Weak (4,2)-Chain Complexes
of Simplicial Complexes

Ajet Ahmeti
University of Prishtina, Prishtina, Kosovo
ajet_ahmeti@yahoo.com

Dončo Dimovski
University Ss Cyril and Methodius, Skopje, Macedonia
donco@pmf.ukim.mk

Copyright © 2013 Ajet Ahmeti and Dončo Dimovski. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

For a given simplicial complex K, we give a construction of a weak (4,2)-chain complex wC(K), of commutative (4,2)-groups. Using wC(K), we produce three chain complexes C_2(K), C_+(K) and C_*(K) of abelian groups with homology groups H_{n,2}(K), H_{n,+}(K) and H_{n,*}(K).

Keywords: commutative (4,2)-group, weak (4,2)-chain complex, simplicial complex

1 Preliminaries

The (n,m)-groups were introduced in [8], and their properties were examined for example in [1] and [10]. The free (n,m)-groups are characterized in [2] and [7]. The characterization of (2m,m)-groups is given in [9], and commutative and free commutative (2m,m)-groups are examined in [5] and [6]. In [3] we considered (4,2)-chain complexes, producing (4,2)-homology groups and in [4] we constructed cubical weak (4,2)-chain complexes for topological spaces. In this paper we will construct a weak (4,2)-chain complex for a given simplicial complex.
Let $G$ be a nonempty set. A $(4,2)$-semigroup is a pair $(G,[ ])$, where $[ ] : G^4 \rightarrow G^2$ is a map satisfying the following associativity condition:

$$[[xyzt]uv] = [x[yztu]v] = [x[yztuv]],$$

for each $x,y,z,t,u,v \in G$.

A $(4,2)$-semigroup $(G,[ ])$ induces a semigroup $(G^2,\circ)$, where “$\circ$” is the binary operation on $G^2$ defined by:

$$x \circ y = [xy], \text{ for } x,y \in G^2, \text{ i.e. } (a,b) \circ (c,d) = [abcd] \text{ for } a,b,c,d \in G.$$ \(A\) $(4,2)$-semigroup $(G,[ ])$ is called a commutative $(4,2)$-group if $(G^2,\circ)$ is a commutative group.

The existence of a $(4,2)$-operation $[ ]$ on $G$ is equivalent to the existence of two operations: $[ ]_1, [ ]_2 : G^4 \rightarrow G$, such that $[xyzt] = ([xyzt]_1, [xyzt]_2)$.

The commutative $(4,2)$-groups, together with the $(4,2)$-homomorphisms, form a category, denoted by $(4,2)$-$\text{Ab}$, where a $(4,2)$-homomorphisms from $(G,[ ])$ to $(H,[ ])$ is a map $f : G \rightarrow H$, such that $f([xyzt]_j) = [f(x)f(y)f(z)f(t)]_j$, $j=1,2$.

We state some properties for the commutative $(4,2)$-groups, shown in [9] and [5].

**Proposition 1.1.** A $(4,2)$-semigroup $(G,[ ])$ is a commutative $(4,2)$-group if and only if there exist an $e \in G$ and a map $g : G \rightarrow G$ such that, $g((g(x)) = x$ and:

(a) $[eexy] = (x,y)$, for each $x,y \in G$;
(b) $[xxg(x)g(x)] = (e,e)$, for each $x \in G$;
(c) $[xyzt] = [ztxy]$, for each $x,y,z,t \in G$.

In a commutative $(4,2)$-group $(G,[ ])$, the element $e$ will be called “zero” and will be denoted by $0$, and for each $x \in G$, the element $g(x)$ will be denoted by $x'$.

**Proposition 1.2.** For a commutative $(4,2)$-group $(G,[ ])$, and $x, y, z, t \in G$:

(a) $[xyzt] = [zxty] = [zyx] = [zt]xy$;
(b) $[xyzt] = (u,v) \Leftrightarrow [yxtz] = (v,u)$ i.e. $[xyzt]_1 = [yxtz]_2 = [tyxz]_2$;
(c) $[00xy] = [y00x] = [0y0x] = (x,y)$;
(d) $[xxx'x'] = [xx'xx] = [x'xxx'] = (0,0)$;
(e) $[[xy'y'],[zxy'y']_j] = [xy'y'],[zxy'y']_j$;
(f) $[xx'] = (u,v) \Rightarrow u = v$ i.e. $[xx']_1 = [xx']_2$;
(g) $[xx'y'] = (u,v) \Rightarrow v = u'$ i.e. $[xx'y']_2 = [xx'y']_1$;
(h) $D = \{(x,x) \mid x \in G\} \subseteq G^2$ is a subgroup of the group $(G^2,\circ)$; and
(i) $K = \{(x,x') \mid x \in G\} \subseteq G^2$ is a subgroup of the group $(G^2,\circ)$.

Next as a corollary we give generalizat ions of some of the above properties needed later. If $U = x_1x_2...x_t$ is a word (sequence) of elements in $G$, we say that $U$ has length $t$ and write $|U| = t$. For such a word $U$ we will use the notations:

$$\hat{U} = x_1x_{t-1}...x_2x_1, \ U' = (x_1)'(x_2)'...(x_{t-1})'(x_t)' \text{ and } \overline{U} = (x_1)'(x_2)'...(x_{t-1})'...(x_2)'(x_1)'.
With these notations, \( \widetilde{U} = U = U' = (U')', \widetilde{U} = U = U', (\widetilde{U}') = \widetilde{U}. \)

**Corollary 1.3.** For a commutative \((4,2)\)-group \((G, [\ ]^{})\), for any \(x, y, z \in G\) and any words \(U, V, W\):

(a) \([UxVxyW] = [UVxyW]\), for \(|V|\) even and \(|U| + |W|\) even;

(b) \([UVW] = [U\widetilde{V}W]\), for \(|V|\) odd and \(|U| + |W|\) odd;

(c) \([UxVxyW] = [Uxy\widetilde{V}W]\), for \(|V|\) even and \(|U| + |W|\) even;

(d) \([UxVzyW] = [UxyVW] = [UzVxyW]\), for \(|V|\) odd and \(|U| + |W|\) even;

(e) \([U] = [\widetilde{U}]_2\), for \(|U|\) even; and

(f) \([(UU)_1]^y = [U\widetilde{U}]_2\), i.e. \([U \widetilde{U}]_1, [U \widetilde{U}]_1, [U \widetilde{U}]_2, [U \widetilde{U}]_2\) = (0, 0). \(\Box\)

Proposition 1.2. implies the existence of three covariant functors, denoted by \(\Phi_2, \Phi_+\) and \(\Phi_*\), from the category \((4,2)\)-Ab to the category Ab of abelian groups.

For a commutative \((4,2)\)-group \(G = (G, [\ ]):\)

\(\Phi_2(G)\) is the group \((G^2, \circ);\)

\(\Phi_+(G) = (G, +)\) where \(x + y = a \iff [xyy] = (a, a);\) and

\(\Phi_*(G) = (G, \ast),\) where \(x \ast y = a \iff [xx'yy'] = (a, a').\)

A weak \((4,2)\)-chain complex, denoted by \(wK\), is a sequence

\[0 \leftarrow \cdots (G_n, [\ ]^{}) \leftarrow \cdots (G_{n-1}, [\ ]^{}) \leftarrow \cdots (G_n, [\ ]^{}) \leftarrow \cdots (G_{n+1}, [\ ]^{}) \leftarrow \cdots\]

of commutative \((4,2)\)-groups \((G_n, [\ ]^{})\) and \((4,2)\)-homomorphisms \(\partial_n, n \geq 0,\) such that \(\partial_n \partial_{n+1} = 0,\) i.e. for each \(x \in G_{n+1}, \partial_n \partial_{n+1}(x) = 0.\) The \((4,2)\)-chain maps are defined in the usual way. The weak \((4,2)\)-chain complexes as objects and the \((4,2)\)-chain maps as morphisms form a category of \((4,2)\)-\(\mathcal{wC}\), called the **category of weak \((4,2)\)-chain complexes**.

Using the functors \(\Phi_2, \Phi_+\) and \(\Phi_*\), three covariant functors \(F_2, F_+\) and \(F_*\) from the category \((4,2)\)-\(\mathcal{wC}\) to the usual category \(\mathcal{C}\) of chain complexes of commutative groups are obtained. The compositions of these three functors with the homology functors \(H_n\) produce three functors from \((4,2)\)-\(\mathcal{wC}\) to \(\text{Ab}: H_n \circ F_2\) denoted by \(H_n,2\), \(H_n \circ F_+\) denoted by \(H_n,\ast\) and \(H_n \circ F_*\) denoted by \(H_n,*\).

2 Weak \((4,2)\)-chain complexes of simplicial complexes

Let \(K\) be a simplicial complex, i.e. a set of simplices that satisfies the following:

1) Any face of a simplex in \(K\) is a simplex in \(K;\) and

2) The intersection of any two simplices \(\sigma, \tau\) of \(K\) is a face of both \(\sigma\) and \(\tau.\)
We denote by $K^0$ the set of all the 0-dimensional simplices (vertices) of $K$. We choose a well ordering $<$ for the set $K^0$. For an n-simplex of $K$, we order its vertices by the chosen well ordering as $v_0<v_{j-1}<v_j<v_{j+1}<...<v_n$, and denote the simplex by $s=(v_0v_1...v_iv_n)$. An (n-1)-dimensional face of $s$, opposite the vertex $v_i$, will be denoted by $s_i=(v_0v_1...v_{i-1}v_{i+1}...v_n)= (v_0v_1...v_{i-1}\hat{v}_iv_{i+1}...v_n)$ (where $\hat{v}_i$ means that $v_i$ is not there). An (n-2)-dimensional face of $s$, opposite the vertices $v_i$ and $v_j$, will be denoted by \[ s_{ij}=(v_0v_1...v_{i-1}\hat{v}_iv_{i+1}...v_{j-1}\hat{v}_jv_{j+1}...v_n). \]

Directly from the above notions, it follows that:

a) Two n-simplices $s=(v_0v_1...v_nv_n)$ and $t=(w_0w_1...w_n)$ are equal if and only if for each $0 \leq t \leq n$, $v_t=w_t$; and

b) For any $s=(v_0v_1...v_nv_n)$ and any $0 \leq i < j \leq n$, $s_{ij}=s_{ji}$.

For any n>0, let $K^n$ be the set of all the n-simplices of $K$ with the above mentioned orderings of their vertices. For any n, let $(S_n(K),[\cdot])$ be the free commutative $(4,2)$-group generated by $K^n$, such that the sets $K^n$ and the set \( \{s|s \in K^n\} \) are disjoint (see Proposition 1.1. and the notation $g(z)=z$). Thus, in $S_0(K)$, for any $s \in K^n$, \([sss's']=(0,0)\), and in $S_0(K)$, for any $v \in K^0$, \([v\hat{v}_v'=v]\).

We define a map $\partial_0:(S_0(K),[\cdot]) \rightarrow 0$ by $\partial_0(x)=0$ for any $x \in S_0$.

For any n>0, we will define a (4,2)-homomorphism $\partial_n:(S_n(K),[\cdot]) \rightarrow (S_{n-1}(K),[\cdot])$ first by defining it on $K^n \cup (K^n)'$ and then extending it on $(S_n(K),[\cdot])$.

For an $s=(v_0v_1...v_{j-1}v_jv_{j+1}...v_n) \in K^n$ we define:

\[ \partial_n(s)=[s_0s_1,...,s_{n-1}s_n (s_n)'(s_{n-1})'...(s_1)'(s_0)]_{1}=[\overline{U}], \]

where $U=s_0s_1...s_{n-1}s_n$.

Directly from the definition and Corollary 1.3., it follows that

\[ \partial_{n-1}\partial_n(s)=0 \] for any n>0 and any oriented n-simplex s. For n=1 it is obvious that $\partial_{n-1}\partial_n(s)=0$.

Let n=2 and $s=(v_0v_1v_2)$, where $v_0<v_1<v_2$. Let $W=v_2v_1$. Then:

$s_0=(v_1v_2), s_1=(v_0v_2), s_2=(v_0v_1), s_{0,1}=v_2, s_{0,2}=v_1, s_{1,2}=v_0$ and by using Corollary 1.3. we obtain:

\[ \partial_0\partial_1(s) = \partial_1([s_0s_2s_2(s_2)'(s_1)'(s_0)]_{1}) = \partial_1[s_0(s_0)'(s_1)'s_1s_2(s_2')], \]

where $U=s_0s_2s_2(s_2)'(s_1)'(s_0)$. Then:

$[s_0s_2s_2(s_2)'(s_1)'(s_0)]_{1}, [s_0s_2s_2(s_2)'(s_1)'(s_0)]_{2}$ and $[s_0s_2s_2(s_2)'(s_1)'(s_0)]_{3}$.
Weak (4,2)-chain complexes of simplicial complexes

In the above discussion, we can see that \( \partial_i \partial_2 (s) = [M] \), where \( M \) is a word of the elements \( s_{ij} \) and \( (s_{ij})' \), \( 0 \leq i < j \leq 2 \), where \( s_{ij} \) appears twice in \( M \) and \( (s_{ij})' \) appears twice in \( M \). Moreover, after using Corollary 1.3., we replaced these appearances in \( M \) by 00, which leads to the conclusion that \( \partial_i \partial_2 (s) = 0 \). Next we will show that we can apply the same process for any \( n \), showing that \( \partial_n \partial_2 (s) = 0 \).

Let \( s \) be an \( n \)-simplex. Using Corollary 1.3, \( \partial_n (s) \) can be represented as:

\[
\partial_n (s) = [r_0, r_0] [r_1, r_1] [r_2, r_2] \ldots [r_n, r_n]
\]

where \( r_{2i} = s_{2i} \) and \( r_{2i+1} = (s_{2i+1})' \), i.e. \( r_i = s_i \) for \( t \) even, and \( r_i = (s_i)' \) for \( t \) odd. So,

\[
\partial_{n-1} \partial_n (s) = [\partial_{n-1} (r_0)] [\partial_{n-1} (r_1)] [\partial_{n-1} (r_2)] \ldots [\partial_{n-1} (r_n)].
\]

For any \( t \), \( \partial_{n-1} (s_i) = [U_t, U_t]_2 \) and \( \partial_{n-1} ((s_i)') = [U_t, (U_t)']_2 \), where \( U_t = s_{t,0} s_{t,1} \ldots s_{t, 2i-1} s_{t, 2i} \ldots s_{t, n} \).

If \( r_i = s_i \), then

\[
\partial_{n-1} (r_i) [\partial_{n-1} ((r_i)')] = [U_t, U_t]_2 [U_t, (U_t)']_2 = [U_t, U_t]_2;
\]

and if \( r_i = (s_i)' \) then

\[
\partial_{n-1} (r_i) [\partial_{n-1} ((r_i)')] = [U_t, U_t]_2 [U_t, (U_t)']_2 = [(U_t)' [U_t]']_2 = [(U_t)' U_t]_2.
\]

All this implies that

\[
\partial_{n-1} \partial_n (s) = [V_0, V_0] [V_1, V_1] \ldots [V_{t}, V_{t}]_2 [V_{t+1}, V_{t+1}]_2 \ldots [V_{n-1}, V_{n-1}]_2 [V_n, V_n]_2 = [M],
\]

where \( V_t = U_t \) for \( t \) even and \( V_t = (U_t)' \) for \( t \) odd, and \( M \) is a word of the elements \( s_{ij} \) and \( (s_{ij})' \) for all \( 0 \leq i < j \leq n \). Moreover, for any \( 0 \leq i < j \leq n \), \( s_{ij} \) appears twice in \( M \), once in \( V_i, V_i \) and once in \( V_j, V_j \) and similarly \( (s_{ij})' \) appears twice in \( M \), once in \( V_i, (V_j) \) and once in \( (V_i)' V_j \).

Let \( 0 \leq i < j \leq n \). Using Proposition 1.2. and Corollary 1.3, we will rearrange the word \( M \) to a new word \( N \) of the form \( N = N_1 s_{ij} s_{ij} (s_{ij})' (s_{ij})' N_2 \). This will imply that we can replace \( s_{ij} \) and \( (s_{ij})' \) by 0. By doing this for any
\[ 0 \leq i < j \leq n, \text{ at the end we will obtain that } \partial_{n-1} \partial_n (s) = 0. \]

First we rearrange the word \( M \) to the word \( V_i \nabla_i V_j \nabla_j M_i \), and next we look only at the word \( A = V_i \nabla_i V_j \nabla_j \).

For \( s_i \) and \( s_j \) we write \( U_i \) and \( U_j \) as:

\[
U_i = s_i^0 s_{i,1} \ldots s_{i,j-1} s_{i,j} s_{i,j+1} \ldots s_{i,n} = U_{i1} s_{i,j} U_{i2} \quad \text{and} \quad U_j = s_j^0 s_{j,1} \ldots s_{j,j-1} s_{j,j} s_{j,j+1} \ldots s_{j,n} = U_{j1} s_{j,j} U_{j2}.
\]

With this notation, \( |U_{i1}| = j - 1, \quad |U_{i2}| = n - j, \quad |U_{j1}| = i, \quad |U_{j2}| = n - i - 1. \)

Now, we have to discuss the parity of \( i \) and \( j \), and so we consider the following four cases.

**Case 1.** \( i \) and \( j \) even. Then, \( |U_{i1}| + |U_{j1}| = j + i - 1 \) is odd, and so:

\[
\mathcal{A} = [U_i \nabla_i U_j \nabla_j] = [U_{i1} s_{i,j} U_{i2} \nabla_{i1} (s_{i,j})' \nabla_{i1} U_{j1} s_{j,j} U_{j2} \nabla_{j1} (s_{j,j})' \nabla_{j1}]
\]

\[
= [U_{i1} (U_{i2})' \nabla_{i1} s_{i,j} (s_{i,j})' \nabla_{i1} U_{j1} s_{j,j} (s_{j,j})' (U_{j2})' \nabla_{j1}]
\]

\[
= [U_{i1} (U_{i2})' \nabla_{i1} s_{i,j} (s_{i,j})' \nabla_{i1} U_{j1} s_{j,j} (s_{j,j})' (U_{j2})' \nabla_{j1}]
\]

\[
= [C_{1s_{i,j}s_{j,j}} (s_{i,j})' (s_{j,j})' B_2].
\]

**Case 2.** \( i \) odd, \( j \) even. Then, \( |U_{i1}| + |U_{j1}| = j + i - 1 \) is even, and so:

\[
\mathcal{A} = [(U_{i1})' \nabla_i U_j \nabla_j] = [(U_{i1})' (s_{i,j})'(U_{i2})' \nabla_{i1} s_{i,j} \nabla_{i1} U_{j1} s_{j,j} U_{j2} \nabla_{j1} (s_{j,j})' \nabla_{j1}]
\]

\[
= [(U_{i1})' U_{i2} \nabla_{i1} s_{i,j} (s_{i,j})' \nabla_{i1} U_{j1} s_{j,j} (s_{j,j})' (U_{j2})' \nabla_{j1}]
\]

\[
= [(U_{i1})' U_{i2} \nabla_{i1} s_{i,j} (s_{i,j})' \nabla_{i1} U_{j1} s_{j,j} (s_{j,j})' (U_{j2})' \nabla_{j1}]
\]

\[
= [C_{1s_{i,j}s_{j,j}} (s_{i,j})'(s_{j,j})' B_2].
\]

**Case 3.** \( i \) even, \( j \) odd. Then, \( |U_{i1}| + |U_{j1}| = j + i - 1 \) is even, and so:

\[
\mathcal{A} = [U_i \nabla_i (U_j) \nabla_j] = [U_{i1} s_{i,j} U_{i2} \nabla_{i1} (s_{i,j})' \nabla_{i1} U_{j1} (s_{j,j})' (U_{j2})' \nabla_{j1} s_{j,j} \nabla_{j1}]
\]

\[
= [U_{i1} (U_{i2})' \nabla_{i1} s_{i,j} (s_{i,j})' \nabla_{i1} U_{j1} (s_{j,j})' (U_{j2})' \nabla_{j1}]
\]

\[
= [U_{i1} (U_{i2})' \nabla_{i1} s_{i,j} (s_{i,j})' \nabla_{i1} U_{j1} (s_{j,j})' (U_{j2})' \nabla_{j1}]
\]

\[
= [C_{2s_{i,j}s_{j,j}} (s_{i,j})' (s_{j,j})' (B_2)'].
\]

**Case 4.** \( i \) odd, \( j \) odd. Then, \( |U_{i1}| + |U_{j1}| = j + i - 1 \) is odd, and so:

\[
\mathcal{A} = [(U_{i1})' \nabla_i (U_j) \nabla_j]
\]

\[
= [(U_{i1})' (s_{i,j})'(U_{i2})' \nabla_{i1} s_{i,j} (s_{i,j})'(U_{j2})' \nabla_{j1} s_{j,j} \nabla_{j1}]
\]

\[
= [(U_{i1})' U_{i2} \nabla_{i1} s_{i,j} (s_{i,j})' \nabla_{i1} U_{j1} (s_{j,j})' (s_{j,j})' \nabla_{j1}]
\]

\[
= [(U_{i1})' U_{i2} \nabla_{i1} s_{i,j} (s_{i,j})' \nabla_{i1} U_{j1} (s_{j,j})' (s_{j,j})' \nabla_{j1}]
\]

\[
= [C_{3s_{i,j}s_{j,j}} (s_{i,j})' (s_{j,j})' (B_2)]'.
\]
Weak (4,2)-chain complexes of simplicial complexes

With the above discussion we have completed the proof of the following:

**Theorem 2.1.** For a simplicial complex \( K \), let \( (S_n(K),[ ]) \) and \( \partial_n \) be defined as above. Then the sequence

\[
0 \leftarrow (S_0(K),[ ]) \leftarrow (S_1(K),[ ]) \leftarrow ... \leftarrow (S_n(K),[ ]) \leftarrow (S_{n+1}(K),[ ]) \leftarrow ... \]

is a weak (4,2)-chain complex.

At the end we give the following example.

**Example 2.2.** Let \( K \) be the simplicial complex having three vertices and three 1-simplices, i.e. \( K^0 = \{v_0,v_1, v_2\} \) and \( K^1 = \{(v_0,v_1), (v_0,v_2), (v_1,v_2)\} \). The weak (4,2)-chain \( wC(K) \) is:

\[
0 \leftarrow (S_0(K),[ ]) \leftarrow (S_1(K),[ ]) \leftarrow 0 \leftarrow 0 \leftarrow 0 \leftarrow ... .
\]

It can be checked that the (4,2)-homology group \( H_{1,*}(K) \) contains as a subgroup the ordinary homology group \( H_1(K) \) but \( H_{1,*}(K) \) and \( H_1(K) \) are not isomorphic.

**References**

[1] D. Dimovski; Examples of vector valued groups, Contributions, Macedonian Academy of Science and Arts, VI, 2, Skopje 1985, 105-144.


[6] D. Dimovski, S.Ilic; Free commutative \((2m,m)\)-groups, Matematički bilten, 13, Skopje 1989, 25-34.


Received: June 11, 2013