Calculus Space

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Abstract

This is sequel to our previous work [5] in which we introduced the notion of region algebra. It has been justified in [5] with the help of several examples that many of the simple results, formula, equalities, identities, rules etc. of elementary algebra are not valid in general in any group, ring, field, module, linear space, algebra over a field, associative algebra over a field, division algebra, or in any existing algebraic system, but in region algebra [5]. The minimum platform required for practicing elementary algebra is the region algebra [5]. In this paper we introduce the notion of ‘calculus space’ as the minimal structured mathematical space where a new calculus can be developed. A calculus space is a real region subject to fulfillment of four conditions which are explained here. The classical calculus developed independently by Newton and Leibniz is based on the platform of RR region as the calculus space. The work is initiated with the prior assumption that in our giant universe or multiverse (if exists), it may happen that the classical calculus may not be applicable at somewhere in the space. But whatever be the appropriate calculus, it can only be developed over a calculus space for it.

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1 Introduction

The Universe is commonly defined as the totality of existence as far as the people on the earth can think about. The present universe appears to be expanding at an accelerating rate. There are many competing theories about the ultimate fate of the universe. Scientific observation of the universe has led to inferences of its earlier stages too. Physicists remain unsure about what, if anything, preceded the Big Bang. Many refuse to speculate, doubting that any information from any such prior state could ever be accessible. There are various multiverse hypotheses, in which physicists have suggested that this universe might be one among many universes that likewise exist. But one question arises: Whether every space of this universe is being governed by the same physical laws and constants throughout most of its extent and history? If the speculation about the existence of multiverse be accepted to be true, then whether every universe of the multiverse is being governed by the same physical laws and constants throughout most of its extent and history? We consider here the mathematical system ‘Calculus’ (developed independently by Newton and Leibniz). John von Neumann said: "The calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance. I think it defines more unequivocally than anything else the inception of modern mathematics, and the system of mathematical analysis, which is its logical development, still constitutes the greatest technical advance in exact thinking”. But, can we accept the hypothesis that this classical Calculus is valid in every planet of our solar system or at every space of our universe or at every universe of the multiverse (if exists)? Is our classical Calculus an absolute calculus for everywhere in our universe or multiverse? Does it not get influenced by the facts of relativity. Does it not get influenced at different solar systems or at different spaces of the universe or at different universes of the multiverse where the concept of ‘time’ and ‘distance’ are different? In this paper we do not (can not) propose any answer to these
questions, but we propose the hypothesis that there could be a generalized calculus of which our classical calculus is a particular case just. We introduce the notion of a new mathematical object called by “Calculus Space”. A calculus space is a base-platform on which one can develop a new calculus. A calculus can not be developed without a calculus space, called a base-platform of the calculus. For this a preliminary study about the ‘Theory of Regions’ [5] is recommended. In [5], it is unearthed that the elementary algebra (various rules, formulas, equalities, identities, solution methods, results, etc.) can not be fluently practiced in a group, ring, field, module, linear space, algebra over a field, associative algebra over a field, division algebra, or in any existing algebraic system [4], in general. The minimum platform required for practicing elementary algebra is the region algebra [5]. In the same work [5], the ‘Theory of Objects’ has also been introduced and it has been justified and analysed that the ‘Theory of Numbers’ is a particular case of the topic ‘Theory of Objects’.

In the next section we make some characterization of regions and introduce the notion of calculus space. We use the following notations in our work here:
\[
\begin{align*}
R &= \text{set of all real numbers}, \\
R^+ &= \text{set of all positive real numbers}, \\
R^- &= \text{set of all negative real numbers}, \\
R^{\geq 0} &= \text{set of all non-negative real numbers}.
\end{align*}
\]

2 Calculus Space over a Real Region

Consider any region $A$. A variable $x$ which can take values from the region $A$ is called an object variable.

**Definition 2.1 Extended Region**

Consider a region $A$. If we include two more objects $\infty_+ A$ and $-\infty A$ in $A$, where $+\infty_+ A = \frac{x}{0_A}$ where $x (\neq 0_A)$ is any positive object, and $-\infty A = \frac{z}{0_A}$ where $z (\neq 0_A)$ is any negative object, then the set $A \cup \{+\infty_+ A, -\infty A\}$ is called to be an extended region. Note that an extended region is not a region. But if we
say that A is an extended region, it implies that A is a region and two infinities are included to it.

**Definition 2.2  2-to-1 Bijective Mapping**

Consider two non-null sets X and Y. A function $f : X \to Y$ is said to be a ‘2-to-1 Bijective Mapping’ if

(i) $f$ is onto

(ii) $\forall y \in Y, \exists$ two and only two distinct (not same) elements $x_1$ and $x_2$ in X such that $f(x_1) = y = f(x_2)$.

For example, the function $f : \mathbb{R} - \{0\} \to \mathbb{R}^+$ given by $f(x) = x^2$ is a 2-to-1 Bijective Mapping.

**Definition 2.3  Calculus Space**

A non-empty set A of objects forms a Calculus Space if the following conditions are true:

(i) A is an extended real region.

(ii) A is a normed complete metric space with respect to a norm $\| \cdot \|$ and the corresponding induced metric $\rho(x, y) = \| x - y \|$, (i.e. $\| x \| = \rho(x, 0_A)$).

(iii) The norm $\| \cdot \|$ is 2-to-1 bijective mapping from $A - \{0_A\}$ to $\mathbb{R}^+$.

(iv) A is a chain w.r.t. the total order relation $\leq$.

**Definition 2.4  Absolute Partitioning of a Real Region**

Consider a real region A. Consider a partition of the real region A into three(3) mutually disjoint non-null sets $A^+, A^-$ and $\{0_A\}$ such that

(i) $A^+ = \{ a : a \in A \text{ and } 0_A < a \}$

(ii) $A^- = \{ a : a \in A \text{ and } a < 0_A \}$.

Clearly, $\forall a \in A^+, \sim a \in A^-$ and $\forall b \in A^-, \sim b \in A^+$.

(Nota: we say that $a < b$ iff $a \leq b$ and $a \neq b$).

This partition, once made, must be regarded as an ‘absolute partition’ for the real region A over which one desires to develop a calculus. It is called to be absolute in the sense that it generates the sign of every object of A, positive or negative,
which will remain absolute for the complete literature of the corresponding calculus.

**Definition 2.5 Positive Object, Negative Object and Object Line**

The elements of $A^+$ are said to be positive objects and the elements of $A^-$ are said to be negative objects. The object $0_A$ is neither in $A^+$ nor in $A^-$, and so we say that $0_A$ is neither a positive object nor a negative object. The attribute of being positive or negative is called the sign of the object, and $0_A$ is not considered to have a sign.

For a given calculus space, a line (linear or even curvilinear) can be drawn with positive objects to the right, and negative objects to the left of $0_A$. Thus the ‘positive direction’ of X-axis and the ‘negative direction’ of X-axis can be well understood and the line which the objects of the region $A$ is considered to lie upon is called the **Object Line** (see Figure 1 and Figure 2 below).

![Figure 1](image1)  
**Figure 1.** Object line of the region $A$ with consecutive equi-spaced points.

![Figure 2](image2)  
**Figure 2.** Objects line of the region $A$, a general view.

We use the following notations in our work here:

$A^+ = \text{set of all positive objects of } A$, \hspace{1em} $A^- = \text{set of all negative objects of } A$ \hspace{1em} $A^{\geq 0} = \text{set of all non-negative objects of } A$
For developing a calculus, be it in a two dimensional coordinate system, or in an n-dimensional coordinate system, at least one calculus space is required. Consider the object line and the corresponding X-axis. Since A is complete, there are no "points missing" from it (inside or at the boundary). Since A is a chain, every object has a unique address on this linear continuum X. Consider a point x on the X-axis. Then for the infinitesimal small positive object Δx, the point (x + Δx) will be at a distance $\|\Delta x\|$ from the point x along the positive direction of X-axis and the point (x - Δx) will be at a distance $\|\Delta x\|$ from the point x along the negative direction of X-axis.

**Example 2.1**

If we choose the real region A to be the RR region and $\|x\| = |x|$ in RR where $\rho (x, y) = \|x-y\| = |x-y|$ and the RR region is a chain w.r.t. the crisp order relation “≤”, then the corresponding calculus happens to be the classical calculus (developed independently by Newton and Leibniz).

It is a wrong concept mentioned in many of the existing books/literature that the classical calculus is based on the field R of real numbers (of course, considering the extended real-axis). Actually it is neither the field R nor the division algebra [5], but it is the region R (which is called by RR region in region algebra). Interestingly, the field R (or the division algebra R) satisfies few additional properties trivially (beyond the requirement to be a field or to be a division algebra) by which it qualifies to become a real region; and consequently the classical calculus never faced any computational constraints or invalidity even assuming R to be a field just. For a clear understanding of this point, one needs a serious study of the region algebra [5].

### 3 Developing a new Calculus

Suppose that we want to develop a new calculus. We have already had a complete idea of the development and growth of the classical calculus since its inception. In
an analogous way, the basic concepts of any new calculus (new differential calculus) are limit, continuity, differentiability of a function of objects, etc.

**Definition 3.1** What do you mean by “$x \rightarrow a$”?

Consider an object variable $x$ over the calculus space $A$. Let $a \in A$ be a fixed object. Suppose that $x$ assumes successive values $(a \oplus 0.1 \cdot 1_A)$, $(a \oplus 0.01 \cdot 1_A)$, $(a \oplus 0.001 \cdot 1_A)$, $(a \oplus 0.0001 \cdot 1_A)$, …… as shown in Figure 3. Obviously, as $x$ passes through these successive values, the value $\rho(x, a)$ becomes less and less and become so small that for any positive real number $\varepsilon$, no matter however small, $\rho(x, a) < \varepsilon$ is satisfied. Let us express this situation using the notation “$x \rightarrow a^+$” which means that the variable $x$ approaches the fixed object $a$ from the right hand side of $a$.

![Figure 3. On the object line, $x \rightarrow a^+$](image)

Similarly, suppose that $x$ assumes successive values $(a \sim 0.1 \cdot 1_A)$, $(a \sim 0.01 \cdot 1_A)$, $(a \sim 0.001 \cdot 1_A)$, $(a \sim 0.0001 \cdot 1_A)$, …… as shown in Figure 4. Obviously, as $x$ passes through these successive values, the value $\rho(x, a)$ becomes less and less and become so small that for any positive real number $\varepsilon$, no matter however small, $\rho(x, a) < \varepsilon$ is satisfied. Let us express this situation using the notation “$x \rightarrow a^-$” which means that the variable $x$ approaches the fixed object $a$ from the left hand side of $a$.
By the expression “x tends to a” symbolically written as “x → a” we mean given any ε>0 no matter however small, the successive values of x ultimately satisfy the inequality 0 < ρ(x, a) < ε. It is to be noted that if “x → a” then ρ(x, a) ≠ 0, i.e. x ≠ a.

**Definition 3.2 Neighborhood of a Point on the Object Line**

Consider an object a on the object line of the real region A. Let δ > 0 be a real number. Then the δ-neighborhood of the object a is defined by the set N_δ(a) of objects given by N_δ(a) = { x : x ∈ A and ρ(x, a) < δ }.

**Definition 3.3 Limit of a Function**

Consider a calculus space A. Let X and Y be two subsets of A and let f be a function f : X → Y which is actually an object valued function of object variable. Then f(x) is said to have a limit l in Y if for any pre-assigned real number ε>0, no matter however small, ∃ a real number δ > 0 such that ρ( f(x), l) < ε whenever 0 < ρ(x, a) < δ.

We write symbolically as Limit f(x) = l, i.e. f(x)→l as x→a.

**Example 3.1**

Show that Limit 5 • x = 10 • 1_A in the calculus space A.

**Solution** Given ε>0, no matter however small, we need to find out δ > 0 such that
Calculus space

\[ \rho (5 \cdot x, 10 \cdot 1_A) < \varepsilon \quad \text{whenever} \quad 0 < \rho (x, 2 \cdot 1_A) < \delta. \]

i.e. \[ \|5 \cdot x - 10 \cdot 1_A\| < \varepsilon \quad \text{whenever} \quad 0 < \|x - 2 \cdot 1_A\| < \delta. \]

i.e. \[ 5 \cdot \|x - 2 \cdot 1_A\| < \varepsilon \quad \text{whenever} \quad 0 < \|x - 2 \cdot 1_A\| < \delta. \]

Now if we choose \( \delta = \varepsilon / 5 \), our definition is satisfied.

Hence, \( \lim_{x \to 2 \cdot 1_A} 5 \cdot x = 10 \cdot 1_A \) in the calculus space \( A \).

**Example 3.2**

Show that \( \lim_{x \to 3 \cdot 1_A} \frac{(x^2 - 9 \cdot 1_A)}{(x - 3 \cdot 1_A)} = 6 \cdot 1_A \) in the calculus space \( A \).

**Solution:** Given \( \varepsilon > 0 \), no matter however small, we need to find out \( \delta > 0 \) such that

\[ \rho \left(\frac{(x^2 - 9 \cdot 1_A)}{(x - 3 \cdot 1_A)}, 6 \cdot 1_A\right) < \varepsilon \quad \text{whenever} \quad 0 < \rho (x, 3 \cdot 1_A) < \delta. \]

i.e. \[ \left\|\frac{(x^2 - 9 \cdot 1_A)}{(x - 3 \cdot 1_A)} - 6 \cdot 1_A\right\| < \varepsilon \quad \text{whenever} \quad 0 < \left\|x - 3 \cdot 1_A\right\| < \delta. \]

Since \( x \to 3 \cdot 1_A \) therefore \( x \neq 3 \cdot 1_A \) and hence \( (x - 3 \cdot 1_A) \neq 0_A \).

Therefore, Cancellation Laws of region algebra [5] can be applied to get the following result:

\[ \left\|\frac{x}{3 \cdot 1_A} - 6 \cdot 1_A\right\| < \varepsilon \quad \text{whenever} \quad 0 < \left\|x - 3 \cdot 1_A\right\| < \delta. \]

i.e. \[ \|x - 3 \cdot 1_A\| < \varepsilon \quad \text{whenever} \quad 0 < \|x - 3 \cdot 1_A\| < \delta. \]

Now if we choose \( \delta = \varepsilon \), our definition is satisfied. Hence the result.

**4 Conclusion**

The classical calculus developed independently by Newton and Leibniz is based on the set \( R \) of real numbers, extended with two infinities (and then took its shape further with functions of complex variables, vector calculus, tensor calculus, etc.).

The growth of classical calculus at every stage required fluent applications of various operations and results which are valid by virtue of the properties owned by the real numbers. In many of the existing literatures, \( R \) is assumed to be just a
field or a division algebra. But this assumption is not true because using the properties of a ‘field’ or a ‘division algebra’ or any existing algebra other than region algebra [5], the classical calculus can not have the validity of its all fluent results. A careful study of the region algebra [5] will clarify that many of the results, formulas, equalities, identities, rules, etc. of elementary algebra (say, the algebra practiced at high school level) are not valid in the fields or in division algebras or in any existing algebras in general, but in regions only. Fortunately the set R is a trivial example of real region. In this paper we have identified ‘What are the minimum properties which need to be satisfied by a set A so that a calculus can be developed over A?’. Consequently we define a ‘calculus space’ as a general minimal platform on which a calculus can be developed. It has been explained how the platform R of classical calculus forms a particular instance of calculus space. For a non-example, the set of all triangular fuzzy numbers [2,6] do not form a real region [5] with respect to its commonly used operations, and hence can not open any platform to develop any fuzzy differential calculus and fuzzy integral calculus over it in the style of the classical calculus.

This work is initiated with a prior intuitionistic assumption that the classical calculus may not be applicable successfully at everywhere of our universe system or of the multiverse system (if exists). We presume here that our future computations (be it in this solar system or in other, be it in this universe or in other of the multiverse) may not be sufficiently covered by or compatible with our classical calculus. Consequently, the very first job is to define the general structure of a space which is a minimum requirement for making an attempt to define any new calculus over it.

References


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