated with the cyclically presented group. This polynomial often coincides with the Alexander polynomial of the fundamental group of certain knots. In this paper, we consider cyclically presented groups that are isomorphic to the fundamental groups of certain Takahashi Manifolds introduced by Takahashi[9]. Consider the family of cyclically presented groups

$$G_n(b) = G_n((x_{i+1}^{-\ell}x_i^\ell)^m x_{i+1}^{-\ell}(x_{i+1}^{-\ell}x_i^\ell)^m), b = m\ell$$

These groups are defined in Cavicchioli et al[1]. Kim and Vesin[6] obtained the following theorem.

Theorem 1.2 If $p_i = -\epsilon, \ell_i = \ell, q_i = 1$ and $m_i = m$ for each $i = 1, \ldots, n$, then the fundamental group of the Takahashi manifold $M_n(p_i/\ell_i; q_i/m_i)$ is isomorphic to $G_n(b)$

In this article we are concern with the cases where the cyclically presented groups $G_n(b)$ have associated polynomials of the form $f(t) = bt^2 - (2b - 1)t + b$. Each factorization of $b$ leads to new family of cyclically presented group as in Cavicchioli and Spaggiari[2]. Taking $b = 1$ we get that $S_n = \langle x_{i+2}x_i,x_{i+1} \rangle_n$ corresponding to the Sieradski groups with associate polynomial $f(t) = t^2 - t + 1$. When $b = 2$, we see that these groups represent a generalization of the groups $G_n(x_{i+1}^{-1}x_{i+2}x_i^{-1}x_{i+1}x_{i+2}x_i^{-1}x_{i+1}x_i)$ with Alexander polynomial $2t^2 - 3t + 2$ of the knot with 5 crossing, denoted by $5_2$ in Rolfsen[5], also see solan[8] and Solan and Stoddart[7]. In what follows we let $A_n(b)$ denote the maximal Abelian quotient of $G_n(b)$.

2 The order of $A_n(b)$ is effectively computable

It is of interest to know the order of $A_n(b)$? The prevailing questions are when is the order of $A_n(b)$ finite? When is the order infinite? Is it possible to effectively compute the order of $A_n(b)$? A well known result for the order of $A_n(b)$ is the following theorem, found in Johnson[1].

Theorem 2.1 Let $f$ be the polynomial associated with $G_n(b)$. If $A_n(b)$ is the Maximal Abelian quotient of $G_n$ with order $\circ(A_n(b))$, then $\circ(A_n(b)) = | \prod_{\xi^n=1} f(\xi) |$. 
The difficulty in obtaining the orders lies in the enumeration. In fact, even for \( f(t) = 2t^2 - 3t + 2 \), Johnson, Kim and O’Brien[4] remarked that there is no nice formula for the order of the Maximal Abelian quotient. Solan and Stoddart[7] found what can be considered as a nice formula for order enumeration. The work which follows generalizes work done by Solan and Stoddart[7] and Solan[8].

Using \( f(t) = bt^2 - (2b - 1)t + 1 \), we immediately get that \( \circ(A_1(b)) = 1, \circ(A_2(b)) = 4b - 1 \) and \( \circ(A_3(b)) = (3b - 1)^2 \). However, these computations depend on the product of evaluations of the polynomial \( f(t) \) at the roots of the polynomial \( g(x) = x^n - 1 \). In the following theorem we prove that order of \( A_n(b) \) is effectively computable, that is, its computation depends only on \( n \) and \( b \). In other words, the order of \( A_n(b) \) exist as a function of \( n \).

**Theorem 2.2** Let \( f(t) = bt^2 - (2b - 1)t + b \) be the polynomial associated with the cyclically presented group \( G_n(b) \) and let \( \circ(A_n(b)) \) be the order of the Maximal Abelian quotient \( A_n(b) \). Then

\[
\circ(A_n(b)) = 2b^n(1 - \cos n\theta),
\]

where \( \theta = \arccos\left(\frac{2b - 1}{2b}\right) \).

**Proof:** Let \( g_n = x^n - 1 \) and \( \alpha = e^{(\frac{2\pi i}{n})} \). From Theorem 2.1 we have that

\[
\circ(A_n(b)) = \pm \prod_{k=0}^{n-1} f(\alpha^k) = \pm \prod_{k=0}^{n-1} b[\alpha^k - \frac{2b - 1}{b} \alpha^k + 1]
\]

\[
= \pm b^n \prod_{k=0}^{n-1} \frac{1}{b} f(\alpha^k) = \pm b^n \prod_{k=0}^{n-1} (\alpha^k - \gamma_1)(\alpha^k - \gamma_2)
\]

\[
= \pm b^n \text{Res}(g_n, \frac{1}{b} f) = \pm b^n g(\gamma_1) g_n(\gamma_2)
\]

\[
= \pm b^n [(\gamma_1^n \gamma_2^n) - (\gamma_1^n + \gamma_2^n) + 1]
\]

\[
= b^n [2 - (\gamma_1^n + \gamma_2^n)]
\]

\[
= b^n [2 - 2 \cos n\theta]
\]

\[
= 2b^n [1 - \cos n\theta],
\]

where \( \cos \theta = \frac{2b - 1}{2b} \) and \( \gamma_1, \gamma_2 \) are roots of \( \frac{1}{b} f(t) \).

We will have need for the following lemma in the next theorem.

**Lemma 2.3** Suppose that \( b \geq 1 \) is an integer and \( \cos \theta = \frac{2b - 1}{2b} \). Then the following holds.

(a) \( b^2(3b - 1) \sin \theta - b^2(3b - 1) \sin 2\theta + b^3 \sin 3\theta = 0 \)
(b) \[-b^2(3b - 1) \cos \theta + b^2(3b - 1) \cos 2\theta + b^3 \cos 3\theta = -b^3\]

We are now in a position to give a recurrence relation for the order of \(A_n(b)\) in the following theorem.

**Theorem 2.4** Let \(b \geq 1\) be an integer and \(A_n(b)\) be the Maximal Abelian quotient of the cyclically presented group \(G_n(b)\). If \(a_n = \circ(A_n(b))\), then \(a_1 = 1, a_2 = 4b - 1, a_3 = (3b - 1)^2\) and \(a_n = (3b - 1)a_{n-1} - b(3b - 1)a_{n-2} + b^3a_{n-3}\) for all \(n \geq 4\).

**Proof:** From Theorem 2.2, \(\circ(A_n(b)) = 2b^n (1 - \cos n\theta)\), giving \(a_1 = 1, a_2 = 4b - 1\) and \(a_3 = (3b - 1)^2\). We obtain the result by directly applying Lemma 2.3 and verifying that

\[
(3b - 1) [2b^{n-1}(1 - \cos(n - 1)\theta)] - b(3b - 1) [2b^{n-2}(1 - \cos(n - 2)\theta)] + b^3 [2b^{n-3}(1 - \cos(n - 3)\theta)] = 2b^n(1 - \cos n\theta).
\]

3 The group \(A_n\) is 2-generated

In the next theorem presents an algebraic proof that the family of groups \(A_n(b)\) is 2-generated. We will need the following Lemma.

**Lemma 3.1** Let \(A_n(b)\) be the Maximal Abelian quotient of the cyclically presented group \(G_n(b)\). Suppose that \(b \geq 1\) is an integer and \(\cos \theta = \frac{2b-1}{2b}\). Then

\[
\circ(A_n(b)) \equiv \begin{cases} 
1 \mod b, & \text{if } n \text{ is odd} \\
-1 \mod b, & \text{if } n \text{ is even}
\end{cases}
\]

**Proof:** Recall that \(\cos n\theta = 2 \cos \theta \cos (n-1)\theta - \cos (n-2)\theta\). Thus, there are integers \(A_k\) such that \(2^{k-1}\) divides \(A_k\) and

\[
\cos n\theta = \begin{cases} 
2^{n-1}\cos^n \theta + \sum_{k=1}^{n-1} A_k \cos^k \theta, & \text{if } n \text{ is odd} \\
2^{n-1}\cos^n \theta + \sum_{k=1}^{n-1} A_k \cos^k \theta \pm 1, & \text{if } n \text{ is even}
\end{cases}
\]

Therefore, \(2b^n A_k \cos^k \theta = 2b^n A_k \left(\frac{2b-1}{2b}\right)^k \equiv 0 \mod b\) when \(k < n\). Hence,

\[
\circ(A_n(b)) = 2b^n (1 - \cos n\theta) \equiv -2^n b^n \cos^n \theta \equiv -2^n b^n \left(\frac{2b-1}{2b}\right)^n \equiv -(2b - 1)^n \mod b.
\]
Theorem 3.2 Let $b = \ell k \geq 1$ be an integer and consider the cyclically presented group $G_n(b) = G_n((x_i^{-\ell}, x_i^{\ell})^k x_{i+1} x_{i+1}^{-\ell} x_i^k)$. Suppose that $A_n(b)$ is the Maximal Abelian quotient of $G_n(b)$. Then, for each $n$, the group $A_n(b)$ is 2-generated.

**Proof:** Let $a_n = o(A_n(b))$. From Lemma 3.1 we have that $a_n \equiv \pm 1 \mod b$ depending on whether $n$ is odd or even. From the relations of $A_n(b)$ we have that $x_i x_{i+2} = x_{i+1}^{[2b-1] \pm a_n}$. This implies that $x_i = x_{i+1}^{\alpha} x_{i+2}^{\beta}$, for all $i$, where $\alpha$ and $\beta$ are generated by the recurrence relation $y_n = ky_{n-1} - y_{n-2}$. Hence $A_n(b)$ is two generated.

In our final theorem we give a direct sum decomposition of the family of groups $A_n(b)$.

**Theorem 3.3** Let $h \geq 1$ be an integer and suppose that $A_n(b)$ is the Maximal Abelian quotient of $G_n(b)$. For each $n$, set $o(A_n(b)) = a_n$. Let $b_n = \sqrt{\frac{a_n}{4b-1}}$ when $n$ is even and $c_n = \sqrt{a_n}$ when $n$ is odd. Then

$$A_n(b) \cong \begin{cases} \mathbb{Z}_{(4b-1)b_n} \oplus \mathbb{Z}_{b_n}, & \text{if } n \text{ is even} \\ \mathbb{Z}_{c_n} \oplus \mathbb{Z}_{c_n}, & \text{if } n \text{ is odd} \end{cases}$$

where $b_2 = 1, b_4 = 2b - 1, c_1 = 1, c_3 = 3b - 1$ and both $b_n$ and $c_n$ are obtained from the absolute values of $a_n$ in the recurrence relation $a_n = (2b-1)a_{n-1} - b^2a_{n-2}$.

We obtain the decomposition of $A_n(b)$ via the Smith-Normal form of the exponent-sum matrix of $A_n(b)$ and the Basis Theorem. We also note that the last theorem corrects an error in recurrence relation of Theorem 3.1 of Cavicchioli and Spaggiari[2].

**References**


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