A Conjecture about the Gelfand-Kirillov Dimension of the Universal Algebra of $A \otimes E$ in Positive Characteristic

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Abstract

In this paper we discuss the PI-equivalence of $M_{a,b}(E) \otimes E$ and $M_{c,d}(E) \otimes E$, when $a + b = c + d$ in positive characteristic $p > 2$. We also launch a conjecture about the Gelfand-Kirillov dimension of the universal algebra of rank $m$, $U_m(A \otimes E)$ in the variety generated by $A \otimes E$, in positive characteristic $p > 2$.

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1 Introduction

In characteristic zero the algebra $M_{a,b}(E) \otimes E$ appear in the list of non-trivial verbally-prime algebras. Verbally-prime algebras play a prominent role in Kemer’s results about structure of the varieties of associative algebras in characteristic zero (see [8] for a detailed account of it). Recently, some new results have been obtained about the behavior of polynomial identities satisfied by the verbally-prime algebras, giving a new proof that the so called Kemer’s Tensor Product Theorem is in part no longer valid in positive characteristic (see [11, Introduction] for a detailed account of this research). Berele in [4] constructed the generic algebras for $M_n(E)$ and $M_{a,b}(E)$ and computed their GK dimensions, while Regev in [6] studied various properties of $T(E)$, $T(M_n(E))$ and $T(M_{a,b}(E))$ when char $K = p > 2$. This work is based in the papers [5] and [13] and we hope that it will contribute to this subject.

2 Preliminary Notes

We recall some of the main definitions and notations that will be used in what follows. Unless stated otherwise, we consider associative unitary algebras, $K$ is a fixed infinite field with char $K = p \neq 2$. The algebras, vector spaces and tensor products will be considered over $K$. We denote by $K\langle X \rangle$ the free associative algebra of infinite rank freely generated over $K$ by the set $X = \{x_1, x_2, \ldots \}$. We denote by $\mathcal{A} \sim \mathcal{B}$ the PI-equivalence of the algebras $\mathcal{A}$ and $\mathcal{B}$. We refer to the books of Drensky and Procesi [2, 9] for background information on PI-algebras.

Let $E$ be the Grassmann (or exterior) algebra of a vector space $V$ with basis $\{e_1, e_2, \ldots \}$. It is well known that $E = E_0 \oplus E_1$, that is $E$ is $\mathbb{Z}_2$-graded, where $E_0 = \text{span}\{1, e_{i_1} \cdots e_{i_k} | i_1 < \cdots < i_{2k}, k \geq 1\}$ and $E_1 = \text{span}\{e_{i_1} \cdots e_{i_{2k+1}} | i_1 < \cdots < i_{2k+1}, k \geq 0\}$.

Let $M_n(E)$ be the algebra of $n \times n$ matrices over $E$. We denote by $\Delta_0 = \{(i, j) \in \mathbb{N} \times \mathbb{N} | 1 \leq i, j \leq a \text{ or } a + 1 \leq i, j \leq a + b\}$ and $\Delta_1 = \{(i, j) \in \mathbb{N} \times \mathbb{N} | 1 \leq i \leq a, a + 1 \leq j \leq a + b \text{ or } 1 \leq j \leq a, a + 1 \leq j \leq a + b\}$. Then $M_{a,b}(E)$ is the subalgebra of $M_{a+b}(E)$ of matrices $(a_{ij})$ such that $a_{ij} \in E_\beta$ if $(i, j) \in \Delta_\beta$.

Definition 2.1 The relatively free (also called universal) algebra of rank $m$ of an algebra $\mathcal{A}$ is $U_m(\mathcal{A}) = K\langle x_1, x_2, \ldots, x_n \rangle / T(\mathcal{A}) \cap K\langle x_1, x_2, \ldots, x_n \rangle$ where $T(\mathcal{A})$ is the $T$-ideal of the identities of the algebra $\mathcal{A}$.

Models for the universal algebra of rank $m$ of $M_n(E)$ and $M_{a,b}(E)$ were constructed by Berele in [4] and the same was done for $A_{a,b}$ and $M_{k,k}(E) \otimes E$ by Alves in [10, 1].

We recall briefly the definition of the GK dimension of an algebra $\mathcal{A}$. 
Definition 2.2 Let $A$ be an algebra generated by $\{r_1, \ldots, r_m\}$, and set $V^n = \text{span}\{r_1 \ldots r_i/r_{ij} = 1, \ldots, m\}$; $n = 1, 2, \ldots$ and $V^0 = K$. Define

$$g_V(n) = \dim_K (V^0 + \ldots + V^n) ; n = 1, 2, \ldots.$$ 

The Gelfand-Kirillov dimension of the algebra $A$ is:

$$\text{GKdim} (A) = \limsup_{n \to \infty} \log n [g_V(n)] = \limsup_{n \to \infty} \{\frac{\log [g_V(n)]}{\log(n)}\}.$$ 

We refer the reader to [4, 3, 7] for further details about the GK dimension of an algebra and PI algebras.

The following lemma is straightforward:

Lemma 2.3 Let $A, B$ be PI-algebras such that $\text{GKdim } U_m(A) \neq \text{GKdim } U_m(B)$. Then $A$ and $B$ are not PI-equivalent.

The following open question was stated in [5]:

We know that $T(M_{a,b}(E) \otimes E) = T(M_{c,d}(E) \otimes E)$ whenever $a + b = c + d$ and $\text{char } K = 0$. Is this true when $\text{char } K = p > 2$?

In this paper we answer the above question. It turns out that the answer is negative. We use the following theorem, proved in [13]:

Theorem 2.4 If $\text{char } K = p > 2$ then

$$\text{GKdim } U_m(M_{a,b}(E) \otimes E) = (m - 1)(a^2 + b^2) + 2.$$ 

We will also launch a conjecture about the Gelfand-Kirillov dimension of the universal algebra of rank $m$, $U_m(A \otimes E)$ in the varietie generated by $A \otimes E$, in positive characteristic $p > 2$.

3 Main Result

It is well known that in $\text{char } K = 0$, $M_{a,b}(E) \otimes E$ and $M_{c,d}(E) \otimes E$ are PI-equivalent, whenever $a + b = c + d$.

In this section we will establish that $M_{a,b}(E) \otimes E$ and $M_{c,d}(E) \otimes E$ are not PI-equivalent, when $a + b = c + d$ and $a \neq c$ in $\text{char } K = p > 2$.

Theorem 3.1 Let $\text{char } K = p > 2$, $a \neq c$ and $a + b = c + d$. Then:

$M_{a,b}(E) \otimes E$ and $M_{c,d}(E) \otimes E$ are not PI-equivalent.
Proof. According to Theorem 2.4, \(\text{GKdim} U_m(M_{a,b}(E) \otimes E) = (m-1)(a^2 + b^2) + 2\) and \(\text{GKdim} U_m(M_{c,d}(E) \otimes E) = (m-1)(c^2 + d^2) + 2\). As \(a \neq c\) and \(a + b = c + d\) then \(b \neq d\) and \(a^2 + b^2 \neq c^2 + d^2\) which implies:

\[
(m-1)(a^2 + b^2) + 2 \neq (m-1)(c^2 + d^2) + 2
\]

\(\text{GKdim} U_m(M_{a,b}(E) \otimes E) \neq \text{GKdim} U_m(M_{c,d}(E) \otimes E)\).

Now, applying Lemma 2.3 we have that the algebras \(M_{a,b}(E) \otimes E\) and \(M_{c,d}(E) \otimes E\) can not be PI-equivalent, as desired.

\[\blacksquare\]

4 The conjecture

In this section we introduce a conjecture about the Gelfand-Kirillov dimension of the universal algebras, concerning the tensorial product of verbally prime algebras by Grassmann algebra.

Our conjecture is justified by the following results, in char \(K = p > 2\):

In [4], Berele computed \(\text{GKdim} [U_m(M_n(E))]\) which coincides with \(\text{GKdim} [U_m(M_n(K))]\). Noting that \(M_n(E) \simeq M_n(K) \otimes E\) we conclude that:

\[\text{GKdim} [U_m(M_n(K) \otimes E)] = \text{GKdim} [U_m(M_n(K))].\]

In [11], Alves and Koshlukov proved that:

\[\text{GKdim} [U_m(E \otimes E)] = \text{GKdim} [U_m(E)].\]

In [13], the authors proved that:

\[\text{GKdim} [U_m(M_{a,b}(E) \otimes E)] = \text{GKdim} [U_m(M_{a,b}(E))].\]

In [14] the author proved that:

\[\text{GKdim} [U_m(M_n(E) \otimes E)] = \text{GKdim} [U_m(M_n(E))].\]

The above leads us to believe in the following:

**Conjecture** Let \(A\) be a verbally prime PI-algebra, non trivial, over a infinite field of char \(K = p > 2\). Then:

\[\text{GKdim} [U_m(A \otimes E)] = \text{GKdim} [U_m(A)].\]

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References


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