On Projective Ideals in Commutative Rings

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Abstract

Let R be a commutative with identity, let a be a nonzero element of R as in [1], the principal ideal I = <a> of R is projective if, and only if there exists a′∈ R such that a = a a′ and ann(a)=ann(a′). We prove the following result among others, the principal ideal I=<a>is projective ideal in R if, and only if <a/a'> is projective in Rs_a if, and only if <a/a'> is projectivein Rs_a[x] where S_a={a′∈R: a=aa′}. [2]

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1. Introduction

Let \( R \) be a commutative ring with identity, let \( a \in R \) as in [1] the principal ideal \( I=\langle a \rangle \) is projective in \( R \) if, and only if there exists \( a' \in R \) such that:
\[
a = a \cdot a' \quad (1)
\]
and
\[
\text{ann}(a) = \text{ann}(a'). \quad (2)
\]
Where \( \text{ann}(a) : \text{Annihilator of } a = \{ x \in R : ax = 0 \} \). \quad (3)

Example:
The principal ideal \( I = \langle 2 \rangle \) is projective ideal in \( \mathbb{Z}_6 \) while the principal ideal \( I = \langle 3 \rangle \) is not projective ideal in \( \mathbb{Z}_6 \).

As in [1], the finitely generated ideal \( I = \langle a_1, a_2, \ldots, a_n \rangle \) of \( R \) is projective ideal if, and only if there exists an \( nxn \) matrix \( M = (m_{ij}) \) with elements in \( R \) such that:
\[
(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_n) M \quad (4)
\]
and
\[
\text{U}^\perp = \text{ann} (M) \quad (5)
\]
where
\[
\text{U}^\perp = \{ x = (x_1, x_2, \ldots, x_n) \in R^n : Ux^t = a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \} \quad (6)
\]
and
\[
\text{ann}(M) = \{ x = (x_1, x_2, \ldots, x_n) \in R^n : Mx^t = 0^t \} \quad (7)
\]
Throughout this work we use the following notations:

- \( R \): Commutative ring with identity.
- \( N(R) \), the nil radical of \( R = \{ x \in R : x^n = 0 \text{ for some } n \in \mathbb{Z}_+ \} \)
- \( S_a = \{ a' \in R : a = aa' \} \) \quad [2] \quad (8)
- \( R_s = \{ a/a' : a \in R, a' \in S_a \} \) \quad [2] \quad (9)
- \( R[x] \): The ring of polynomials over \( R \) in the indeterminate \( x \).
- \( R_{sa}[x] \): The ring of polynomials over \( R_{sa} \) in the indeterminate \( x \).
- \( R[x_1, x_2, \ldots, x_n] \): The ring of polynomials over \( R \) in the indeterminate \( x_1, x_2, \ldots, x_n \).
2. Main Results

In this section we state down the following results:

**Theorem (1):** Let R be a commutative ring with identity such that N(R) = 0 let a be a nonzero element in R then \(<a>\) is projective ideal in R if, and only if \(<a^n>\) is projective.

**Theorem (2):** Let R be a commutative ring with identity, let a be a nonzero element in R, b is a nonzero divisor of R the principal ideal \(<ab>\) is projective ideal in R if, and only if \(<a>\) is projective.

**Theorem (3):** Let R be a commutative with identity, a be a nonzero element of R, the principal ideal \(<a>\) of R is projective in R if, and only if \(<a>\) is projective in \(R[x]\).

**Theorem (4):** Let R be a commutative with identity, a be a nonzero element of R, the principal ideal \(<a>\) is projective in R if, and only if \(<a>\) is projective in \(R[x,y]\).

**Corollary (1):** Let R be a commutative with identity, a be a nonzero element of R, the principal ideal \(<a>\) is projective in R if, and only if \(<a/a'>\) is projective in \(R_{Sa}\).

**Theorem (5):** Let R be a commutative with identity, a be a nonzero element of R, the principal ideal \(<a>\) is projective in R if, and only if \(<a/a'>\) is projective in \(R_{Sa}[x_1, x_2, \ldots, x_n]\).

**Corollary (2):** Let R be a commutative with identity, a be a nonzero element of R, \(<a/a'>\) is projective in \(R_{Sa}\) if, and only if \(<a/a'>\) is projective in \(R_{Sa}[x_1, x_2, \ldots, x_n]\).

**Theorem (6):** Let R be a commutative with identity, a be a nonzero element of R, \(a_1, a_2, \ldots, a_n\) are nonzero elements of R, the finitely generated ideal \(I=\langle a_1, a_2, \ldots, a_n\rangle\) is projective in R if and only if \(I=\langle a_1, a_2, \ldots, a_n\rangle\) is projective in \(R[x]\).

**Corollary (3):** Let R be a commutative with identity, \(a_1, a_2, \ldots, a_n\) are nonzero elements of R, the finitely generated ideal \(I=\langle a_1, a_2, \ldots, a_n\rangle\) is projective in R if, and only if \(I=\langle a_1, a_2, \ldots, a_n\rangle\) is projective in \(R[x_1, x_2, \ldots, x_n]\).
**Theorem (7):** Let \( R \) be a commutative ring with identity. If the principal ideals \(<a_1>, <a_2>\ldots<a_n>\) are projective ideals in the ring \( R \) then the ideal \(<a_1, a_2\ldots a_n>\) is projective ideal in the ring \( R^n=R\oplus R\oplus\ldots\oplus R \).

### 3. Proofs

In this section we prove our main results.

**Proof of theorem (1)**

Assume that \(<a>\) is projective ideal, using (1) and (2) there exists \( a'\in R \) such that:

\[
a = aa' \tag{11}
\]

and \( \text{ann}(a)=\text{ann}(a') \tag{12} \)

Using (11) we get \( a^n = a^n a' \)

Now, to show that:

\( \text{ann}(a^n) = \text{ann}(a') \)

let \( x \in \text{ann}(a^n) \) this implies that \( xa^n = 0 \), clearly we get \( x^n a^n = 0 \).

Since \( N(R) = 0 \), it follows that \( xa = 0 \) therefore \( x\in\text{ann}(a) = \text{ann}(a') \) and hence

\( \text{ann}(a^n) \subseteq \text{ann}(a') \)

To prove the converse, let \( x \in \text{ann}(a') \) using (12) it follows that \( x\in\text{ann}(a) \) this implies that

\( xa = 0 \) and hence \( xa^n = 0 \) therefore \( x\in\text{ann}(a^n) \), we get \( \text{ann}(a') \subseteq \text{ann}(a^n) \).

To prove the other half of the theorem. Suppose that \(<a^n>\) is projective, using (1) and (2) there exists \( a'\in R \) such that:

\[
a^n = a^n a' \tag{13}
\]

and \( \text{ann}(a^n) = \text{ann}(a') \tag{14} \)

Using (13) we get \( a^n (1-a') = 0 \), this implies that \( a^n (1-a')^n = 0 \), since \( N(R) = 0 \) we get

\( a = aa' \).
As it is shown in the first part of this theorem we get that:
\[ \text{ann}(a) = \text{ann}(a^n) = \text{ann}(a') \] 
therefore \(<a>\) is projective.

**Proof of theorem (2)**

Since \(<ab>\) is projective using (1) and (2) there exists \(c' \in R\) such that:
\[ ab = abc' \]
and \( \text{ann}(ab) = \text{ann}(c') \)

Since \(b\) is nonzero divisor, we get \(a = ac'\)
and \( \text{ann}(ab) = \text{ann}(a) = \text{ann}(c') \) therefore \(<a>\) is projective

**Proof of theorem (3)**

The first part can be shown easily. Now to prove the converse assumes that \(<a>\) is projective ideal in \(R[x]\), using (1) and (2) there exists \(g = a_0 + a_1x + \cdots + a_nx^n \in R[x]\) such that:
\[ a = ag \]  \hspace{1cm} (15)
and \( \text{ann}(a) = \text{ann}(g) \) \hspace{1cm} (16)

Using (16), it follows from [3] that \(a = aa_0\)

To show that: \( \text{ann}(a) = \text{ann}(a_0) \)

Let \(b \in \text{ann}(a)\) using (3), it follows that that \(ba = 0\) using (16) we get \(bg = 0\) this implies that \(ba_0 = 0\) \hspace{1cm} [3]

Thus \(b \in \text{ann}(a_0)\).

Now, let \(b \in \text{ann}(a_0)\), using (3), it follows that \(ba_0 = 0\) directly \(baa_0 = 0\)

Using (17) we get \(ba = 0\), therefore \(b \in \text{ann}(a)\).

Thus, \( \text{ann}(a) = \text{ann}(a_0) \)

**Proof of Theorem (4):**

The first part can be shown easily. Next, let \(<a>\) be projective ideal in \(R[x,y]\), using definition of projective ideal, there exists
\[ f = \sum_{j=0}^{n} \sum_{i=0}^{n} a_{ij} x^i y^j \in R[x, y] \]

Such that:  
\[ a = af \]  \hspace{1cm} (18)

and  
\[ \text{ann}(a) = \text{ann}(f) \]  \hspace{1cm} (19)

Using (18), it follows from [4] that:
\[ a = aa_{00} \]

Now to show that \( \text{ann}(a) = \text{ann}(aa_{00}) \)

Let \( b \in \text{ann}(a) \) using (19), it follows that \( bf = 0 \) using (3) we get that \( b \in \text{ann}(f) \) this implies that \( ba_{00} = 0 \) \hspace{1cm} [4].

Now, let \( b \in \text{ann}(aa_{00}) \), using (3) it follows that \( ba_{00} = 0 \) directly \( baa_{00} = 0 \).

Since \( a = aa_{00} \), we get that \( ba = 0 \). This completes the proof.

Proof of corollary (1)

The proof of this corollary comes directly from the definition of \( R[x_1, x_2, \ldots, x_n] \) which is equal to \( R[x_1, x_2, \ldots, x_{n-1}] [x_n] \) \hspace{1cm} [5] and by using induction on theorem (2).

Proof of theorem (5):

Assume that \( <a> \) is projective in \( R \), using (1) and (2) it follows that there exists \( c \in R \) such that:
\[ a = ac', \text{ hence } c' \in s_a \]

and \( \text{ann}(a) = \text{ann}(c') \)  \hspace{1cm} (20)

It’s clear that there exists \( c'/1 \in R_{s_a} \) such that:
\[ a/a' = a/a'.c'/1 \]

To show that \( \text{ann}(a/a') = \text{ann}(c'/1) \), let \( x/x' \in \text{ann}(a/a') \) using (3) it follows that
\[ x/x'.a/a' = 0 \]

This means that there exists \( b \in s_a \) such that \( b'xa = 0 \) \hspace{1cm} [4]

Using definition of the multiplicatively closed set \( s_a \), it follows that \( xa = 0 \) using (20) we get \( xc' = 0 \) and therefore \( x/x'.c'/1 = 0 \).
Thus \( x/x'\epsilon\text{ann}(c'/1) \)

Now, let \( x/x'\epsilon\text{ann}(c'/1) \), using (3) it follows that \( x/x'.c'/1=0 \) this means that there exists \( b'\epsilon s_a \) such that \( b'xc'=0 \) \[4\]

Directly, we get that \( b'xc'=0 \) since \( a=ac' \), we get \( b'xa=0 \) therefore \( xa=0 \) this implies that \( x/x'.a/a'=0 \) thus we prove that \( x/x'\epsilon\text{ann}(a/a') \).

**Proof of corollary (2)**

The proof of this corollary comes directly by using corollary (1).

**Proof of theorem (6)**

The first part can be proved easily.

To prove the converse, suppose that the finitely generated ideal \( I=<a_1, a_2\ldots a_n> \) is projective in \( R[x] \) using (3) and (4) there exists a matrix

\[
M = \left(\begin{array}{ccc}
 f_{11} & \cdots & f_{1n} \\
 \vdots & \ddots & \vdots \\
 f_{n1} & \cdots & f_{nn}
\end{array}\right),
\]

where

\[
f_{ij} = b_{ij}x^i + \cdots + b_{ijn}x^n \in R[x], \quad 1 \leq i,j \leq n
\]

Such that:

\[
( a_1, a_2\ldots a_n) = ( a_1, a_2\ldots a_n) M
\]

and

\[
\text{ann}( a_1, a_2\ldots a_n) = \text{ann} M
\]

using (21) it follows from [4] that:

\[
( a_1, a_2\ldots a_n) = ( a_1, a_2\ldots a_n) \left(\begin{array}{ccc}
 b_{11} & \cdots & b_{1n} \\
 \vdots & \ddots & \vdots \\
 b_{n1} & \cdots & b_{nn}
\end{array}\right)
\]

Using (22) it follows from [4] that

\[
\text{ann}( a_1, a_2\ldots a_n) = \text{ann} \left(\begin{array}{ccc}
 b_{11} & \cdots & b_{1n} \\
 \vdots & \ddots & \vdots \\
 b_{n1} & \cdots & b_{nn}
\end{array}\right)
\]

**Proof of theorem (7)**

Since \(<a_i> \) is projective in \( R \) for each \( i=1,2\ldots,n \) using (1) and (2), there exist \( a'_i \epsilon R \) \( i=1,2\ldots,n \) such that:
\[ a_i = a_i', \quad i = 1, 2, \ldots, n \]
and
\[ \text{ann } a_i = \text{ann } a_i', \quad i = 1, 2, \ldots, n \]

It is clear that there exists a matrix 
\[ M = \begin{pmatrix} a_1' & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n' \end{pmatrix} \]
whose entries belong to \( R \)
such that:
\[ (a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_n) M \]
and
\[ \text{ann } (a_1, a_2, \ldots, a_n) = \text{ann } M \]

## References


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