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Weak (4,2)-Chain Complexes of Simplicial Complexes

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Abstract

For a given simplicial complex K, we give a construction of a weak (4,2)-chain complex wC(K), of commutative (4,2)-groups. Using wC(K), we produce three chain complexes $C_2(K)$, $C_+(K)$ and $C_*(K)$ of abelian groups with homology groups $H_{n,2}(K)$, $H_{n,+}(K)$ and $H_{n,*}(K)$.

Keywords: commutative (4,2)-group, weak (4,2)-chain complex, simplicial complex

1 Preliminaries

The (n,m)-groups were introduced in [8], and their properties were examined for example in [1] and [10]. The free (n,m)-groups are characterized in [2] and [7]. The characterization of (2m,m)-groups is given in [9], and commutative and free commutative (2m,m)-groups are examined in [5] and [6]. In [3] we considered (4,2)-chain complexes, producing (4,2)-homology groups and in [4] we constructed cubical weak (4,2)-chain complexes for topological spaces. In this paper we will construct a weak (4,2)-chain complex for a given simplicial complex.

Let G be a nonempty set. A **(4,2)-semigroup** is a pair (G,[ ]) ,where [ ]:$G^4 \to G^2$ is a map satisfying the following associativity condition:

$$[[xyzt]uv] = [x[yztu]v] = [xy[ztuv]], \quad \text{for each } x,y,z,t,u,v \in G.$$

A (4,2)-semigroup (G,[ ]) induces a semigroup $(G^2, \circ)$, where "$\circ$" is the binary operation on G^2 defined by:

$\mathbf{x} \circ \mathbf{y} = [\mathbf{xy}]$, for $\mathbf{x}, \mathbf{y} \in G^2$, i.e. $(a,b) \circ (c,d) = [abcd]$ for $a,b,c,d \in G$.

A (4,2)-semigroup (G,[ ]) is called a **commutative (4,2)-group** if $(G^2, \circ)$ is a commutative group .

The existence of a (4,2)-operation [ ] on G is equivalent to the existence of two operations: $[\ ]_1$, $[\ ]_2 : G^4 \to G$, such that $[xyzt]=([xyzt]_1, [xyzt]_2)$.

The commutative (4,2)-groups, together with the (4,2)-homomorphisms, form a category, denoted by **(4,2)-Ab**, where a (4,2)-homomorphisms from (G,[ ]) to (H,[ ]) is a map f: G $\to$ H, such that $f([xyzt]_j) = [f(x)f(y)f(z)f(t)]_j$, j=1,2.

We state some properties for the commutative (4,2)-groups, shown in [9] and [5].

Proposition 1.1. A (4,2)-semigroup (G,[ ]) is a commutative (4,2)-group if and only if there exist an $e \in G$ and a map g: $G \to G$ such that, $g((g(x))=x$ and:

(a) $[eexy]=(x,y)$, for each $x,y \in G$;

(b) $[xxg(x)g(x)] = (e,e)$, for each $x \in G$;

(c) $[xyzt] = [ztxy]$, for each $x,y,z,t \in G$. □

In a commutative (4,2)-group (G,[ ]), the element e will be called "zero" and will be denoted by 0, and for each $x \in G$, the element g(x) will be denoted by z'.

Proposition 1.2. For a commutative (4,2)-group $(G,[\ ])$, and $x, y, z, t \in G$:

(a) $[xyzt] = [xtzy] = [zyxt] = [ztxy]$;

(b) $[xyzt] = (u,v) \Leftrightarrow [yxtz] = (v,u)$ i.e. $[xyzt]_1 = [yxtz]_2 = [tzyx]_2$;

(c) $[00xy] = [x00y] = [xy00] = [0yx0] = (x,y)$;

(d) $[xxx'x'] = [xx'x'x] = [x'xxx'] = (0,0)$;

(e) $([xyy'x']_1)' = [xyy'x']_2$;

(f) $[xxyy] = (u,v) \Rightarrow u = v$ i.e. $[xxyy]_1 = [xxyy]_2$;

(g) $[xx'yy'] = (u,v) \Rightarrow v = u'$ i.e. $[xx'yy']_2 = [xx'yy']_1$;

(h) $D = \{(x,x) \mid x \in G\} \subseteq G^2$ is a subgroup of the group $(G^2, \circ)$; and

(i) $K = \{(x,x') \mid x \in G\} \subseteq G^2$ is a subgroup of the group $(G^2, \circ)$. □

Next as a corollary we give generalizations of some of the above properties needed later. If $U = x_1 x_2 ... x_t$ is a word (sequence) of elements in G, we say that U has length t and write |U| = t . For such a word U we will use the notations: $\widetilde{U} = x_t x_{t-1} ... x_2 x_1$, $U' = (x_1)'(x_2)'...(x_{t-1})'(x_t)'$ and $\overline{U} = (x_t)'(x_{t-1})'...(x_2)'(x_1)'$.

With these notations, $\tilde{\tilde{U}} = U = \overline{\overline{U}} = (U')'$, $\overline{\tilde{\tilde{U}}} = \tilde{\overline{U}} = U'$, $(\overline{U})' = \tilde{U}$,.

Corollary 1.3. For a commutative (4,2)-group $(G,[\ ])$, for any $x, y, z \in G$ and any words U,V,W:

(a) $[UxyVW] = [UVxyW]$, for |V| even and |U|+|W| even;

(b) $[UVW] = [U\tilde{V}W]$, for |V| odd and |U|+|W| odd;

(c) $[UxVyW] = [Uxy\tilde{V}W] = [U\tilde{V}xyW]$, for |V| even and |U|+|W| even;

(d) $[UxVzyW] = [UxyzVW] = [UzVxyW]$, for |V| odd and |U|+|W| even;

(e) $[U]_1 = [\tilde{U}]_2$, for |U| even; and

(f) $([U\overline{U}]_1)' = [U\overline{U}]_2$, i.e. $[[U\overline{U}]_1[U\overline{U}]_1[U\overline{U}]_2[U\overline{U}]_2] = (0,0)$. □

Proposition 1.2. implies the existence of three covariant functors, denoted by Φ_2, Φ_+ and Φ_* , from the category (4,2)-Ab to the category Ab of abelian groups.

For a commutative (4,2)-group $\underline{G} = (G,[\ ])$:

$\Phi_2(\underline{G})$ is the group $(G^2, \circ)$;

$\Phi_+(\underline{G}) = (G,+)$ where $x + y = a \Leftrightarrow [xxyy] = (a,a)$; and

$\Phi_*(\underline{G}) = (G,*)$, where $x * y = a \Leftrightarrow [xx'yy'] = (a,a')$.

A **weak (4,2)-chain complex** , denoted by wK, is a sequence

$$0 \xleftarrow{\partial_0} (G_0,[\ ]) \xleftarrow{\partial_1} \dots (G_{n-1},[\ ]) \xleftarrow{\partial_n} (G_n,[\ ]) \xleftarrow{\partial_{n+1}} (G_{n+1},[\ ]) \longleftarrow \dots$$

of commutative (4,2)-groups $(G_n,[\ ])$ and (4,2)-homomorphisms ∂_n, $n \geq 0$, such that $\partial_n \partial_{n+1} = 0$, i.e. for each $x \in G_{n+1}$, $\partial_n \partial_{n+1}(x) = 0$. The (4,2)-chain maps are defined in the usual way. The weak (4,2)-chain complexes as objects and the (4,2)-chain maps as morphisms form a category of (4,2)-w∂C, called the **category of weak (4,2)-chain complexes.**

Using the functors Φ_2, Φ_+ and Φ_*, three covariant functors F_2, F_+ and F_* from the category (4,2)-w∂C to the usual category ∂C of chain complexes of commutative groups are obtained. The compositions of these three functors with the homology functors H_n, produce three functors from (4,2)-w∂C to Ab: $H_n \circ F_2$ denoted by $H_{n,2}$, $H_n \circ F_+$ denoted by $H_{n,+}$ and $H_n \circ F_*$ denoted by $H_{n,*}$.

2 Weak (4,2)-chain complexes of simplicial complexes

Let K be a simplicial complex, i.e. a set of simplices that satisfies the following:

1) Any face of a simplex in K is a simplex in K; and

2) The intersection of any two simplices σ,τ of K is a face of both σ and τ.

We denote by K^0 the set of all the 0-dimensional simplices (vertices) of K. We choose a well ordering < for the set K^0. For an n-simplex of K, we order its vertices by the chosen well ordering as $v_0<\ldots v_{j-1}<v_j<v_{j+1}<\ldots<v_n$, and denote the simplex by $s=(v_0v_1...v_i...v_n)$. An (n–1)-dimensional face of s, opposite the vertex v_i, will be denoted by $s_i=(v_0v_1...v_{i-1}v_{i+1}...v_n)=(v_0v_1...v_{i-1}\hat{v}_iv_{i+1}...v_n)$ (where $\hat{v}_i$ means that v_i is not there). An (n–2)-dimensional face of s, opposite the vertices v_i and v_j, will be denoted by $s_{i,j}=(v_0v_1...v_{i-1}\hat{v}_iv_{i+1}...v_{j-1}\hat{v}_jv_{j+1}...v_n)$. Directly from the above notions, it follows that:

a) Two n-simplices $s=(v_0v_1...v_i...v_n)$ and $t=(w_0w_1...w_i...w_n)$ are equal if and only if for each $0\le t\le n$, $v_t=w_t$; and

b) For any $s=(v_0v_1\cdots v_i\cdots v_n)$ and any $0\le i<j\le n$, $s_{i,j}=s_{j,i}$.

For any n>0, let K^n be the set of all the n-simplexes of K with the above mentioned orderings of their vertices. For any n, let ($S_n(K)$, [ ]) be the free commutative (4,2)-group generated by K^n, such that the sets K^n and the set $(K^n)'=\{s'|\, s\in K^n\}$ are disjoint (see Proposition 1.1. and the notation $g(z)=z'$). Thus, in $S_n(K)$, for any $s\in K^n$, $[sss's']=(0,0)$, and in $S_0(K)$, for any $v\in K^0$, $[vvv'v']=(0,0)$.

We define a map $\partial_0:(S_0(K),[\ ])\to 0$ by $\partial_0(x)=0$ for any $x\in S_0$.

For any n>0, we will define a (4,2)-homomorphism

$$\partial_n:(S_n(K),[\ ])\to(S_{n-1}(K),[\ ])$$

first by defining it on $K^n\cup(K^n)'$ and then extending it on $(S_n(K),[\ ])$.

For an $s=(v_0v_1\cdots v_{j-1}v_jv_{j+1}\cdots v_n)\in K^n$ we define:

$$\partial_n(s)=[s_0s_1....s_{n-1}s_n(s_n)'(s_{n-1})'....(s_1)'(s_0)']_1=[U\overline{U}]_1\text{, and}$$

$$\partial_n(s')=[s_0s_1....s_{n-1}s_n(s_n)'(s_{n-1})'....(s_1)'(s_0)']_2=[U\overline{U}]_2\text{,}$$

where $U=s_0s_1....s_{n-1}s_n$.

Directly from the definition and Corollary 1.3., it follows that

$$\partial_n(s')=[\overline{U}U]_1=(\partial_n(s))'.$$

Next we will show that $\partial_{n-1}\partial_n(s)=0$ for any n>0 and any oriented n-simplex s. For n=1 it is obvious that $\partial_{n-1}\partial_n(s)=0$.

Let n=2 and $s=(v_0v_1v_2)$, where $v_0<v_1<v_2$. Let $W=v_2v_1$. Then: $s_0=(v_1v_2),\ s_1=(v_0v_2),\ s_2=(v_0v_1),\ s_{0,1}=v_2,\ s_{0,2}=v_1,\ s_{1,2}=v_0$ and by using Corollary 1.3. we obtain:

$$\begin{aligned}\partial_1\partial_2(s)&=\partial_1([(s_0s_1s_2(s_2)'(s_1)'(s_0)']_1=\partial_1[s_0(s_0)'(s_1)'s_1s_2(s_2)']_1)\\&=[\partial_1(s_0)\partial_1(s_0)'\partial_1(s_1)'\partial_1(s_1)\partial_1(s_2)\partial_1(s_2)']_1\\&=[[s_{0,1}s_{0,2}(s_{0,2})'(s_{0,1})']_1[s_{0,1}s_{0,2}(s_{0,2})'(s_{0,1})']_2\partial_1(s_1)'\partial_1(s_1)\partial_1(s_2)\partial_1(s_2)']_1\end{aligned}$$

$$= [[s_{0,1}s_{0,2}(s_{0,2})'(s_{0,1})']\partial_1(s_1)'\partial_1(s_1)\partial_1(s_2)\partial_1(s_2)']_1$$
$$= [s_{0,1}s_{0,2}(s_{0,2})'(s_{0,1})'s_{1,2}s_{1,0}(s_{1,0})'(s_{1,2})'s_{2,0}s_{2,1}(s_{2,1})'(s_{2,0})']_1$$
$$= [W\overline{W}s_{1,2}s_{0,1}(s_{0,1})'(s_{1,2})'s_{0,2}s_{1,2}(s_{1,2})'(s_{0,2})']_1$$
$$= [W\overline{W}(s_{0,1})'s_{0,1}s_{1,2}(s_{1,2})'(s_{1,2})'s_{1,2}s_{0,2}(s_{0,2})']_1$$
$$= [W\overline{W}(s_{0,1})'s_{0,1}00s_{0,2}(s_{0,2})']_1 = [W\overline{W}(s_{0,1})'s_{0,1}s_{0,2}(s_{0,2})']_1$$
$$= [W\overline{W}s_{0,2}s_{0,1}(s_{0,1})'(s_{0,2})']_1 = [W\overline{W}\widetilde{W}W']_1 = 0\,.$$

In the above discussion, we can see that $\partial_1\partial_2(s) = [M]_1$ where M is a word of the elements $s_{i,j}$ and $(s_{i,j})'$, $0 \le i < j \le 2$, where $s_{i,j}$ appears twice in M and $(s_{i,j})'$ appears twice in M. Moreover, after using Corollary 1.3., we replaced these appearances in M by 00, which leads to the conclusion that $\partial_1\partial_2(s) = 0$. Next we will show that we can apply the same process for any n, showing that $\partial_{n-1}\partial_n(s) = 0$.

Let s be an n-simplex. Using Corollary 1.3, $\partial_n(s)$ can be represented as:

$$\partial_n(s) = [r_0(r_0)'r_1(r_1)'...r_t(r_t)'....r_n(r_n)']_1$$

where $r_{2t} = s_{2t}$ and $r_{2t+1} = (s_{2t+1})'$, i.e. $r_t = s_t$ for t even, and $r_t = (s_t)'$ for t odd. So,

$$\partial_{n-1}\partial_n(s) = [\partial_{n-1}(r_0)\partial_{n-1}((r_0)')...\partial_{n-1}(r_t)\partial_{n-1}((r_t)')....\partial_{n-1}(r_n)\partial_{n-1}((r_n)')]_1\,.$$

For any t, $\partial_{n-1}(s_t) = [U_t\,\overline{U}_t]_1$ and $\partial_{n-1}((s_t)') = (\partial_{n-1}(s_t))' = [U_t\,\overline{U}_t]_2$, where $U_t = s_{t,0}s_{t,1}...s_{t,t-1}s_{t,t+1}...s_{t,n}$.

If $r_t = s_t$, then

$$\partial_{n-1}(r_t)\partial_{n-1}((r_t)') = [U_t\,\overline{U}_t]_1[U_t\,\overline{U}_t]_2 = [U_t\,\overline{U}_t];$$

and if $r_t = (s_t)'$ then

$$\partial_{n-1}(r_t)\partial_{n-1}((r_t)') = [U_t\,\overline{U}_t]_2[U_t\,\overline{U}_t]_1 = [(U_t)'\widetilde{U}_t]_1[(U_t)'\widetilde{U}_t]_2 = [(U_t)'\widetilde{U}_t].$$

All this implies that

$$\partial_{n-1}\partial_n(s) = [V_0\,\overline{V}_0\,V_1\,\overline{V}_1....V_t\,\overline{V}_t....V_{n-1}\,\overline{V}_{n-1}\,V_n\,\overline{V}_n]_1 = [M]_1,$$

where $V_t = U_t$ for t even and $V_t = (U_t)'$ for t odd, and M is a word of the elements $s_{i,j}$ and $(s_{i,j})'$ for all $0 \le i < j \le n$. Moreover, for any $0 \le i < j \le n$, $s_{i,j}$ appears twice in M, once in $V_i\,\overline{V}_i$ and once in $V_j\,\overline{V}_j$ and similarly $(s_{i,j})'$ appears twice in M, once in $V_i\,\overline{V}_i$ and once in $V_j\,\overline{V}_j$.

Let $0 \le i < j \le n$. Using Proposition 1.2. and Corollary 1.3, we will rearrange the word M to a new word N of the form $N = N_1 s_{i,j}s_{i,j}(s_{i,j})'(s_{i,j})'N_2$. This will imply that we can replace $s_{i,j}$ and $(s_{i,j})'$ by 0. By doing this for any

$0 \le i < j \le n$, at the end we will obtain that $\partial_{n-1}\partial_n(s) = 0$.

First we rearrange the word M to the word $V_i \overline{V}_i V_j \overline{V}_j M_1$, and next we look only at the word $A = V_i \overline{V}_i V_j \overline{V}_j$.

For s_i and s_j we write U_i and U_j as:

$U_i = s_{i,0}s_{i,1}\ldots s_{i,i-1}s_{i,i+1}\ldots s_{i,j-1}s_{i,j}s_{i,j+1}\ldots s_{i,n} = U_{i1}s_{i,j}U_{i2}$ and

$U_j = s_{j,0}s_{j,1}\ldots s_{j,i-1}s_{j,i}s_{j,i+1}\ldots s_{j,j-1}s_{j,j+1}\ldots s_{j,n} = U_{j1}s_{j,i}U_{j2}$.

With this notation, $|U_{i1}| = j-1,\ |U_{i2}| = n-j,\ |U_{j1}| = i,\ |U_{j2}| = n-i-1$.

Now, we have to discuss the parity of i and j, and so we consider the following four cases.

Case 1. i and j even. Then, $|U_{i1}| + |U_{j1}| = j+i-1$ is odd, and so:

$$\begin{aligned}[A] &= [U_i \overline{U}_i U_j \overline{U}_j] = [U_{i1} s_{i,j}U_{i2} \overline{U}_{i2}(s_{i,j})' \overline{U}_{i1} U_{j1} s_{j,i}U_{j2} \overline{U}_{j2}(s_{j,i})' \overline{U}_{j1}] \\ &= [U_{i1} (U_{i2})' \widetilde{U}_{i2} s_{i,j}(s_{i,j})'\overline{U}_{i1} U_{j1} s_{j,i}(s_{j,i})' (U_{j2})' \widetilde{U}_{j2} \overline{U}_{j1}] \\ &= [U_{i1} (U_{i2})' \widetilde{U}_{i2} \widetilde{U}_{j1} (U_{i1})'(s_{i,j})'s_{i,j}s_{j,i}(s_{j,i})' (U_{j2})' \widetilde{U}_{j2} \overline{U}_{j1}] \\ &= [B_1 s_{i,j}s_{i,j}(s_{i,j})'(s_{i,j})'B_2].\end{aligned}$$

Case 2. i odd, j even. Then, $|U_{i1}| + |U_{j1}| = j+i-1$ is even, and so:

$$\begin{aligned}[A] &= [(U_i)'\widetilde{U}_i U_j \overline{U}_j] = [(U_{i1})'(s_{i,j})'(U_{i2})'\widetilde{U}_{i2} s_{i,j} \widetilde{U}_{i1} U_{j1} s_{j,i}U_{j2} \overline{U}_{j2}(s_{j,i})'\overline{U}_{j1}] \\ &= [(U_{i1})'U_{i2} \overline{U}_{i2} (s_{i,j})'s_{i,j}\widetilde{U}_{i1} U_{j1} s_{j,i}(s_{j,i})' (U_{j2})' \widetilde{U}_{j2} \overline{U}_{j1}] \\ &= [(U_{i1})'U_{i2} \overline{U}_{i2} \widetilde{U}_{i1} U_{j1}(s_{i,j})'s_{i,j}s_{j,i}(s_{j,i})'(U_{j2})'\widetilde{U}_{j2} \overline{U}_{j1}] \\ &= [C_1 s_{i,j}s_{i,j}(s_{i,j})'(s_{i,j})'B_2].\end{aligned}$$

Case 3. i even, j odd. Then, $|U_{i1}| + |U_{j1}| = j+i-1$ is even, and so:

$$\begin{aligned}[A] &= [U_i \overline{U}_i (U_j)'\widetilde{U}_j] = [U_{i1} s_{i,j}U_{i2} \overline{U}_{i2}(s_{i,j})'\overline{U}_{i1} (U_{j1})'(s_{j,i})'(U_{j2})'\widetilde{U}_{j2} s_{j,i} \widetilde{U}_{j1}] \\ &= [U_{i1} (U_{i2})'\widetilde{U}_{i2} s_{i,j}(s_{i,j})'\overline{U}_{i1} U_{j1} (s_{j,i})'s_{j,i} U_{j2} \overline{U}_{j2} \widetilde{U}_{j1}] \\ &= [U_{i1} (U_{i2})'\widetilde{U}_{i2} \overline{U}_{i1} U_{j1} s_{i,j}(s_{i,j})'(s_{j,i})'s_{j,i} U_{j2} \overline{U}_{j2} \widetilde{U}_{j1}] \\ &= [C_2 s_{i,j}s_{i,j}(s_{i,j})'(s_{i,j})'(B_2)'].\end{aligned}$$

Case 4. i odd, j odd. Then, $|U_{i1}| + |U_{j1}| = j+i-1$ is odd, and so:

$$\begin{aligned}[A] &= [(U_i)'\widetilde{U}_i (U_j)'\widetilde{U}_j] \\ &= [(U_{i1})'(s_{i,j})'(U_{i2})'\widetilde{U}_{i2}s_{i,j} \widetilde{U}_{i1} (U_{j1})'(s_{j,i})'(U_{j2})'\widetilde{U}_{j2} s_{j,i} \widetilde{U}_{j1}] \\ &= [(U_{i1})'U_{i2} \overline{U}_{i2} (s_{i,j})'s_{i,j}\widetilde{U}_{i1} (U_{j1})'(s_{j,i})'s_{j,i} U_{j2} \overline{U}_{j2} \widetilde{U}_{j1}] \\ &= [(U_{i1})'U_{i2} \overline{U}_{i2} \overline{U}_{j1} U_{i1} s_{i,j}(s_{i,j})'(s_{j,i})'s_{j,i} U_{j2} \overline{U}_{j2} \widetilde{U}_{j1}] \\ &= [C_3 s_{i,j}s_{i,j}(s_{i,j})'(s_{i,j})'(B_2)'].\end{aligned}$$

With the above discussion we have completed the proof of the following:

Theorem 2.1. For a simplicial complex K, let $(S_n(K),[\,])$ and ∂_n be defined as above. Then the sequence

$$0 \overset{\partial_0}{\leftarrow} (S_0(K),[\,]) \overset{\partial_1}{\leftarrow} (S_1(K),[\,]) \overset{\partial_2}{\leftarrow} \;\ldots\; \overset{\partial_n}{\leftarrow} (S_n(K),[\,]) \overset{\partial_{n+1}}{\leftarrow} (S_{n+1}(K),[\,]) \overset{\partial_{n+2}}{\leftarrow} \ldots.$$

is a weak (4,2)-chain complex . □

At the end we give the following example.

Example 2.2. Let K be the simplicial complex having three vertices and three 1-simplices, i.e. $K^0 = \{v_0, v_1, v_2\}$ and $K^1 = \{(v_0v_1), (v_0v_2), (v_1v_2)\}$. The weak (4,2)-chain wC(K) is:

$$0 \overset{\partial_0}{\leftarrow} (S_0(K),[\,]) \overset{\partial_1}{\leftarrow} (S_1(K),[\,]) \overset{\partial_2}{\leftarrow} 0 \;\ldots\; \overset{\partial_n}{\leftarrow} 0 \overset{\partial_{n+1}}{\leftarrow} 0 \overset{\partial_{n+2}}{\leftarrow} \ldots. \;.$$

It can be checked that the (4,2)-homology group $H_{1,*}(K)$ contains as a subgroup the ordinary homology group $H_1(K)$ but $H_{1,*}(K)$ and $H_1(K)$ are not isomorphic.

References

[1] D. Dimovski; Examples of vector valued groups, Contributions, Macedonian Academy of Science and Arts, VI, 2, Skopje 1985, 105-144.

[2] D. Dimovski; Free (n+1,n)-groups, Vector Valued Semigroups and Groups, Maced. Acad. of Sci. and Arts, Skopje, 1988, 103-122.

[3] D. Dimovski, A.Ahmeti: *(4,2)-Homology groups,* Contributions, Macedonian Academy of Science and Arts, Skopje, 2000, 33-46.

[4] D. Dimovski, A.Ahmeti: Cubical weak (4,2)-chain complexes for topological spaces, Mathematica Macedonica Vol.5., 2007, 25-36.

[5] D.Dimovski , S.Ilic; Commutative (2m,m)-groups, Vector Valued Semigroups and Groups, Maced. Acad. of Sci. and Arts, Skopje, 1988, 79-90.

[6] D. Dimovski, S.Ilic; Free commutative (2m,m)-groups, Matematički bilten, 13, Skopje 1989, 25-34.

[7] D. Dimovski, B. Janeva, S. Ilic: Free (n,m)-groups, Communications in Algebra, 19 (3), 1991, 965-979.

[8] G. Čupona; Vector valued semigroups, Semigroup forum, Vol. 26, 1983, 65-74.

[9] G. Čupona, D.Dimovski; On a class of vector valued groups; Proc. of Conf. Alg. and Logic, Zagreb, (1984), 29-373.

[10] G. Cupona, N. Celakoski, S. Markovski. D. Dimovski: Vector valued qroupoids, semiqroups and groups, Vector Valued Semigroups and Groups, Maced. Acad. of Sci. and Arts, Skopje, 1988. 1-79.

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