On Multiplicative Rough Modules

Esra ÖZTÜRK and Senol EREN

Department of Mathematics, Faculty of Science and Art
Ondokuz Mayıs University, 55139, Samsun, Turkey

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Abstract

In recent several years, the rough set theory has been rapidly developed. The theory of rough set deal with the approximation of an arbitrary subset of a universe by two definable subsets called lower and upper approximations. A pair \((U, \theta)\) is called an approximation space if \(U \neq \emptyset\) and \(\theta\) is an equivalence relation on a universal set \(U\). Let \(S\) be a multiplicative subset of a commutative ring \(R\) with unity and \(M\) be an \(R\)-module. In this work we consider \(M \times S\) as a universal set, introduce the notion of lower and upper approximations and give some properties of them. We also study on rough approximations of an \(S^{-1}M, S^{-1}R\)-module.

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1 Introduction

Rough set theory was initiated by Pawlak [14] for dealing with vagueness and granularity in information systems. This theory is an extension of set theory in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. A key notion in Pawlak rough set model is an equivalence relation. The equivalence classes are the building blocks for the construction of lower and upper approximations. The lower approximation of a given set is the union of all the equivalence classes which have a non-empty intersection with the set. Rough set theory since its introduction in

2 Preliminaries

Definition 2.1 Let $R$ be a ring, a left $R$-module is an additive abelian group together with a function $R \times A \rightarrow A$ (the image of $(r,a)$ being denoted by $ra$) such that for all $r, s \in R$ and $a, b \in A$:

i $r(a + b) = ra + rb$

ii $(r + s)a = ra + sa$

iii $r(sa) = (rs)a$

Definition 2.2 Let $R$ be a ring, $A$ be an $R$-module and $B$ be a non-empty subset of $A$. $B$ is a submodule of $A$ provided that $B$ is an additive subgroup of $A$ and $rb \in B$ for all $r \in R, b \in B$.

Let $R$ be a commutative ring with identity element $M$ be an $R$-module. If now $P$ is a prime ideal and $S$ is its complement in $R$, then $S$ is a non-empty multiplicatively closed subset of $R$. We may therefore use $S$ to form the ring $S^{-1}R$ and at the same time construct the $S^{-1}R$-module $S^{-1}M$. We shall often describe this by saying that we are localizing at $P$.

Lemma 2.3 Let $X, Y$ be submodules of an $R$-module $M$ and $S^{-1}X, S^{-1}Y$ be considered as $S^{-1}R$-submodules of $S^{-1}M$. Then, $S^{-1}X = S^{-1}Y \iff X = Y$ for every maximal ideal $P$ of $R$. 
Definition 2.4 A pair of \((U, \theta)\) where \(U \neq \emptyset\) is a universal set and \(\theta\) is an equivalence relation on \(U\) is called an approximation space.

Definition 2.5 For an approximation space \((U, \theta)\) by a rough approximation in \((U, \theta)\) we mean a mapping \(\text{Apr} : \mathcal{P}(U) \rightarrow \mathcal{P}(U) \times \mathcal{P}(U)\) defined by for every \(X \in \mathcal{P}(U), \text{Apr}(X) = (\underline{\text{Apr}}(X), \overline{\text{Apr}}(X))\) where 
\[
\underline{\text{Apr}}(X) = \{x \in U \mid [x]_{\theta} \subseteq X\}, \overline{\text{Apr}}(X) = \{x \in U \mid [x]_{\theta} \cap X \neq \emptyset\}.
\]
\(\underline{\text{Apr}}(X)\) and \(\overline{\text{Apr}}(X)\) are called the lower and upper approximations respectively in space \((U, \theta)\). A pair \((A, B) \in \mathcal{P}(U) \times \mathcal{P}(U)\) is called a rough subset in \((U, \theta)\) if and only if \((A, B) = \text{Apr}(X)\) for some \(X \in \mathcal{P}(U)\). Note that a rough subset is called a rough set.

3 Main Results

In this section we study on a commutative ring with unity. We choose the universal set as a multiplicative set and investigate rough multiplicative set.

Definition 3.1 \(S\) is called a multiplicative subset of a ring \(R\) if \(S\) is a multiplicatively closed subset of \(R\) with \(1 \in S\) and \(0 \in S\).

Definition 3.2 Let \(S\) be a multiplicative subset of a ring \(R\) and \(M\) be an \(R\)-module. Define a relation ” \(\sim\) ” on \(M \times S\) by 
\[(m, s) \sim (m', s') \iff \exists u \in S, u(s'm - sm') = 0\]

This is an equivalence relation on \(T\), and we denote equivalence classes of \((m, s)\) by \(\frac{m}{s}\). Let \(S^{-1}M\) denote the set of all equivalence classes of \(M \times S\) with respect to this relation, i.e. \(S^{-1}M = \left\{ \frac{m}{s} \mid m \in M, s \in S \right\}\). For \(X \subseteq M \times S\), the sets
\[
\underline{S}(X) = \left\{ (m, s) \in M \times S \mid \frac{m}{s} \subseteq X \right\}, \overline{S}(X) = \left\{ (m, s) \in M \times S \mid \frac{m}{s} \cap X \neq \emptyset \right\}
\]
are called respectively lower and upper rough multiplicative approximation of \(X\) in approximation space \((M \times S, \sim)\).

Theorem 3.3 For every approximation space \((M \times S, \sim)\) and every subset \(A, B \subseteq M\), we have:

1. \(\underline{S}(A) \subseteq A \subseteq \overline{S}(A)\)
2. \(\underline{S}(\emptyset) = \emptyset = \overline{S}(\emptyset)\)
\[
\begin{align*}
3 & \quad \overline{S}(M \times S) = M \times S = \overline{S}(M \times S) \\
4 & \quad A \subseteq B \implies \overline{S}(A) \subseteq \overline{S}(B) \wedge \overline{S}(A) \subseteq \overline{S}(B) \\
5 & \quad \overline{S}(\overline{S}(A)) = \overline{S}(A) \\
6 & \quad \overline{S}(\overline{S}(A)) = \overline{S}(A) \\
7 & \quad \overline{S}(\overline{S}(A)) = \overline{S}(A) \\
8 & \quad \overline{S}(\overline{S}(A)) = \overline{S}(A) \\
9 & \quad \overline{S}(A) = (\overline{S}(A^c))^c \\
10 & \quad \overline{S}(A) = (\overline{S}(A^c))^c \\
11 & \quad \overline{S}(A \cap B) = \overline{S}(A) \cap \overline{S}(B) \\
12 & \quad \overline{S}(A \cap B) \subseteq \overline{S}(A) \cap \overline{S}(B) \\
13 & \quad S(\overline{S}(A \cup B)) \supseteq S(A \cup B) \\
14 & \quad \overline{S}(A \cup B) = \overline{S}(A) \cup \overline{S}(B) \\
15 & \quad \forall (m, s) \in M \times S, \overline{S}(\frac{m}{s}) = \overline{S}(\frac{m}{s}) \\
16 & \quad \overline{S}(A) = \overline{S}(B) \subseteq \overline{S}(A - B) \\
17 & \quad \overline{S}(A) \cap \overline{S}(B) \subseteq \overline{S}(A \cap B)
\end{align*}
\]

**Proof.** The proof is similar to theorem 2.1 of [10] and also see [3].

The following examples show that the converse of 12, 13, 16, 17 in proposition (3.3) are not true.

**Example 1.** Let \( R = M = \mathbb{Z}_4 \). The set \( S = \{1, 3\} \) is a multiplicative subset of \( R \). Let \( A = \{(3, 1)\} \) and \( B = \{(1, 3), (2, 3)\} \) be subsets of \( M \times S \). Then,

\[
\begin{align*}
\overline{S}(A) &= \{(1, 3), (3, 1)\} \\
\overline{S}(B) &= \{(1, 3), (3, 1), (2, 3), (2, 1)\} \\
\overline{S}(A) \cap \overline{S}(B) &= \{(1, 3), (3, 1)\} \\
\overline{S}(A \cap B) &= \emptyset \text{ since } A \cap B = \emptyset.
\end{align*}
\]

This shows that \( \overline{S}(A) \cap \overline{S}(B) \subseteq \overline{S}(A \cap B) \) is not true in general.

**Example 2.** Let \( R = M = \mathbb{Z}_4 \). The set \( S = \{1, 2\} \) is a multiplicative subset of \( R \). Let \( C = \{(1, 1)\} \) and \( D = \{(2, 2)\} \) be subsets of \( M \times S \). Then,

\[
\begin{align*}
\overline{S}(C) &= \emptyset = \overline{S}(D) \\
\overline{S}(C \cup D) &= \{(1, 1), (2, 2)\}
\end{align*}
\]
Example 3 Let $R = M = \mathbb{Z}_5$. The set $S = \{1, 2, 3, 4\}$ is a multiplicative subset of $R$. Let $A = \{(1, 4), (2, 3)\}$ and $B = \{(2, 3)\}$ be subsets of $M \times S$. Then,

- $S(A) = \emptyset$
- $S(B) = \{(4, 1), (1, 4), (2, 3), (3, 2)\}$
- $S(A) \cap S(B) = \emptyset$
- $S(A \cap B) = \{(4, 1), (1, 4), (2, 3), (3, 2)\}$

This shows that $S(A \cap B) \subseteq S(A) \cap S(B)$ is not true in general.

This shows that $S(A - B) \subseteq S(A) - S(B)$ is not true in general.

Lemma 3.4 Let $M$ be an $R$-module, $A$ and $B$ be two ideals of $R$. If $X$ is a non-empty subset of $M \times AB$, then

\begin{align*}
  i & \quad A \cap B(X) \subseteq \overline{AB}(X) \\
  ii & \quad \overline{AB}(X) \subseteq A \cap B(X)
\end{align*}

Proof.

i

\begin{align*}
A \cap B(X) &= \{(m, s) \in M \times AB \mid \frac{m}{s} = \overline{m, s}_{A \cap B} \subseteq X\} \\
\overline{AB}(X) &= \{(m, s) \in M \times AB \mid \frac{m}{s} = \overline{m, s}_{AB} \subseteq X\}
\end{align*}

We know the fact $AB$ is an ideal of $R$ which implies $AB \subseteq A \cap B$. In this situation for $(m, s) \in A \cap B(X), (m, s)_{AB} \subseteq (m, s)_{A \cap B}$. Actually assume that $(x, y)$ is an arbitrary element of $(m, s)_{AB}$, then, there exists $u \in AB$ which implies that $u(\overline{sx - ym}) = 0$. Since $AB$ is a subset of $A \cap B$, we get $u \in A \cap B$ and $u(\overline{sx - ym}) = 0$. And so $(x, y)$ is an element of $(m, s)_{A \cap B}$.

ii

\begin{align*}
\overline{AB}(X) &= \{(m, s) \in M \times AB \mid \frac{m}{s} = \overline{m, s}_{AB} \cap X \neq \emptyset\} \\
\overline{A \cap B}(X) &= \{(m, s) \in M \times AB \mid \frac{m}{s} = \overline{m, s}_{A \cap B} \cap X \neq \emptyset\}
\end{align*}

For an arbitrary element $(m, s) \in \overline{AB}(X), \overline{m, s}_{AB} \cap X \neq \emptyset$ and $AB \subseteq A \cap B$, we have $(m, s)_{A \cap B} \cap X \neq \emptyset$. Therefore we get $(m, s) \in \overline{A \cap B}(X)$ namely $\overline{AB}(X) \subseteq \overline{A \cap B}(X)$. 


Lemma 3.5 Let $R$ be an integral domain, $M$ be an $R$-module, $A$ and $B$ be two ideals of $R$ such that $R = A + B$. Then,

\[ i \ A \cap B(X) = AB(X) \]

\[ ii \ \overline{AB(X)} = \overline{A \cap B(X)} \]

Proof. (Straightforward)

\[ \Box \]

Lemma 3.6 Let $M$ be an $R$-module, $A$ and $B$ be two multiplicative subsets of $R$ such that $A \subseteq B$ and $\emptyset \neq X \subseteq M \times B$. Then,

\[ i \ B(X) \subseteq A(X) \]

\[ ii \ \overline{A(X)} \subseteq \overline{B(X)} \]

Proof. (Straightforward)

\[ \Box \]

Lemma 3.7 Let $M$ be an $R$-module, $A$ and $B$ be two multiplicative subsets of $R$ such that $A \subseteq B$ and $\emptyset \neq X \subseteq M \times (A \cap B)$. Then,

\[ i \ A(X) \cap B(X) \subseteq A \cap B(X) \]

\[ ii \ A \cap B(X) \subseteq \overline{A(X)} \cap \overline{B(X)} \]

Proof.

\[ i \ (m, s) \in \overline{A(X)} \cap \overline{B(X)} \implies (m, s) \in A(X) \cap B(X) \implies (m, s)_A \subseteq X \land (m, s)_B \subseteq X; (since \ A \cap B \subseteq A) \implies (m, s)_{A \cap B} \subseteq (m, s)_A \subseteq X \implies (m, s) \in A \cap B(X) \]

\[ ii \ (m, s) \in A \cap B(X) \implies (m, s)_{A \cap B} \cap X \neq \emptyset \implies \exists (x, y) \in (m, s)_{A \cap B} \cap X \implies (x, y) \in (m, s)_A \land (x, y) \in (m, s)_B \land (x, y) \in X; (since \ A \cap B \subseteq A). \ Therefore, (m, s)_A \cap X \neq \emptyset \land (m, s)_B \cap X \neq \emptyset. \ Finally, (m, s) \in A(X) \cap B(X) \]

\[ \Box \]

Lemma 3.8 Let $M$ be an $R$-module, $A$ and $B$ be two submodules of $M$ and $S$ be a multiplicative set of $R$. Then $S^{-1}\overline{A}_{B}(A) = S^{-1}\overline{A}_{S^{-1}B}(S^{-1}A)$
Proof.

\[ S^{-1} \text{Apr}_B(A) = \left\{ \frac{x}{s} \in S^{-1}M \mid (x + B) \subseteq A, s \in S \right\} \]

\[ \text{Apr}_{S^{-1}B}(S^{-1}A) = \left\{ \frac{x}{s} \in S^{-1}M \mid \left( \frac{x}{s} + S^{-1}B \right) \subseteq S^{-1}A, s \in S \right\} \]

If \( \frac{x}{s} \in S^{-1} \text{Apr}_B(A) \), then \((x + B) \subseteq A\) and so \( S^{-1}(x + B) \subseteq S^{-1}A \). Since \( \frac{x}{s} + S^{-1}B \subseteq S^{-1}x + S^{-1}B \subseteq S^{-1}A \), \( \frac{x}{s} \in \text{Apr}_{S^{-1}B}(S^{-1}A) \). The converse of the theorem is clear from lemma 2.3. □

References


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