Relative Character Graphs Related to 

\( GL(2, q) \) and its Subgroups

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Abstract

Let \( G \) be a finite group and \( H \) be a subgroup of \( G \). T. Gnanaseelan and C. Selvaraj defined a relative character graph \( \Gamma(G, H) \) by using the irreducible characters of \( G \) and the restrictions of these characters to \( H \). In this paper, we study these graphs for \( GL(2, q) \) and its subgroups.

Mathematics Subject Classification: 20C99

Keywords: Relative character graph, irreducible character

1 Introduction

In this paper, all groups are finite unless otherwise stated. All characters of a group are assumed to be complex-valued. Let \( K_n \) be the complete graph with \( n \) vertices. For a group \( G \), let \( \text{Irr}(G) \) denote the set of all irreducible characters of \( G \). Let \( H \) be a subgroup of \( G \) and \( \chi \) be a character of \( G \). \( \chi_H \) is
its restriction to $H$. In [2], a graph called relative character graph is defined as follows:

**Definition 1.1** If $G$ is a finite group and $H$ is a subgroup of $G$, then the relative character graph denoted by $\Gamma(G, H)$ has the vertex set $V = \text{Irr}(G)$. Two vertices $\chi$ and $\psi$ are joined by an edge if $\chi_H$ and $\psi_H$ have at least one irreducible character of $H$ as a common constituent.

Its properties and some examples were studied in [2], [5], [6] and [7]. In this article, we focus on examples of relative character graphs for $GL(2, q)$ and its subgroups, where $q$ is an odd prime.

## 2 Character Theory of $GL(2, q)$ and its subgroups

### 2.1 Notations and Prerequisites

We shall follow the notations used in [4]. Let $\mathbb{F}_q$ be the finite field of order $q$.

For each $s \in \mathbb{F}_q^*$, let

$$sI = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}, \quad u_s = \begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix}.$$ 

For $s, t \in \mathbb{F}_q^*$ where $s \neq t$, let

$$d_{s,t} = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}.$$ 

Let $\mathbb{F}_{q^2}$ be the finite field of order $q^2$ and let $S = \{s \in \mathbb{F}_{q^2} \mid s^q = s\}$. Then $S$ is a subfield of $\mathbb{F}_{q^2}$ of order $q$ and hence is isomorphic to $\mathbb{F}_q$. From now on we shall identify the subfield $S$ of $\mathbb{F}_{q^2}$ with the field $\mathbb{F}_q$. Note that if $r \in \mathbb{F}_{q^2}$, then $r^{1+q}$ and $r + r^q$ are both in the subfield $\mathbb{F}_q$ of $\mathbb{F}_{q^2}$, see [1] for further reference.

Take $r \in \mathbb{F}_{q^2}^* \setminus \mathbb{F}_q^*$, let

$$v_r = \begin{pmatrix} 0 & 1 \\ -r^{1+q} & r + r^q \end{pmatrix}.$$ 

The conjugacy classes of $G$ can be described by Proposition 28.4 in [4]:

**Proposition 2.1** ([4]) There are $q^2 - 1$ conjugacy classes in $GL(2, q)$, described as follows:

<table>
<thead>
<tr>
<th>class rep. $g$</th>
<th>$sI$</th>
<th>$u_s$</th>
<th>$d_{s,t}$</th>
<th>$v_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>C_G(g)</td>
<td>$</td>
<td>$(q^2 - 1)(q^2 - q)$</td>
<td>$(q - 1)q$</td>
</tr>
<tr>
<td>No. of classes</td>
<td>$q - 1$</td>
<td>$q - 1$</td>
<td>$(q - 1)(q - 2)/2$</td>
<td>$(q^2 - q)/2$</td>
</tr>
</tbody>
</table>
The family of conjugacy class representatives $sI$ and $u_s$ are indexed by the elements $s \in \mathbb{F}_q^*$. The family of conjugacy class representatives $d_{s,t}$ is indexed by unordered pairs $\{s,t\}$ of distinct elements of $\mathbb{F}_q^*$. The family of conjugacy class representatives $v_r$ is indexed by unordered pairs $\{r,r^q\}$ of elements in $\mathbb{F}_q^{*2}\setminus\mathbb{F}_q^*$.

Let $\epsilon$ be a generator of the cyclic group $\mathbb{F}_q^{*2}$ and let $\omega = e^{2\pi i/(q^2-1)}$. Suppose that $r \in \mathbb{F}_q^{*2}$ and $r = \epsilon^m$ for some $m$. Let $\tau = \omega^m$. Then $r \mapsto \tau$ is an irreducible character of $\mathbb{F}_q^{*2}$. Note that every irreducible character of $\mathbb{F}_q^{*2}$ is of the form $r \mapsto r^j$ for some $j$.

### 2.2 Characters of $GL(2,q)$

The character table of $GL(2,q)$ is described by Theorem 28.5 in [4].

**Theorem 2.2** ([4]) By the notations in Section 2.1, the irreducible characters of $GL(2,q)$ are given by $\lambda_i$, $\psi_i$, $\psi_{i,j}$, $\chi_i$ as follows.

<table>
<thead>
<tr>
<th></th>
<th>$sI$</th>
<th>$u_s$</th>
<th>$d_{s,t}$</th>
<th>$v_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_i$</td>
<td>$\mathbb{F}_q^{*2}$</td>
<td>$\mathbb{F}_q^{*2}$</td>
<td>$(st)^i$</td>
<td>$\tau^{i(1+q)}$</td>
</tr>
<tr>
<td>$\psi_i$</td>
<td>$q\mathbb{F}_q^{*2}$</td>
<td>0</td>
<td>$(st)^i$</td>
<td>$-\tau^{i(1+q)}$</td>
</tr>
<tr>
<td>$\psi_{i,j}$</td>
<td>$(q+1)\mathbb{F}_q^{*2}$</td>
<td>$\mathbb{F}_q^{*2}$</td>
<td>$\tau^{i+j}$</td>
<td>$\tau^j + \tau^q$</td>
</tr>
<tr>
<td>$\chi_i$</td>
<td>$(q-1)\mathbb{F}_q^{*2}$</td>
<td>$-\mathbb{F}_q^{*2}$</td>
<td>0</td>
<td>$-(\tau^j + \tau^q)$</td>
</tr>
</tbody>
</table>

For $\lambda_i$, $\psi_i$, we have $0 \leq i \leq q-2$. For $\psi_{i,j}$, we have $0 \leq i < j \leq q-2$. For $\chi_i$, we first consider the set of integers $j$ with $0 \leq j \leq q^2-1$ such that $q+1$ does not divide $j$. If $j_1$, $j_2$ belong to this set and $j_1 \equiv j_2 \mod (q^2-1)$, we choose precisely one of $j_1$ and $j_2$ to belong to the indexing set for the character $\chi_i$. Hence there are $(q^2-q)/2$ characters $\chi_i$.

### 2.3 Characters of $B$

Let

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, c \in \mathbb{F}_q^*, b \in \mathbb{F}_q \right\}$$

We call $B$ the Borel subgroup of $GL(2,q)$. Note that $|B| = (q-1)^2 q$. The author is not aware of any reference for the conjugacy classes and character table of $B$. Hence they will be derived as follows.

It is clear that $sI$ commutes with every element in $B$ and hence each $sI$ forms an orbit of size 1 under the action of $B$ on itself by conjugation. There are $q-1$ of them.
For \( u_s, s \in \mathbb{F}_q^* \), an element belongs to the centralizer \( C_B(u_s) \) if and only if the matrix is of the form:

\[
\begin{pmatrix}
  a & b \\
  0 & a
\end{pmatrix}
\]

Hence \( |C_B(u_s)| = (q-1)q \) and each conjugacy class represented by \( u_s \) has size \( q - 1 \). There are \( q - 1 \) of such conjugacy classes.

For \( d_{s,t}, s \neq t \) and \( s, t \in \mathbb{F}_q^* \), an element belongs to its centralizer if and only if the matrix is of the form:

\[
\begin{pmatrix}
  a & 0 \\
  0 & d
\end{pmatrix}
\]

Hence \( |C_B(d_{s,t})| = (q-1)^2 \). Hence each class is of size \( q \). Note that the following equation

\[
\begin{pmatrix}
  0 & 1 \\
  1 & 0
\end{pmatrix}^{-1} d_{s,t} \begin{pmatrix}
  0 & 1 \\
  1 & 0
\end{pmatrix} = d_{t,s}
\]

holds in \( GL(2, q) \) but not in \( B \). The classes represented by \( d_{s,t} \) and \( d_{t,s} \) are different conjugacy classes in \( B \). There are \( (q-1)(q-2)/2 \) of them respectively.

The sum of the numbers of all elements from the conjugacy classes represented by \( sI, u_s, d_{s,t} \) and \( d_{t,s} \) is

\[(q-1)(1) + (q-1)(q-1) + (q-1)(q-2)(q) = (q-1)^2q.
\]

It is equal to \( |B| \) and it is the class equation on \( B \). Hence we have found all conjugacy classes of \( B \). We summarize the result as follows:

**Lemma 2.3** By the notations in Section 2.1, there are \( (q-1)q \) conjugacy classes in \( B \subset GL(2, q) \), described as follows:

<table>
<thead>
<tr>
<th>class rep. ( g )</th>
<th>( sI )</th>
<th>( u_s )</th>
<th>( d_{s,t} )</th>
<th>( d_{t,s} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>C_B(g)</td>
<td>)</td>
<td>( (q-1)^2q )</td>
<td>( (q-1)q )</td>
</tr>
<tr>
<td>No. of classes</td>
<td>( (q-1) )</td>
<td>( (q-1) )</td>
<td>( (q-1)(q-2)/2 )</td>
<td>( (q-1)(q-2)/2 )</td>
</tr>
</tbody>
</table>

where \( 1 \leq s < t \leq q - 1 \).

Then we shall look at the character table of \( B \).

Let \( B' \) be the derived subgroup of \( B \). We have

\[ B/B' = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{F}_q^* \right\} \]

This quotient group has order \( (q - 1)^2 \) and hence there are \( (q - 1)^2 \) irreducible linear characters of \( B \).

For \( 0 \leq i, j \leq q - 2 \), define \( \alpha_{i,j} : B \to C \) by sending \( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \) to \( \overline{a} \overline{d}^j \). Hence \( \alpha_{i,j} \) has the following values on each conjugacy class of \( B \):
Relative character graphs related to $GL(2, q)$ and its subgroups

<table>
<thead>
<tr>
<th>$\alpha_{i,j}$</th>
<th>$sI$</th>
<th>$u_s$</th>
<th>$d_{s,t}$</th>
<th>$d_{t,s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overline{s^{i+j}}$</td>
<td>$\overline{s^{i+j}}$</td>
<td>$\overline{s^{i}}$</td>
<td>$\overline{s^{t}}$</td>
<td></td>
</tr>
</tbody>
</table>

It is clear that $\alpha_{i,j}$ is a linear character. (Indeed, these are the characters lifted from $B/B'$.)

Note that the complex conjugate of $s$ is $(s)^{-1}$. A simple calculation shows that $\langle \alpha_{i,j}, \alpha_{i,j} \rangle_B = 1$ and hence $\alpha_{i,j}$, $0 \leq i, j \leq q - 2$ are all irreducible linear characters of $B$.

Let

$$D = \left\{ \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} \mid s, t \in F_q^* \right\}$$

For $0 \leq i \leq q - 2$, let $f_i : D \to C$ be the functions

$$f_i : g \mapsto (\det g)^i$$

Let $\beta_i = (f_i)^B$, the induced character of $f_i$ from $D$ to $B$. We use Proposition 21.23 in [4] to calculate $\beta_i$ for each conjugacy class representative as follows.

For $g = sI$, we have

$$\beta_i(g) = \frac{|C_B(g)|}{|C_D(g)|} f_i(g) = \frac{(q - 1)^2 q}{(q - 1)^2} f_i(g) = q f_i(g)$$

For $g = u_s$, we have

$$\beta_i(g) = 0$$

For $g = d_{s,t}$,

$$\beta_i(g) = \frac{|C_B(g)|}{|C_D(g)|} f_i(g) = \frac{(q - 1)^2}{(q - 1)^2} f_i(g) = f_i(g)$$

To conclude, $\beta_i$ has the following values on each conjugacy class:

<table>
<thead>
<tr>
<th>$\beta_i$</th>
<th>$sI$</th>
<th>$u_s$</th>
<th>$d_{s,t}$</th>
<th>$d_{t,s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q s^{2i}$</td>
<td>0</td>
<td>$\overline{s^i}$</td>
<td>$\overline{s^t}$</td>
<td></td>
</tr>
</tbody>
</table>

The following calculations

$$\langle \beta_i, \beta_i \rangle_B = \frac{1}{|B|} \sum_{sI} \sum_{g \in sI} \beta_i(g) \overline{\beta_i(g)} + \frac{1}{|B|} \sum_{d_{s,t}} \sum_{g \in d_{s,t}} \beta_i(g) \overline{\beta_i(g)}$$

$$= \frac{1}{(q - 1)^2 q^2} q^2(q - 1) + \frac{1}{(q - 1)^2 q} q(q - 1)(q - 2)$$

$$= \frac{q}{q - 1} + \frac{q - 2}{q - 1} = 2$$
show that $\beta_i$ is not irreducible. We calculate $\langle \beta_i, \alpha_{i,i} \rangle_B$ as follows.

$$\langle \beta_i, \alpha_{i,i} \rangle_B = \frac{1}{|B|} \sum_{sI} \sum_{g \in sI} \beta_i(g) \alpha_{i,i}(g) + \frac{1}{|B|} \sum_{d_{s,t}} \sum_{g \in d_{s,t}} \beta_i(g) \alpha_{i,i}(g)$$

$$= \frac{1}{(q-1)^2} q(q-1) + \frac{1}{(q-1)^2} q(q-1)(q-2)$$

$$= \frac{1}{q-1} + \frac{q-2}{q-1} = \frac{q-1}{q-1} = 1$$

Set $\gamma_i = \beta_i - \alpha_{i,i}$. Then we have

$$\langle \gamma_i, \gamma_i \rangle_B = \langle \beta_i, \beta_i \rangle_B - \langle \beta_i, \alpha_{i,i} \rangle_B - \langle \alpha_{i,i}, \beta_i \rangle_B + \langle \alpha_{i,i}, \alpha_{i,i} \rangle_B$$

$$= 2 - 1 - 1 + 1 = 1$$

Hence, $\gamma_i$ is an irreducible character for each $i$ in $0 \leq i \leq q-2$. $\gamma_i$ has the following values on each conjugacy class:

<table>
<thead>
<tr>
<th>$sI$</th>
<th>$u_s$</th>
<th>$d_{s,t}$</th>
<th>$d_{t,s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_i$</td>
<td>$(q-1)^{2i}$</td>
<td>$-s^{2i}$</td>
<td>0</td>
</tr>
</tbody>
</table>

However, $\gamma_i$ are not all different. More precisely, let $k = (q-1)/2$. For $0 \leq i \leq (q-3)/2$, $\gamma_i$ and $\gamma_{i+k}$ are the same characters since

$$s^{2(i+k)} = s^{2i+2k} = s^{2i+(q-1)} = s^{2i}$$

as $s \in \mathbb{F}_q^*$ and hence $s^{q-1} = 1$. It implies that the set $\{\gamma_i | 0 \leq i \leq q-2\}$ gives $(q-1)/2$ different non-linear irreducible characters of degree $(q-1)$. If we define $\delta_i$ by the following table:

<table>
<thead>
<tr>
<th>$sI$</th>
<th>$u_s$</th>
<th>$d_{s,t}$</th>
<th>$d_{t,s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_i$</td>
<td>$(q-1)^{2i}$</td>
<td>$-s^{2i}$</td>
<td>0</td>
</tr>
</tbody>
</table>

It is clear that all $\delta_i$ are distinct. The following calculation

$$\langle \delta_i, \delta_i \rangle_B = \frac{1}{|B|} \sum_{sI} \sum_{g \in sI} \delta_i(g) \delta_i(g) + \frac{1}{|B|} \sum_{u_s} \sum_{g \in u_s} \delta_i(g) \delta_i(g)$$

$$= \frac{1}{(q-1)^2} (q-1)(q-1)^2 + \frac{1}{(q-1)^2} (q-1)(q-1) = 1$$

shows that $\delta_i$ is irreducible for all $0 \leq i \leq q-2$. Hence we have $(q-1)$ different irreducible non-linear characters of $B$. Note that $\delta_i$ for even $i$ is the same as $\gamma_{i/2}$. Based on $\alpha_{i,j}$ and $\delta_i$, we have derived a total of $(q-1)^2 + (q-1) = (q-1)q$ irreducible characters of $B$. We summarize the result as follows:
Theorem 2.4 By the notations in Section 2.1, there are \((q-1)q\) irreducible characters in \(B\), described as follows:

<table>
<thead>
<tr>
<th>(sI)</th>
<th>(u_s)</th>
<th>(d_{s,t})</th>
<th>(d_{t,s})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha_{i,j})</td>
<td>(\bar{s}^i j)</td>
<td>(\bar{s}^i j)</td>
<td>(\bar{s}^i j)</td>
</tr>
<tr>
<td>(\delta_i)</td>
<td>((q-1)\bar{s}^i)</td>
<td>(-\bar{s}^i)</td>
<td>0</td>
</tr>
</tbody>
</table>

where \(0 \leq i, j \leq q - 2\).

The following definition will be used in the last section.

Definition 2.5 If \(k < 0\), then we define \(\delta_k\) as a character of \(B\) by

\(\delta_k \equiv \delta_{k'}\)

where \(0 \leq k' \leq q - 2\) and \(k' \equiv k \mod (q - 1)\).

If \(k > q - 2\), then we define \(\delta_k\) as a character of \(B\) by

\(\delta_k \equiv \delta_{k''}\)

where \(0 \leq k'' \leq q - 2\) and \(k'' \equiv k \mod (q - 1)\).

2.4 Characters of \(D\)

For

\[D = \left\{ \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} \mid s, t \in \mathbb{F}_q^* \right\},\]

we call \(D\) the diagonal subgroup of \(GL(2,q)\). Note that \(D\) is an abelian subgroup of \(GL(2,q)\) and is of order \((q - 1)^2\). Hence there are \((q - 1)^2\) irreducible linear characters of \(D\). All conjugacy classes are of size 1 and, by the notations above, are given by \(sI\), \(d_{s,t}\) and \(d_{t,s}\) where \(1 \leq s < t \leq q - 1\). We have the following character table for \(D\).

Lemma 2.6 By the notations in Section 2.1, there are \((q-1)^2\) irreducible linear characters of \(D\) which are given by \(f_{i,j}\) as follows.

<table>
<thead>
<tr>
<th>(f_{i,j})</th>
<th>(sI)</th>
<th>(d_{s,t})</th>
<th>(d_{t,s})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\bar{s}^i j)</td>
<td>(\bar{s}^i j)</td>
<td>(\bar{s}^i j)</td>
<td>(\bar{s}^i j)</td>
</tr>
</tbody>
</table>

where \(0 \leq i, j \leq q - 2\).

The following definition will be used in the last section.

Definition 2.7 If \(k < 0\), then we define \(f_{i,k}\) as a character of \(D\) by

\(f_{i,k} \equiv f_{i,k'}\)

where \(0 \leq k' \leq q - 2\) and \(k' \equiv k \mod (q - 1)\).

If \(k > q - 2\), then we define \(f_{i,k}\) as a character of \(D\) by

\(f_{i,k} \equiv f_{i,k''}\)

where \(0 \leq k'' \leq q - 2\) and \(k'' \equiv k \mod (q - 1)\).
3 Relative character graphs

This section is devoted to the study of certain subgraphs of $\Gamma(GL(2, q), B)$ and $\Gamma(B, D)$.

3.1 $\Gamma(GL(2, q), B)$

To illuminate our discussion, let

$$S = \text{Irr}(GL(2, q)) \setminus \{\chi_i, \forall i \text{ in the index set}\}.$$  \hspace{1cm} (1)

Then let $\Gamma_S(GL(2, q), D)$ be the subgraph of $\Gamma(GL(2, q), D)$ spanned by the vertices in $S$. Let $K_p$ be the complete graph of $p$ vertices. Let $K'_p$ be the graph obtained by removing an edge from $K_p$. The following theorem was shown in [6].

**Theorem 3.1** ([6]) Let $G = GL(2, q)$ and $D$ be the diagonal subgroup of $G$. Let $S$ be the set defined in (1). Let $\Gamma_S(G, D)$ be the subgraph of $\Gamma(G, D)$ spanned by the vertices in $S$. Then $\Gamma_S(G, D)$ has $(q - 1)$ connected components. Of these, $(q - 1)/2$ are isomorphic to $K_{(q-1)/2}$ and $(q - 1)/2$ are isomorphic to $K'_{(q+5)/2}$, where $K'_{(q+5)/2}$ is a graph obtained by removing an edge from $K_{(q+5)/2}$.

We shall look at $\Gamma_S(GL(2, q), B)$.

Since $D \leq B \leq GL(2, q)$, $\Gamma_S(GL(2, q), B)$ is a subgraph of $\Gamma_S(GL(2, q), D)$ by Lemma 2.4 in [2]. First, we calculate the values of restrictions of the characters of $GL(2, q)$ to $B$.

**Lemma 3.2** By the notations in Section 1, Definition 2.5, Theorem 2.2 and Theorem 2.4,

$$(\lambda_i)_B = \alpha_{i,i}$$

$$(\psi_i)_B = \alpha_{i,i} + \delta_{2i}$$

$$(\psi_{i,j})_B = \alpha_{i,j} + \alpha_{j,i} + \delta_{i+j}$$

$$(\chi_i)_B = \delta_i$$

**Proof.** By Theorem 2.2, we have the following table,

<table>
<thead>
<tr>
<th></th>
<th>$sI$</th>
<th>$u_s$</th>
<th>$d_{s,t}$</th>
<th>$d_{t,s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\lambda_i)_B$</td>
<td>$\overline{s^i}$</td>
<td>$\overline{s^i}$</td>
<td>$(st)^i$</td>
<td>$(st)^i$</td>
</tr>
<tr>
<td>$(\psi_i)_B$</td>
<td>$q\overline{s^{2i}}$</td>
<td>0</td>
<td>$(st)^i$</td>
<td>$(st)^i$</td>
</tr>
<tr>
<td>$(\psi_{i,j})_B$</td>
<td>$(q + 1)s^{i+j}$</td>
<td>$s^{i+j}$</td>
<td>$\overline{s^{i+j}} + \overline{s^{i+j}}$</td>
<td>$\overline{s^{i+j}} + \overline{s^{i+j}}$</td>
</tr>
<tr>
<td>$(\chi_i)_B$</td>
<td>$(q - 1)\overline{s^i}$</td>
<td>$-\overline{s^i}$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Comparing this table with the character table of $B$ in Theorem 2.4, it is clear that the four equations in the lemma holds true and hence the lemma.
**Definition 3.3** If $n$ is odd, let $Q_{(n+5)/2}$ be the graph obtained by adjoining two vertices $w_1, w_2$ and two edges to $K_{(n+1)/2}$ in the following way: Choose a pair of adjacent vertices in $K_{(n+1)/2}$, call them $v_1, v_2$. Join $w_1$ with $v_1$ by an edge and join $w_2$ with $v_2$ by an edge.

Up to isomorphism, the definition of $Q_{(n+5)/2}$ is independent of the choice of the adjacent vertices $v_1, v_2$ in $K_{(n+1)/2}$.

**Remark 3.4** Note that $Q_{(q+5)/2}$ is a subgraph of $K_3$, where $K_3$ is the graph obtained by removing an edge from $K_{(q+5)/2}$.

We are in the position to describe $\Gamma_S(GL(2, q), B)$.

**Theorem 3.5** Let $G$ be the group $GL(2, q)$ and $B$ be the Borel subgroup of $G$. Let $S$ be the set defined in (1). Let $\Gamma_S(G, B)$ be the subgraph of $\Gamma(G, B)$ spanned by the vertices in $S$. Then $\Gamma_S(G, B)$ has $(q-1)$ connected components. Of these, $(q-1)/2$ are isomorphic to $K_{(q-1)/2}$ and $(q-1)/2$ are isomorphic to $Q_{(q+5)/2}$, where $Q_{(q+5)/2}$ is the graph defined in Definition 3.3.

**Proof.** Let $S_\lambda = \{\lambda_i | 0 \leq i \leq q-2\} \subset V(\Gamma(G, B))$, then the subgraph $\Gamma_{S_\lambda}(G, B)$ spanned by the vertices in $S_\lambda$ has $q-1$ isolated vertices since $(\lambda_i)_B$ and $(\lambda_{i'})_B$ have no common irreducible constituent for $i \neq i'$, by Lemma 3.2.

Next, let $S_{\lambda, \psi} = \{\lambda_i, \psi_i | 0 \leq i \leq q-2\} \subset V(\Gamma(G, B))$. $(\lambda_i)_B$ and $(\psi_i)_B$ have $\alpha_{i,i}$ as the common irreducible constituent for each $i$ by Lemma 3.2 and hence they are connected by an edge in $\Gamma_{S_{\lambda, \psi}}(G, B)$. Furthermore, for $0 \leq i \leq (q-3)/2$, let $k = (q-1)/2$. Then $\delta_{2i} = \gamma_i$ is the same as $\delta_{2(i+k)} = \gamma_{(i+k)}$ as explained in Section 2.3. Hence $\psi_i$ and $\psi_{i+k}$ are joined by an edge in $\Gamma_{S_{\lambda, \psi}}(G, B)$. Hence there is a path that connects $\lambda_i$, $\psi_i$, $\psi_{i+k}$ and $\lambda_{i+k}$ in this order for each $i = 0, 1, \ldots, (q-3)/2$. It implies that, in $\Gamma_{S_{\lambda, \psi}}(G, B)$, there are $(q-1)/2$ connected component. Each of them contains 4 vertices and there is a non-closed path that connects these 4 vertices.

We consider $\Gamma_S(G, B)$. First, note that $(\psi_{i,j})_B$ has no common irreducible constituent with $(\lambda_i)_B$ since $i \neq j$. There are $(q-1)(q-2)/2$ pairs of $\{i, j\}$ such that $0 \leq i < j \leq q-2$. We call a pair odd if $i + j$ is odd and even otherwise. There are $(q-1)^2/4$ odd pairs and $(q-1)(q-3)/4$ even pairs. If $\{i, j\}$ is odd and $\{i', j'\}$ is even, then obviously there is no common irreducible constituent between $(\psi_{i,j})_B$ and $(\psi_{i', j'})_B$. In this case, $(\psi_{i,j})_B$ also does not have any common irreducible constituent with any vertex in the set $S_{\lambda, \psi}$. For two different odd pairs $\{i, j\}$ and $\{i'', j''\}$, without loss of generality, we assume that $i + j < i'' + j''$. $(\psi_{i,j})_B$ and $(\psi_{i'', j''})_B$ have $\delta_{i+j}$ as a common irreducible constituent if and only if $i + j = i'' + j''$ or $i + j + k = i'' + j''$ where $k = (q-1)$. For an odd $m$ where $1 \leq m \leq q-4$, there are $(q-1)/2$ odd pairs of $\{i, j\}$ such that $i + j = m$ or $i + j = m + k$. Hence they account for
\((q - 3)/2\) disjoint copies of \(K_{(q-1)/2}\) in \(\Gamma_S(G, B)\). There are exactly \((q - 1)/2\) odd pairs of \(\{i, j\}\) such that \(i + j = q - 2\). It accounts for one copy of \(K_{(q-1)/2}\) in \(\Gamma_S(G, B)\). To conclude, after considering all odd pairs \(\{i, j\}\), they account for \((q - 1)/2\) disjoint copies of \(K_{(q-1)/2}\).

For two different even pairs \(\{i, j\}\) and \(\{i'', j''\}\), without loss of generality, we assume that \(i + j < i'' + j''\). For \(k = q - 1\), \((\psi_{(i+j)/2})_B, (\psi_{(i+j+k)/2})_B, (\psi_{i,j})_B\) and \((\psi_{i'',j''})_B\) have \(\delta_{i+j}\) as a common irreducible constituent if and only if \(i + j = i'' + j''\) or \(i + j + k = i'' + j''\). Recall that, by the second paragraph, each connected component in \(\Gamma_{\lambda,\psi}(G, B)\) is a path that connects four vertices where \(\psi_i\) and \(\psi_{i+(q-1)/2}\) are the two middle vertices along this path. There are \((q - 3)/2\) even pairs of \(\{i, j\}\) such that \(i + j = n\) or \(i + j = n + k\) where \(n\) is even and \(0 \leq n \leq q - 3\). Hence, for each connected component, \((q - 3)/2\) vertices are joined with the two middle vertices. So, on each component, there is a copy of \(K_{((q-3)/2)+2} = K_{(q+1)/2}\) as a subgraph. For each such subgraph, the two vertices \(\lambda_i, \lambda_{i+k}\) are connected to \(\psi_{i}\) and \(\psi_{i+k}\) respectively. Note that \(\psi_{i}\) and \(\psi_{i+k}\) are adjacent. Hence each connected component is a copy of \(Q_{((q+1)/2)+2} = Q_{(q+5)/2}\) in \(\Gamma_S(G, B)\). There are \((q - 1)/2\) copies of it by the second paragraph and hence the theorem.

### 3.2 \(\Gamma(B, D)\)

First we look at the restrictions of the irreducible characters of \(B\) to \(D\).

**Lemma 3.6** By the notations in Section 1, Definition 2.7, Theorem 2.4 and Lemma 2.6, we have the following equations:

\[
(\alpha_{i,j})_D = f_{i,j}
\]

\[
(\delta_i)_D = \sum_{x=0}^{q-2} f_{x,i-x}
\]

for \(0 \leq i, j \leq q - 2\).

**Proof.** By Theorem 2.4, we have the following table:

\[
\begin{array}{|c|c|c|}
\hline
& sI & d_{s,t} & d_{t,s} \\
\hline
(\alpha_{i,j})_D & \psi_{i+j} & \psi_{i+j} & \psi_{i+j} \\
\hline
(\delta_i)_D & (q - 1)\psi_i & 0 & 0 \\
\hline
\end{array}
\]

It is clear that

\[
(\alpha_{i,j})_D = f_{i,j}
\]
for all $0 \leq i, j \leq q - 2$. For the second equation, we calculate the values of 
$\sum_{x=0}^{q-2} f_{x,i-x}$ on each conjugacy class. On $sI$, we have

$$\sum_{x=0}^{q-2} f_{x,i-x}(sI) = \sum_{x=0}^{q-2} \overline{s}^{x+i-x} = (q - 1)\overline{s}$$

On $d_{s,t}$,

$$\sum_{x=0}^{q-2} f_{x,i-x}(d_{s,t}) = \sum_{x=0}^{q-2} \overline{s}^{x+t-x} = \overline{t} \sum_{x=0}^{q-2} (\overline{s}/\overline{t})^x$$

$$= \frac{\overline{t}(\overline{s}/\overline{t})^{q-1} - 1}{\overline{t} - 1}.$$ The last equation is valid since $\overline{s} \neq \overline{t}$. It is equal to 0 because $\overline{s}/\overline{t} \in \mathbb{F}_q^*$ and hence $\overline{t}^{q-1} = 1$. A similar calculation shows that $\sum_{x=0}^{q-2} f_{x,i-x}(d_{t,s}) = 0$. Hence we have

$$(\delta_i)_D = \sum_{x=0}^{q-2} f_{x,i-x}$$

and the lemma.

Let $S_n$ be the star graph of $n$ vertices. We are in the position to describe

$$\Gamma(B, D).$$

**Theorem 3.7** Let $B$ be the Borel subgroup of $GL(2, q)$ and $D$ be the diagonal subgroup of $GL(2, q)$. The graph $\Gamma(B, D)$ has $(q - 1)$ disjoint copies of $S_q$.

**Proof.** Let $S_\alpha \subset V(\Gamma(B, D))$ be the set

$$S_\alpha = \{\alpha_{i,j} \mid 0 \leq i, j \leq q - 2\}$$

and let $\Gamma_{S_\alpha}(B, D)$ be the subgraph of $\Gamma(B, D)$ spanned by $S_\alpha$. For two different pairs of $(i, j)$ and $(i', j')$, $(\alpha_{i,j})_D$ and $(\alpha_{i',j'})_D$ have no common irreducible constituent by Lemma 3.6. Hence, $\Gamma_{S_\alpha}(B, D)$ has $(q - 1)^2$ isolated vertices.

For $i \neq i'$, $(\delta_i)_D$ and $(\delta_{i'})_D$ have no common irreducible constituent by Lemma 3.6. But each $(\delta_i)_D$ has common irreducible constituents with $(\alpha_{x,i-x})_D$ where $x = 0, 1, ..., q - 2$. (By abuse of notation, if $i - x$ is not in the range $[0, 1, ..., q - 2]$, let $(i - x)'$ represent the value in $[0, q - 2]$ that is in the same equivalence class as $i - x$ modulo $q - 1$. Then we take $(i - x)'$ to be $i - x$ in the notation $\alpha_{x,i-x}$.) Hence, for each $i = 0, 1, ..., q - 2$, $\delta_i$ is connected to $\alpha_{x,i-x}$ by an edge where $x = 0, 1, ..., q - 2$. It accounts for a copy of $S_q$. And there are $q - 1$ of such copies as $i$ ranges from 0 to $q - 2$. Hence the theorem.
References


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