On the Semigroups of Local Homeomorphisms

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Abstract

In this paper we consider the semigroups of local homeomorphisms. Let \( X \) and \( Y \) be topological spaces. A map \( f : X \to Y \) is local homeomorphism if for every point \( x \in X \), there exists an open set \( U \) containing \( x \), such that \( f(U) \) is open in \( Y \) and \( f|_U : U \to f(U) \) is a homeomorphism. We give an abstract characterization of semigroups of local homeomorphisms defined on open sets of Euclidean \( n \)-space.

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1 Introduction

Many researchers have focused their efforts on the characterization of topological spaces by semigroups of continuous, open, closed, quasi-open mappings defined on these spaces [2], [3],[4], [5], [6]. In this paper we investigate semigroups of local homeomorphisms. A map \( f : X \to Y \) is local homeomorphism if for every point \( x \in X \), there exists an open set \( U \subset X \) containing \( x \), such that \( f(U) \) is open in \( Y \) and \( f|_U : U \to f(U) \) is a homeomorphism. Every local homeomorphism is a continuous and open map. If \( f \) and \( g \) are both local homeomorphisms, then the function composition is also local homeomorphism. Let \( LH(X) \) denote the semigroup of local homeomorphisms from a topological space \( X \) into itself with composition of functions as the semigroup operation. If \( X \) and \( Y \) are homeomorphic then the semigroups \( LH(X) \) and \( LH(Y) \) are
isomorphic. If \( \text{LH}(X) \) and \( \text{LH}(Y) \) are isomorphic, must \( X \) and \( Y \) be homeomorphic. In general, the answer is no. Let \( X \) denote any set with more than two elements and \( \xi \in X \). Consider the topological spaces \( Y = (X, \tau_1) \) and \( Z = (X, \tau_2) \) where \( \tau_1 = \{\emptyset, \{\xi\}, X\} \) and \( \tau_2 = \{\emptyset, X \setminus \{\xi\}, X\} \). Evidently \( \text{LH}(Y) \) and \( \text{LH}(Z) \) are isomorphic but \( Y \) and \( Z \) are not homeomorphic.

The purpose of this paper is to give an abstract characterization of semigroups of local homeomorphisms defined on an open set of Euclidean \( n \)-space.

\section{A Characterization of Semigroups of Local Homeomorphisms}

We denote by \( \mathbb{R}^n \) the \( n \)-dimensional Euclidean space with the standard topology. Let \( X \) be an open subset of \( \mathbb{R}^n \) and let \( \text{LH}_0(X) \) denote the set of all local homeomorphisms \( f \) of \( X \) for which \( f(X) \subseteq K_f \) for some compact subset of \( X \). The set \( \text{LH}_0(X) \) is an ideal of \( \text{LH}(X) \).

\textbf{Theorem 1} Let \( X \) and \( Y \) be open subsets of \( \mathbb{R}^n \) and \( \mathbb{R}^m \) respectively, \( (n, m > 1) \). The semigroups \( \text{LH}(X) \) and \( \text{LH}(Y) \) are isomorphic if and only if the spaces \( X \) and \( Y \) are homeomorphic.

\textbf{Proof.} It is obvious that if \( X \) and \( Y \) are homeomorphic then \( \text{LH}(X) \) and \( \text{LH}(Y) \) are isomorphic. Specifically, if \( h : X \to Y \) is a homeomorphism between \( X \) and \( Y \), then \( f \to hfh^{-1} \) is an isomorphism between \( \text{LH}(X) \) and \( \text{LH}(Y) \). The necessity of the condition follows from Lemmas 2-6. \( \blacksquare \)

Throughout this paper, \( \varphi \) denotes an isomorphism between semigroups \( \text{LH}(X) \) and \( \text{LH}(Y) \). Without loss of generality we can assume that \( X \) and \( Y \) are bounded open sets.

\textbf{Lemma 2} Let \( f, g \in \text{LH}(X) \). Then, from \( g(X) \subseteq f(X) \) it follows that \( \varphi g(Y) \subseteq \varphi f(Y) \). In addition, if \( \varphi f \in \text{LH}_0(Y) \) then \( \varphi g(Y) \subseteq \varphi f(Y) \).

\textbf{Proof.} Let \( \varphi f = \tau \) for some \( \varphi, \tau \in \text{LH}(Y) \). Since \( \varphi \) is an isomorphism, there exist \( \varphi, \tau \in \text{LH}(X) \) such that \( \varphi = \varphi \varphi \) and \( \tau = \varphi \tau \). Then \( \varphi \varphi (\varphi f) = (\varphi \tau)(\varphi f) \) and \( \varphi (\varphi f) = \varphi (\tau f) \). Again, because \( \varphi \) is an isomorphism we get \( \varphi f = \tau f \). From \( g(X) \subseteq f(X) \) it follows that for any point \( x_1 \in X \) there is a point \( x_2 \in X \) such that \( g(x_1) = f(x_2) \). Then

\[ \varphi g(x_1) = \varphi (g(x_1)) = \varphi (f(x_2)) = \varphi f(x_2) = \tau f(x_2) = \tau (f(x_2)) = \tau (g(x_1)) = \tau g(x_1) \]

which shows that \( \varphi g = \tau g \). But since \( \varphi \) is an isomorphism we have \( \varphi (\varphi g) = \varphi (\tau g) \) and \( \varphi (\varphi (\varphi f)) = \varphi (\tau (\varphi f)) \). Hence \( \varphi (\varphi f) = \tau (\varphi f) \).

Now suppose that the condition \( \varphi g(Y) \subseteq \varphi f(Y) \) does not hold, i.e. the set \( \varphi g(Y) \setminus \varphi f(Y) \) is not empty. Let \( y' = (\varphi g) y \) be an arbitrary point of
\( \varphi g(Y) \setminus \varphi f(Y) \). Since the set \( \varphi g(Y) \setminus \varphi f(Y) \) is open there exists a closed \( n \)-ball \( E \subset \varphi g(Y) \setminus \varphi f(Y) \) centered at \( y' \). Let \( \phi \) be any homeomorphism of \( E \), which is constant on the boundary but \( \phi(y') \neq y' \). The map \( \gamma : Y \to Y \)
defined by

\[
\gamma(y) = \begin{cases} 
  y, & \text{if } y \in Y \setminus E \\
  \phi(y), & \text{if } y \in E 
\end{cases}
\]

is a homeomorphism of \( Y \).

Now let \( \mathscr{X} \) be a local homeomorphism of \( Y \). Then the map \( \tau' = \mathscr{X}\gamma \) is also a local homeomorphism and for every \( y \in Y \) we have

\[
\tau'(\varphi f)y = \tau'((\varphi f)y) = \mathscr{X}\gamma((\varphi f)y) = \mathscr{X}((\varphi f)y) = (\mathscr{X}(\varphi f))y
\]

But for the point \( y' = (\varphi g)y \in \varphi g(Y) \setminus \varphi f(Y) \) we have \( \mathscr{X}(y') \neq \tau'(y') \). Consequently,

\[
(\mathscr{X}(\varphi g))y = \mathscr{X}((\varphi g)y) = (\mathscr{X}(y')) \neq \tau'(y') = \tau'((\varphi g)y) = (\tau'(\varphi g))y
\]

This contradiction proves the first assertion.

For the second assertion notice that \( \varphi f(Y) \) is a closed and bounded set but \( \varphi g(Y) \) is an open set. ■

**Lemma 3** Let \( f, g \) be arbitrary elements of \( LH_0(X) \) such that \( f(X) \cap g(X) \neq \emptyset \) and \( \varphi f, \varphi g \in LH_0(Y) \). Then

\[
\text{Int} \left[ \overline{(\varphi f)Y} \cap \overline{(\varphi g)Y} \right] \neq \emptyset
\]

**Proof.** Let \( E \) be a closed \( n \)-ball in \( f(X) \cap g(X) \), let \( E_1 \) be a closed \( n \)-ball containing \( X \) and let \( \tau \) denote the homeomorphism from \( E_1 \) onto \( E \). Now let \( \tau \) denote the restriction of this homeomorphism to \( X \). Clearly, \( \tau \in LH_0(X) \) and we have \( \tau(X) \subset E \subset f(X) \), \( \tau(X) \subset E \subset g(X) \). By Lemma 2 we must have \( (\varphi \tau)(Y) \subset (\varphi f)(Y) \) and \( (\varphi \tau)(Y) \subset (\varphi g)(Y) \), i.e., \( (\varphi \tau)(Y) \subset (\varphi f)(Y) \cap (\varphi g)(Y) \). Therefore \( \text{Int} \left[ \overline{(\varphi f)Y} \cap \overline{(\varphi g)Y} \right] \neq \emptyset \). ■

Let \( x \in X \) and \( f_k \in LH(X) \) for \( k = 1, 2, \ldots \). We say that the sequence \( \{f_k\}_k \) of mappings converges to \( x \) if the following three conditions are satisfied

1. \( f_k \in LH_0(X) \), \( (\varphi f_k) \in LH_0(Y) \) for \( k = 1, 2, \ldots \),
2. \( f_{k+1}(X) \subset f_k(X) \) and \( (\varphi f_{k+1})(Y) \subset (\varphi f_k)(Y) \) for \( k = 1, 2, \ldots \),
3. \( \cap_{k=1}^\infty f_k(X) = \{x\} \).

For any \( x \in X \) there exists a sequence \( \{f_k\}_k \) converging to \( x \).
Lemma 4 Let \( x \in X \) and let \( \{f_k\}_{k=1}^{\infty} \) be the sequence converging to \( x \). There exists a unique point \( \theta x \in Y \) such that the sequence \( \{\varphi f_k\}_{k=1}^{\infty} \) converges to \( \theta x \) and the point \( \theta x \) does not depend on the choice of the sequence \( \{f_k\}_{k=1}^{\infty} \).

Proof. The sequence \( \{\varphi f_k\}_{k=1}^{\infty} \) satisfies the conditions 1 and 2. Indeed if \( (\varphi f_k) \in LH_0(Y) \) then \( \varphi^{-1}(\varphi f_k) = f_k \) and therefore \( \varphi^{-1}(\varphi f_k) \in LH_0(X) \). If \( (\varphi f_{k+1}) \subseteq (\varphi f_k) \subseteq (\varphi^{-1}(\varphi f_{k+1}))X \subseteq (\varphi^{-1}(\varphi f_k))X \). Let us show that
\[
\cap_{k=1}^{\infty} (\varphi f_k)(Y) = \cap_{k=1}^{\infty} (\varphi f_k)(Y)
\] (1)

For sake of contradiction, suppose that there exists a point \( y \) such that \( y \in \cap_{k=1}^{\infty} (\varphi f_k)Y \) but \( y \notin \cap_{k=1}^{\infty} (\varphi f_k)Y \). Then there exists a natural number \( m \) such that \( y \in (\varphi f_m)Y \setminus (\varphi f_m)Y \). Since \( (\varphi f_m)Y \subseteq (\varphi f_m)Y \) and \( y \notin (\varphi f_m)Y \) we must have \( y \notin (\varphi f_m)Y \) and hence \( y \notin \cap_{k=1}^{\infty} (\varphi f_k)Y \) contradicting the assumption that \( y \in \cap_{k=1}^{\infty} (\varphi f_k)Y \). The set \( \cap_{k=1}^{\infty} (\varphi f_k)Y \) is not empty as the intersection of nested closed sets and therefore the set \( \cap_{k=1}^{\infty} (\varphi f_k)Y \) is not empty.

Now let \( y \in \cap_{k=1}^{\infty} (\varphi f_k)Y \). Suppose that the set \( \cap_{k=1}^{\infty} (\varphi f_k)(Y) \) contains another point \( y' \). Let \( \{h'_k\}_{k=1}^{\infty} \) be a sequence which converges to \( y' \). Since \( y \in (\varphi f_i)Y \) and \( y \in h'_j(Y) \) for any \( i, j \), it follows that
\[
y \in (\varphi f_i)(Y) \cap h'_j(Y)
\] (2)

for all naturals \( i, j \) and \( \cap_{k=1}^{\infty} h'_k(Y) = \cap_{k=1}^{\infty} h'_k(Y) = \{y\} \). Then there exists a natural number \( m \) such that \( y' \notin h'_m(Y) \) whenever \( k \geq m \). Let \( h_j \) denote the mapping \( \varphi^{-1}(h'_j) \). Then \( (\varphi^{-1}h'_j)X \subseteq (\varphi^{-1}h'_j)X \) and hence \( h_{j+1}(X) \subseteq h_j(X) \). Now suppose that \( x \notin h_jX \) for some natural \( j \). Then \( x \notin h_{j+1}X \) and hence there exists a natural number \( i \) such that
\[
f_i(X) \cap h_{j+1}(X) = \emptyset
\] (3)

From (2) it follows that \( y \in (\varphi f_i)(Y) \cap h'_{j+1}(Y) \) and by Lemma 3 we get \( f_i(X) \cap h_{j+1}(X) \neq \emptyset \) which contradicts (3). Thus for any natural number \( j \) the point \( x \) belongs to \( h_j(X) \). Since the sequence \( \{f_k\}_{k=1}^{\infty} \) converges to \( x \) we must have \( f_k(X) \subseteq h_j(X) \) for every natural number \( k > k_j \), for some \( k_j \). By Lemma 2 we get \( (\varphi f_k)(Y) \subseteq (\varphi f_k)(Y) \) for \( k > k_j \) and therefore \( y' \notin (\varphi f_k)(Y) \) for \( k > k_j, j \geq m \). This contradiction proves that the set \( \cap_{k=1}^{\infty} (\varphi f_k)(Y) \) consists of one point and \( \cap_{k=1}^{\infty} (\varphi f_k)(Y) = \cap_{k=1}^{\infty} (\varphi f_k)(Y) = \{y\} \).

Let us denote the point \( y \) by \( \theta x \) and prove that the point \( \theta x \) does not depend on the choice of the sequence \( \{f_k\}_{k=1}^{\infty} \). Let \( \{g_k\}_{k=1}^{\infty} \) be another sequence converging to \( x \). Then for any natural number \( k \) there exists a natural number \( i_k \) such that \( f_i(X) \subseteq g_k(X) \) whenever \( i \geq i_k \). By Lemma 2 we have \( (\varphi f_i)(Y) \subseteq (\varphi g_k)(Y) \). Since the set \( \cap_{i=1}^{\infty} (\varphi f_i)(Y) \) consists of one point \( y \), then
\[ y \in (\varphi f_i)(Y) \text{ for any natural number } i, \text{ and } y \in \overline{(\varphi g_k)(Y)} \text{ for any natural number } k \text{ as well. We can similarly show that } \]
\[ \cap_{k=1}^{\infty} (\varphi g_k)(Y) = \cap_{k=1}^{\infty} \overline{(\varphi g_k)(Y)} \]

Hence \( \cap_{i=1}^{\infty} (\varphi f_i)(Y) = \cap_{k=1}^{\infty} (\varphi g_k)(Y) = \cap_{k=1}^{\infty} \overline{(\varphi g_k)(Y)} = \{ y \}, \) i.e. the point \( y = \theta(x) \) does not depend on the choice of the sequence \( \{ f_k \}_{k=1}^{\infty} \).

Let \( \theta : X \to Y \) denote the function which maps a point \( x \in X \) to the point \( y = \theta(x) \in Y \).

**Lemma 5** Let \( f \) be an element of \( LH_0(X) \) such that \( \varphi f \in LH_0(Y) \). If \( x \in f(X) \) then \( \theta(x) \in \varphi f(Y) \).

**Proof.** Let \( x \in f(X) \) and \( \{ f_k \}_{k=1}^{\infty} \) be a sequence converging to \( x \). Then there exists a natural number \( i \), such that \( f_k(X) \subset f(X) \) for all \( k \geq i \). The sequence \( \{ f_k \}_{k=1}^{\infty} \) also converges to \( x \) and by Lemma 4 it follows that the sequence \( \{ \varphi f_k \}_{k=1}^{\infty} \) converges to \( y \), where \( y = \theta(x) \in (\varphi f_i)(Y) \). Since \( f_i(X) \subset f(X) \), by Lemma 2 it follows that \( (\varphi f_i)(Y) \subset (\varphi f)(Y) \) and therefore \( y = \theta(x) \in \varphi f(Y) \).

**Lemma 6** The function \( \theta : X \to Y \) is a homeomorphism.

**Proof.** The map \( \theta : X \to Y \) is surjective. Indeed, let \( y \) be any point in \( Y \) and \( \{ f_k' \}_{k=1}^{\infty} \) be a sequence converging to \( y \). Then the sequence \( \{ \varphi^{-1} f_k' \}_{k=1}^{\infty} \) converges to some point \( x \) in \( X \) such that \( \theta x = y \).

Let us show that \( \theta \) is injective. Suppose that \( \theta x_1 = \theta x_2 = y' \) for some \( x_1, x_2 \in X \), \( x_1 \neq x_2 \). Let \( \{ f_k \}_{k=1}^{\infty} \) and \( \{ g_k \}_{k=1}^{\infty} \) be the sequences converging to \( x_1 \) and \( x_2 \), respectively. Then the sequences \( \{ \varphi f_k \}_{k=1}^{\infty} \) and \( \{ \varphi g_k \}_{k=1}^{\infty} \) converge to \( y' \), but then the sequences \( \{ \varphi^{-1} (\varphi f_k) \}_{k=1}^{\infty} = \{ f_k \}_{k=1}^{\infty} \) and \( \{ \varphi^{-1} (\varphi g_k) \}_{k=1}^{\infty} = \{ g_k \}_{k=1}^{\infty} \) must converge to a unique point. Hence \( x_1 = x_2 \) contradicting the assumption that \( x_1 \neq x_2 \).

Let us now show that \( \theta \) and \( \theta^{-1} \) are continuous. Let \( U \) be any open neighborhood of \( \theta x \), \( E \) be a closed \( n \)-ball in \( Y \) centered at \( \theta x \) such that \( E \subset U \), and \( \{ f_k \}_{k=1}^{\infty} \) be a sequence converging to some point \( x \). Then it follows from Lemma 4 that \( \cap_{k=1}^{\infty} (\varphi f_k)(Y) = \{ \theta x \} \) and \( (\varphi f_{k+1})(Y) \subset (\varphi f_k)(Y) \). Therefore \( (\varphi f_k)(Y) \subset E \) for some natural \( k \). Since \( x \in f_k(X) \) and the set \( f_k(X) \) is an open set, there exists an open neighborhood \( V \) of \( x \), such that \( V \subset f_k(X) \). Then it follows from Lemma 5 that \( \theta(V) \subset \theta(f_k X) \subset (\varphi f_k)(Y) \subset E \subset U \). Thus the function \( \theta \) is continuous. A similar proof shows that \( \theta^{-1} \) is continuous and thus \( \theta \) is a homeomorphism.
References


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