Finitely Generated $S_2$ Ideals

Sarab A. Al-Taha

Department of Mathematics
Faculty of Science
Al Zaytoonah University
Amman, Jordan

Abstract

Let $R$ be a commutative ring with identity, let $a_1, a_2 \ldots a_n$ be nonzero elements in $R$. This paper deals with a new definition for the finitely generated ideal $I=<a_1, a_2\ldots a_n>$ that we call finitely generated $S_2$ ideal.

In [1] we give and study the finitely generated $S_1$ ideal, in this work we prove the following result among others, if the finitely generated ideal $I=<a_1, a_2\ldots a_n>$ is $S_1$ ideal then it is $S_2$ ideal.

Keywords: commutative rings, ideals, $S_1$ ideals, $S_2$ ideals

1 Introduction

Let $R$ be a commutative ring with identity, let $a_1, a_2 \ldots a_n$ be nonzero elements in $R$, the finitely generated ideal $I=<a_1, a_2\ldots a_n>$ is called $S_2$ ideal of $R$ if, and only if the following holds:

If $(a_1, a_2\ldots a_n) (b_1b_2\ldots b_n) = (a_1, a_2\ldots a_n)$ where $(a_1, a_2\ldots a_n)$ and $(b_1b_2\ldots b_n) \in \mathbb{R}_1 \oplus \mathbb{R}_2 \oplus \ldots \mathbb{R}_n$

With $b_i \neq 1$ for $i=1, 2, \ldots, n$
There exists \((a_1', a_2', \ldots, a_n') \in \mathbb{R}^n, (a_1', a_2', \ldots, a_n') \neq (0, 0, \ldots, 0)\) such that:

\[
(a_1', a_2', \ldots, a_n') (b_1, b_2, \ldots, b_n) = (a_1', a_2', \ldots, a_n') (a_1, a_2, \ldots, a_n)
\]

This definition will serve as our main tool throughout this work.

Recall that the finitely generated ideal \(I = \langle a_1, a_2, \ldots, a_n \rangle\) is called \(s_1\) ideal if, and only if the following holds:

If \(AM = A\), where

\[A = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n\] and \(M = (m_{ij}) \in M_n(\mathbb{R})\)

Then there exists \(A' = (a_1', a_2', \ldots, a_n') \in \mathbb{R}^n, (a_1', a_2', \ldots, a_n') \neq (0, 0, \ldots, 0)\)

\[A'M = 0. \quad [1]\]

Throughout this work we use the following notations:

- \(\mathbb{R}\): Commutative ring with identity.
- \(\mathbb{R}^n = \mathbb{R}_1 \oplus \mathbb{R}_2 \oplus \ldots \oplus \mathbb{R}_n = \{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{R}, i = 1, 2, \ldots, n\}\).
- \(M_n(\mathbb{R})\): Set of all \(n \times n\) matrices whose entries belong to \(\mathbb{R}\).
- \(\langle a_1, a_2, \ldots, a_n \rangle\): Finitely generated ideal of the ring \(\mathbb{R}\) with generators \((a_1, a_2, \ldots, a_n) \in \mathbb{R}^n\).
- \(\mathbb{R}[x]\): The ring of polynomial over \(\mathbb{R}\) in the indeterminate \(x\).
- \(\mathbb{R}[x_1, x_2, \ldots, x_n]\): The ring of polynomial over \(\mathbb{R}\) in the indeterminates \(x_1, x_2, \ldots, x_n\).

### 2 Main results

In this section we state down the following result:

**Theorem (1):** Let \(\mathbb{R}\) be a commutative ring with identity, let \(a_1, a_2, \ldots, a_n\) be nonzero elements in \(\mathbb{R}\) the finitely generated ideal \(\langle a_1, a_2, \ldots, a_n \rangle\) is \(s_2\) ideal of \(\mathbb{R}\) if, and only if \(\langle a_i \rangle\) is \(s_2\) ideal of \(\mathbb{R}\) for each \(i = 1, 2, \ldots, n\).

**Theorem (2):** Let \(\mathbb{R}\) be a commutative ring with identity, let \(a_1, a_2, \ldots, a_n\) be nonzero elements in \(\mathbb{R}\), if the finitely generated ideal \(\langle a_1, a_2, \ldots, a_n \rangle\) is \(s_1\) ideal of \(\mathbb{R}\) then it is \(s_2\) ideal of \(\mathbb{R}\).

**Theorem (3):** In a Boolean ring \(\mathbb{R}\) any finitely generated ideal \(\langle a_1, a_2, \ldots, a_n \rangle\) is \(s_2\) ideal.
Theorem (4): Let $R$ be a commutative ring with identity, let $a_1, a_2, \ldots, a_n$ be nonzero elements in $R$, the finitely generated ideal $< a_1, a_2, \ldots, a_n >$ is $S_2$ ideal of $R$ if, and only if $< a_1, a_2, \ldots, a_n >$ is $S_2$ ideal in $R[x]$.

Theorem (5): Let $R$ be a commutative ring with identity, let $a_1, a_2, \ldots, a_n$ be nonzero elements in $R$, the finitely generated ideal $< a_1, a_2, \ldots, a_n >$ is $S_2$ ideal of $R[x]$ if, and only if $< a_1, a_2, \ldots, a_n >$ is $S_2$ ideal of $R[x_1, x_2, \ldots, x_n]$.

Corollary (1): Let $R$ be a commutative ring with identity, let $a_1, a_2, \ldots, a_n$ be nonzero elements in $R$, the following are equivalent

(a) The finitely generated ideal $< a_1, a_2, \ldots, a_n >$ is $S_2$ ideal of $R$.

(b) The finitely generated ideal $< a_1, a_2, \ldots, a_n >$ is $S_2$ ideal of $R[x]$.

(c) The finitely generated ideal $< a_1, a_2, \ldots, a_n >$ is $S_2$ ideal in $R[x_1, x_2, \ldots, x_n]$.

Theorem (6) Let $R$ and $S$ be any two commutative rings with identity, let $\phi$ be an isomorphism from $R$ into $S$ then $< a_1, a_2, \ldots, a_n >$ is $S_2$ ideal of $R$ if, and only if $< \phi(a_1), \phi(a_2), \ldots, \phi(a_n) >$ is $S_2$ ideal of $S$.

In this section we prove the main results.

Proof of theorem (1):

Suppose that the finitely generated ideal $< a_1, a_2, \ldots, a_n >$ is $S_2$ ideal of $R$.

In order to show that $< a_i >$ is $S_2$ ideal for each $i = 1, 2, \ldots, n$

Let $a_i b_i = a_i$ for $i = 1, 2, \ldots, n$

This implies that $(a_1, a_2, \ldots, a_n) (b_1, b_2, \ldots, b_n) = (a_1, a_2, \ldots, a_n)$

Since $< a_1, a_2, \ldots, a_n >$ is $S_2$ ideal of $R$, there exists $(a_1', a_2', \ldots, a_n') \in R^n$

$(a_1', a_2', \ldots, a_n') \neq (0, 0, \ldots, 0)$ such that:

$(a_1', a_2', \ldots, a_n') (b_1, b_2, \ldots, b_n) = (a_1', a_2', \ldots, a_n') (a_1, a_2, \ldots, a_n)$

One can see easily that:

$a_i' b_i = a_i' a_i$ for $i = 1, 2, \ldots, n$
Thus $<a_i>$ is $s_2$ ideal of $R$ for $i=1,2...n$.

To prove the other half of the theorem, suppose that $<a_i>$ is $s_2$ ideal of $R$ for each $i=1,2...n$.

To show that $<a_1,a_2...a_n>$ is $s_2$ ideal of $R$, let

$$(a_1,a_2...a_n) (b_1,b_2...b_n) = (a_1,a_2...a_n).$$

This implies that

$$a_ib_i = a_i$$

for $i=1,2...n$.

Since $<a_i>$ is $s_2$ ideal of $R$ there exists $a_i'\in R a_i'\neq 0$ for $i=1,2...n$ such that:

$$a_i'b_i = a_i'a_i$$

This implies that:

$$(a_1',a_2'...a_n') (b_1,b_2...b_n) = (a_1',a_2'...a_n') (a_1,a_2...a_n)$$

Thus $<a_1,a_2...a_n>$ is $s_2$ ideal of $R$.

**Proof of theorem (2):**

Given that the finitely generated ideal $<a_1,a_2...a_n>$ is $s_1$ ideal, to show it is $s_2$ ideal, let

$$(a_1,a_2...a_n) (b_1,b_2...b_n) = (a_1,a_2...a_n).$$

$(b_1,b_2...b_n) \in R^n$

This implies that:

$$(a_1,a_2...a_n) \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & b_n \end{pmatrix} = (a_1,a_2...a_n)$$

Since $<a_1,a_2...a_n>$ is $s_1$ ideal of $R$, there exists $(a_1',a_2'...a_n') \in R^n$ such that

$$(a_1',a_2'...a_n') \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & b_n \end{pmatrix} = 0$$

[1]
Finitely generated $S_2$ ideals

We get that:

$$(a_1', a_2'...a_n') (a_1, a_2...a_n) = (a_1', a_2'...a_n') (a_1, a_2...a_n) = (a_1', a_2'...a_n') (b_1, b_2...b_n)$$

Hence the finitely generated ideal $< a_1, a_2...a_n >$ is $S_2$ ideal.

There exists a principal ideal of $R$ which is $S_2$ but not $S_1$.

**Example:** The ideal $<4>$ in the ring $\mathbb{Z}_{12}$ is $S_2$ ideal which is not $S_1$ ideal. [2]

**Proof of theorem (3):**

Suppose that the ring $R$ is Boolean ring [3]

To show that any finitely generated ideal $I = < a_1, a_2...a_n >$ is $S_2$ ideal of $R$.

Let $(a_1, a_2,...a_n) (b_1, b_2...b_n) = (a_1, a_2...a_n)$ where $(b_1, b_2...b_n) \in R^n$

It is clear that, there exists $(a_1, a_2,...a_n) \neq (0, 0...0) \in R^n$ such that:

$$(a_1, a_2,...a_n) (b_1, b_2...b_n) = (a_1, a_2...a_n) = (a_1, a_2...a_n) (a_1, a_2...a_n)$$

Hence $< a_1, a_2...a_n >$ is $S_2$ ideal.

**Proof of theorem (4):**

Suppose that the finitely generated ideal $< a_1, a_2...a_n >$ is $S_2$ ideal of $R$, to show that $< a_1, a_2...a_n >$ is $S_2$ ideal of $R[x]$.

Let $(a_1, a_2,...a_n) (f_1(x), f_2(x)...f_n(x)) = (a_1, a_2...a_n)$ where $(f_1(x), f_2(x)...f_n(x)) \in R[x]$.

And $f_i(x) = b_0^i + b_1^i(x) + \cdots b_m^i(x)$, where $b_j^i \in R$ for $i=1, 2...n$, $j=1, 2...m$

This implies that:
\[(a_1, a_2, \ldots, a_n) \cdot (b_0^1, b_0^2, \ldots, b_0^n) = (a_1, a_2, \ldots, a_n) \quad \text{[4]}\]

Since \(\langle a_1, a_2, \ldots, a_n \rangle\) is an \(s_2\) ideal of \(R\), there exists \((a_1', a_2', \ldots, a_n') \in R^n\) such that:

\[(a_1', a_2', \ldots, a_n') \cdot (b_0^1, b_0^2, \ldots, b_0^n) = (a_1', a_2', \ldots, a_n') \cdot (a_1, a_2, \ldots, a_n)\]

Now it is clear that:

\[(a_1', a_2', \ldots, a_n') \cdot (f_1(x), f_2(x), \ldots, f_n(x)) = (a_1', a_2', \ldots, a_n') \cdot (a_1, a_2, \ldots, a_n)\]

Hence \(\langle a_1, a_2, \ldots, a_n \rangle\) is an \(s_2\) ideal of \(R[x]\).

To prove the other half of this theorem, suppose that \(\langle a_1, a_2, \ldots, a_n \rangle\) is an \(s_2\) ideal of \(R[x]\).

To show that \(\langle a_1, a_2, \ldots, a_n \rangle\) is an \(s_2\) ideal of \(R\), let

\[(a_1, a_2, \ldots, a_n) \cdot (b_1, b_2, \ldots, b_n) = (a_1, a_2, \ldots, a_n)\]

Since \(\langle a_1, a_2, \ldots, a_n \rangle\) is an \(s_2\) ideal of \(R[x]\), there exists \((f_1'(x), f_2'(x), \ldots, f_n'(x)) \in R^n[x]\), where

\[f_i'(x) = b_0^i + b_1^i(x) + \cdots + b_m^i(x), \text{ with } b_j \in R, \quad j = 0, 1, \ldots, m\]

such that:

\[(f_1'(x), f_2'(x), \ldots, f_n'(x)) \cdot (b_1, b_2, \ldots, b_n) = (f_1'(x), f_2'(x), \ldots, f_n'(x)) \cdot (a_1, a_2, \ldots, a_n)\]

This implies that:

\[(b_0^1, b_0^2, \ldots, b_0^n) \cdot (b_1, b_2, \ldots, b_n) = (b_0^1, b_0^2, \ldots, b_0^n) \cdot (a_1, a_2, \ldots, a_n)\]

And hence \(\langle a_1, a_2, \ldots, a_n \rangle\) is an \(s_2\) ideal of \(R\).

**Proof of theorem (5):**

Follows immediately from theorem (4) and the fact that

\[R[x_1, x_2, \ldots, x_n] = R[x_1, x_2, \ldots, x_{n-1}][x_n] \quad \text{[5]}\]

Proof of corollary (1):

The proof comes direct from theorem (4) and theorem (5).
Proof of theorem (6):

Suppose that \( \langle a_1, a_2, \ldots, a_n \rangle \) is \( s_2 \) ideal of \( R \) if, to show that 
\( \langle xgAAjC(a_1), xgAAjC(a_2), \ldots, xgAAjC(a_n) \rangle \) is \( s_2 \) ideal of \( S \), let
\[
( (a_1), (a_2), \ldots, (a_n) ) = ( (a_1), (a_2), \ldots, (a_n) ) \quad [6]
\]
This implies that:
\[
( a_1, a_2, \ldots, a_n ) (b_1, b_2, \ldots, b_n) = ( a_1, a_2, \ldots, a_n )
\]
Since \( \langle a_1, a_2, \ldots, a_n \rangle \) is \( s_2 \) ideal of \( R \). There exists \( (b_1, b_2, \ldots, b_n) \in R^n \) such that:
\[
(a_1', a_2', \ldots, a_n') (b_1, b_2, \ldots, b_n) = ( a_1', a_2', \ldots, a_n' )
\]
Thus \( \langle xgAAjC(a_1), xgAAjC(a_2), \ldots, xgAAjC(a_n) \rangle \) is \( s_2 \) ideal of \( S \).

To prove the other half of the theorem, assume that \( \langle xgAAjC(a_1), xgAAjC(a_2), \ldots, xgAAjC(a_n) \rangle \) is \( s_2 \) ideal of \( S \).

To show that \( \langle a_1, a_2, \ldots, a_n \rangle \) is \( s_2 \) ideal of \( R \), let
\[
( a_1, a_2, \ldots, a_n ) (b_1, b_2, \ldots, b_n) = ( a_1, a_2, \ldots, a_n )
\]
where \( (b_1, b_2, \ldots, b_n) \in R^n \) this implies that
\[
(a_1', a_2', \ldots, a_n') (b_1, b_2, \ldots, b_n) = ( a_1', a_2', \ldots, a_n' )
\]
Since \( \langle xgAAjC(a_1), xgAAjC(a_2), \ldots, xgAAjC(a_n) \rangle \) is \( s_2 \) ideal of \( S \), there exists \( (a_1', a_2', \ldots, a_n') \in S^n \) such that:
\[
( a_1', a_2', \ldots, a_n' ) (b_1, b_2, \ldots, b_n) = ( a_1', a_2', \ldots, a_n' )
\]
It follows that, there exists \( (a_1', a_2', \ldots, a_n') \in R^n \) such that:
\[
(a_1', a_2', \ldots, a_n') (b_1, b_2, \ldots, b_n) = ( a_1', a_2', \ldots, a_n' ) (a_1, a_2, \ldots, a_n)
\]
Thus \( \langle a_1, a_2, \ldots, a_n \rangle \) is \( s_2 \) ideal of \( R \).
REFERENCES


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