On Faithful Representations of Finite $^1$ Semigroups $|S|$ over the Fields

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Abstract

By a representation of a semigroup $S$ of degree $n$ over a field $F$ we mean a homomorphism $\gamma$ of $S$ into the multiplicative semigroup of the algebra $M_n(F)$ of all $n \times n$ matrices with entries in $F$. A representation is called faithful if it is injective. In this paper we focus our attention to the dimension of the subalgebra of $M_n(F)$ generated by $\gamma(S)$, where $S$ is an $n$-element semigroup and $\gamma$ is a faithful representation of $S$ of degree $n$ over a field $F$. In Section 2 we deal with the case when $S$ and $\gamma$ are arbitrary; in Section 3 we focus our attention to the case when $S$ is left reductive and $\gamma$ is the right regular representation of $S$.

Mathematics Subject Classification: 20M30

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1 Introduction

The representation of semigroups by matrices is a central problem in the theory of semigroups. The literature of this topic is very rich, but here we refer to only the books [1], [6] and the survey [4].

Let $S$ be a semigroup and $F$ a field. By a representation of $S$ of degree $n$ over $F$ we mean a homomorphism $\gamma$ of $S$ into the multiplicative semigroup of

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the algebra $M_n(\mathbb{F})$ of all $n \times n$ matrices with entries in $\mathbb{F}$. If $\gamma$ is injective then the representation is said to be faithful.

In this paper we focus our attention to representations of finite semigroups $S$ of degree $|S|$. We prove theorems about the dimension of the subalgebra of $M_n(\mathbb{F})$ generated by $\gamma(S)$, where $S$ is an $n$-element semigroup and $\gamma$ is a faithful representation of $S$ of degree $n$. We also present some results on couples $(k,n)$ of positive integers $k$ and $n$ with $k \leq n$ which satisfy, for a fixed field $\mathbb{F}$, the following condition: there is an $n$-element semigroup $S$ and a faithful representation $\gamma$ of $S$ of degree $n$ over $\mathbb{F}$ such that the dimension of the subalgebra of $M_n(\mathbb{F})$ generated by $\gamma(S)$ equals $k$. This is equivalent to the condition that the dimension of the kernel of the extension $\gamma^*$ of $\gamma$ to the semigroup algebra $\mathbb{F}[S]$ is $n - k$ (see [1]).

In Section 2, we deal with the general case: the considered finite semigroups $S$ are arbitrary and the representations are their arbitrary faithful representation of degree $|S|$.

In Section 3 we consider a special case: the semigroups $S$ are the finite left reductive semigroups and the representations are their right regular representation.

For notations and notions not defined here, we refer to [1], [3], [5], [6] and [7].

## 2 The case of arbitrary representations

### Definition 2.1
Let $k$ and $n$ be positive integers. We say that $k$ is representable by $n$ (or $n$ represents $k$) over a field $\mathbb{F}$ if $k \leq n$ and there is an $n$-element semigroup $S$ and a faithful representation $\gamma$ of $S$ of degree $n$ over $\mathbb{F}$ such that the dimension of the subalgebra $A(\gamma(S))$ of the matrix algebra $M_n(\mathbb{F})$ generated by $\gamma(S)$ is $k$.

It is clear that $k$ is representable by $n$ if and only if there is an $n$-element semigroup of the multiplicative semigroup of the matrix algebra $M_n(\mathbb{F})$ such that the dimension of the subalgebra of $M_n(\mathbb{F})$ generated by $S$ is $k$.

### Theorem 2.2
Let $n$ be a positive integer. Then every positive integer $k$ with $\frac{n}{2} \leq k \leq n$ is representable by $n$ over every field $\mathbb{F}$ with $\text{char}(\mathbb{F}) \neq 2$.

**Proof.** Let $\mathbb{F}$ be a field with $\text{char}(\mathbb{F}) \neq 2$. Let $n$ and $k$ be positive integers with $\frac{n}{2} \leq k \leq n$. Denote $E_i$ ($i = 1, \ldots, k$) the matrix of $M_n(\mathbb{F})$ defined by the following way: $E_i$ is a diagonal matrix, in which the first $i$ upper elements in the diagonal equal the identity element of the field $\mathbb{F}$ and the other elements are the zero of $\mathbb{F}$. It is easy to see that

$$E_i E_j = E_{\min(i,j)}$$
for every \( i, j \in \{1, \ldots, k\} \). Let \( \mathcal{A} \) denote the subalgebra of the algebra \( M_n(\mathbb{F}) \) generated by the matrices \( E_1, \ldots, E_k \).

As the matrices \( E_1, \ldots, E_k \) are linearly independent over \( \mathbb{F} \),

\[ \dim(\mathcal{A}) = k. \]

Since \( \frac{n}{2} \leq k \), that is, \( n - k \leq k \) then the matrices

\( -E_1, \ldots, -E_{n-k} \)

are in \( \mathcal{A} \). As \( \text{char}(\mathbb{F}) \neq 2 \), the matrices

\[ E_1, \ldots, E_k, -E_1, \ldots, -E_{n-k} \]

are pairwise distinct and

\[ S = \{ E_1, \ldots, E_k, -E_1, \ldots, -E_{n-k} \} \]

is an \( n \)-element subset of \( M_n(\mathbb{F}) \). As

\[ (\pm E_i)(\pm E_j) \in \{ E_{\min(i,j)}, -E_{\min(i,j)} \} \]

for every \( i, j \in \{1, \ldots, k\} \),

\[ S = \{ E_1, \ldots, E_k, -E_1, \ldots, -E_{n-k} \} \]

is an \( n \)-element subsemigroup of the multiplicative semigroup of the algebra \( M_n(\mathbb{F}) \) such that \( S \) generates the subalgebra \( \mathcal{A} \) of \( M_n(\mathbb{F}) \). Since \( \dim(\mathcal{A}) = k \) then \( k \) is representable by \( n \) over \( \mathbb{F} \).

\[ \square \]

**Problem 1.** Is Theorem 2.2 true for arbitrary field?

**Theorem 2.3** If \( k \) is a positive integer which is representable by a positive integer \( n \) over a finite field \( \mathbb{F} \) then \( \log_{|\mathbb{F}|} n \leq k \).

**Proof.** Let \( \mathbb{F} \) be a finite field and \( k \) a positive integer which is representable by a positive integer \( n \). Then there is an \( n \)-element semigroup \( S \) in the multiplicative semigroup of the full matrix algebra \( M_n(\mathbb{F}) \) such that the dimension of the subalgebra \( \mathcal{A} \) of \( M_n(\mathbb{F}) \) generated by \( S \) is \( k \). Then \( n = |S| \leq |\mathcal{A}| = |\mathbb{F}|^k \). Thus \( \log_{|\mathbb{F}|} n \leq k \). \[ \square \]

Let \( n \) be a positive integer and \( \mathbb{F} \) a finite field with \( \text{char}(\mathbb{F}) \neq 2 \). By Theorem 2.2, the integers belonging to the interval \( \left[ \frac{n}{2}, n \right] \) are representable by \( n \) over \( \mathbb{F} \). By Theorem 2.3, the positive integers \( k \) with \( k < \log_{|\mathbb{F}|} n \) are not
representable by \(n\). What can we say about the positive integers belonging to the interval \([\log_{|B/Y|} n, \frac{n}{2}]\).

**Problem 2.** Let \(n\) be a positive integer and \(F\) a finite field with the condition \(\text{char}(F) \neq 2\). Is every positive integer \(k\) belonging to the interval \([\log_{|B/Y|} n, \frac{n}{2}]\) representable by \(n\)?

If the answer was yes, then a positive integer \(k\) would be representable by a positive integer \(n\) over a finite field \(F\) with \(\text{char}(F) \neq 2\) if and only if \(k\) would be in the interval \([\log_{|B/Y|} n, n]\).

**Problem 3.** Is it true that, for a fixed positive integer \(n\) and an arbitrary field \(F\), there is a positive integer \(k_0(n, F) \leq n\) depending on \(F\) and \(n\) such that a positive integer \(k\) is representable by \(n\) over \(F\) if and only if \(k\) belongs to the interval \([k_0(n, F), n]\)?

## 3 The case of the right regular representation

Let \(S\) be a finite semigroup and \(F\) a field. By an \(S\)-matrix over \(F\) we mean a single valued mapping \(A\) of the descartes product \(S \times S\) into \(F\). If we fix an ordering of the elements of \(S\), for example, \(S = \{s_1, \ldots, s_n\}\), then an \(S\)-matrix \(A\) can be written in the usual form: the element of \(A\) being in the \(i\)th row and the \(j\)th column equals \(A((s_i, s_j))\). In most of our proofs we will consider the semigroups \(S\) with a fixed ordering, and the \(S\)-matrices will be written in the usual form detailed above.

Let \(e\) and 0 denote the identity element and the zero element of a field \(F\), respectively. For an arbitrary element \(s\) of a finite semigroup \(S = \{s_1, \ldots, s_n\}\), consider the \(S\)-matrix

\[
R^{(s)} = [r^{(s)}_{i,j}]_{n \times n},
\]

where

\[
r^{(s)}_{i,j} = \begin{cases} 
eq & \text{if } s_is_j = s_j, \\ 0 & \text{otherwise.} \end{cases}
\]

This matrix will be called the right matrix of \(s\) over \(F\).

It is known (see, for example, Exercise 4(b) of §3.5 of [1]) that if \(S\) is a finite \(n\)-element semigroup then

\[
\mathcal{R}_F : s \mapsto R^{(s)}
\]

is a representations of \(S\) of degree \(n\) over \(F\). This representation (which is called the right regular representation of \(S\)) is faithful if and only if \(S\) is left
reductive, that is, for every $a, b \in S$, the assumption "$xa = xb$ for all $x \in S$" implies $a = b$.

For an arbitrary $n$-element semigroup $S$ and an arbitrary field $\mathbb{F}$, let $A(\mathcal{R}_F(S))$ denote the subalgebra of the matrix algebra $M_n(\mathbb{F})$ generated by $\mathcal{R}_F(S)$.

**Definition 3.1** Let $k$ and $n$ be positive integers. We say that $k$ is representable by $n$ (or $n$ represents $k$) over a field $\mathbb{F}$ under the right regular representation $\mathcal{R}_F$ if $k \leq n$ and there is an $n$-element left reductive semigroup $S$ such that the dimension of the subalgebra $A(\mathcal{R}_F(S))$ of the matrix algebra $M_n(\mathbb{F})$ generated by $\mathcal{R}_F(S)$ is $k$.

**Theorem 3.2** If a positive integer $n \leq 4$ represents a positive integer $k$ under the right regular representation $\mathcal{R}_F(S)$ then $k = n$.

**Proof.** In [2], we can find the Cayley-table of all nonisomorphic and nonanti-isomorphic semigroups containing $n$ elements for $2 \leq n \leq 5$. It is a matter of checking to see that the dimension of the subalgebra of $M_n(\mathbb{F})$ generated by $\mathcal{R}_F(S)$ equals $|S|$ for every left reductive semigroup $S$ with $|S| \leq 4$. $\square$

The next example shows that Theorem 3.2 is not true in case $n \geq 5$.

**Example 2.** Let $S = \{1, 2, 3, 4, 5\}$ be a semigroup defined by the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
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<td>4</td>
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<td>2</td>
<td>4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>

(see the Cayley table in the 7th row and the 10th column on page 167 of [2]).

As the columns of the table are pairwise distinct, $S$ is left reductive. It is a matter of checking to see that, for every field $\mathbb{F}$,

$$\mathbf{R}^{(4)} = -\mathbf{R}^{(1)} + \mathbf{R}^{(2)} + \mathbf{R}^{(3)} + 0\mathbf{R}^{(5)}$$

and the matrices

$$\mathbf{R}^{(1)}, \mathbf{R}^{(2)}, \mathbf{R}^{(3)}, \mathbf{R}^{(5)}$$

are linearly independent over $\mathbb{F}$. Thus $\text{dim}A(\mathcal{R}_F(S)) = 4$ and so $4$ is representable by $5$ over every field $\mathbb{F}$ under the right regular representation $\mathcal{R}_F$. 


Theorem 3.3  Let $\mathbb{F}$ be a field and $S_1, S_2$ arbitrary left reductive finite semigroups. Then

$$
\dim[A(\mathcal{R}_\mathbb{F}(S_1))]\dim[A(\mathcal{R}_\mathbb{F}(S_2))] = \dim[A(\mathcal{R}_\mathbb{F}(S_1 \times S_2))].
$$

Proof. Let $S_1 = \{a_i : i = 1, \ldots, |S_1|\}$ and $S_2 = \{b_j : j = 1, \ldots, |S_2|\}$ be arbitrary finite semigroups and $\mathbb{F}$ an arbitrary field. Consider the right regular representations of $S_1$ and $S_2$, respectively. Let $A^{(a_i)}$ and $B^{(b_j)}$ denote the right matrices of the elements $a_i \in S_1$ and $b_j \in S_2$ (corresponding to the above orderings of $S_1$ and $S_2$), respectively. Assume

$$
\dim A(\mathcal{R}_\mathbb{F}(S_1)) = m \quad \text{and} \quad \dim A(\mathcal{R}_\mathbb{F}(S_2)) = n.
$$

Let $B_1$ and $B_2$ denote bases of $A(\mathcal{R}_\mathbb{F}(S_1))$ and $A(\mathcal{R}_\mathbb{F}(S_2))$, respectively. We can suppose that $B_1 = \{A^{(a_1)}, \ldots, A^{(a_m)}\}$ and $B_2 = \{B^{(b_1)}, \ldots, B^{(b_n)}\}$.

It is clear that the direct product $S_1 \times S_2$ is also left reductive. Thus the right regular representation of $S_1 \times S_2$ is faithful. Consider the following ordering of the elements of $S_1 \times S_2$:

$$
S_1 \times S_2 = \{(a_1, b_1); \ldots; (a_1, b_{|S_2|}); \ldots; (a_{|S_1|}, b_1); \ldots; (a_{|S_1|}, b_{|S_2|})\}.
$$

It is a matter of checking to see that the right matrix $C^{(a_i, b_j)}$ of the element $(a_i, b_j) \in S_1 \times S_2$ (corresponding to the above ordering of $S_1 \times S_2$) is a matrix of blocks $C^{(a_i, b_j)}_{k,t}$ ($k, t \in \{1, \ldots, |S_1|\}$) such that

$$
C^{(i,j)}_{k,t} = a^{(a_i)}_{k,t}B^{(b_j)},
$$

where $a^{(a_i)}_{k,t}$ ($k, t = 1, \ldots, |S_1|$) are the elements of the right matrix $A^{(a_i)}$. We show that the right matrices $C^{(a_i, b_j)}$ ($i = 1, \ldots, m; j = 1, \ldots, n$) form a basis of $A(\mathcal{R}_\mathbb{F}(S_1 \times S_2))$.

To show that the matrices $C^{(a_i, b_j)}$ ($i = 1, \ldots, m; j = 1, \ldots, n$) are linearly independent (over $\mathbb{F}$), assume

$$
\sum_{j=1}^n \sum_{i=1}^m \gamma_{j,i} C^{(a_i, b_j)} = 0_{mn \times mn}
$$

for some $\gamma_{j,i} \in \mathbb{F}$. Then, for every $k, t \in \{1, \ldots, |S_1|\}$,

$$
\sum_{j=1}^n \sum_{i=1}^m \gamma_{j,i} C^{(a_i, b_j)}_{k,t} = 0_{n \times n},
$$

that is,

$$
\sum_{j=1}^n \sum_{i=1}^m \gamma_{j,i} a^{(a_i)}_{k,t}B^{(b_j)} = 0_{n \times n}.
$$

Then

$$
\sum_{j=1}^n \left( \sum_{i=1}^m \gamma_{j,i} a^{(a_i)}_{k,t} \right)B^{(b_j)} = 0_{n \times n}.
$$
from which we obtain that, for every \( j = 1, \ldots, n \) (and every \( k, t = 1, \ldots, |S_1| \)),

\[
\sum_{i=1}^{m} \gamma_{j,i} a_{k,t}^{(a_i)} = 0,
\]

because the matrices \( B^{(b_1)}, \ldots, B^{(b_n)} \) are linearly independent. As the coefficients \( \gamma_{j,i} \) do not depend on \( k \) and \( t \), we have

\[
\sum_{i=1}^{m} \gamma_{j,i} A^{(a_i)} = 0_{m \times m}
\]

for every \( j = 1, \ldots, n \). As the matrices \( A^{(a_1)}, \ldots, A^{(a_m)} \) are linearly independent, we get \( \gamma_{j,i} = 0 \) for every \( j = 1, \ldots, n \) and \( i = 1, \ldots, m \).

In the next, we show that the matrices \( C^{(a_i,b_j)} \) (\( i = 1, \ldots, m; j = 1, \ldots, n \)) generate \( A(\mathcal{R}_p(S_1 \times S_2)) \). Let \( (x, y) \in S_1 \times S_2 \) be arbitrary. As \( B_2 \) is a basis of \( A(\mathcal{R}_p(S_2)) \), there are \( \beta_j \in F \) (\( j = 1, \ldots, n \)) such that

\[
B^{(y)} = \sum_{j=1}^{n} \beta_j B^{(b_j)}.
\]

Then, for every \( k, t \in \{1, \ldots, |S_1|\} \),

\[
a_{k,t}^{(x)} B^{(y)} = \sum_{j=1}^{n} \beta_j a_{k,t}^{(x)} B^{(b_j)}.
\]

As \( B_1 \) is a basis of \( A(\mathcal{R}_p(S_1)) \), there are \( \alpha_i \in F \) (\( i = 1, \ldots, m \)) such that

\[
A^{(x)} = \sum_{i=1}^{m} \alpha_i A^{(a_i)},
\]

that is,

\[
a_{k,t}^{(x)} = \sum_{i=1}^{m} \alpha_i a_{k,t}^{(a_i)}
\]

for every \( k, t = 1, \ldots, |S_1| \). Then

\[
a_{k,t}^{(x)} B^{(y)} = \sum_{j=1}^{n} \beta_j \left( \sum_{i=1}^{m} \alpha_i a_{k,t}^{(a_i)} \right) B^{(b_j)} = \sum_{j=1}^{n} \sum_{i=1}^{m} \left( \beta_j \alpha_i \right) (a_{k,t}^{(a_i)} B^{(b_j)})
\]

and so

\[
C^{(x,y)} = \sum_{j=1}^{n} \sum_{i=1}^{m} \left( \beta_j \alpha_i \right) C^{(a_i,b_j)}
\]

for every \( k, t = 1, \ldots, |S_1| \). As the coefficients \( \alpha_i \) (\( i = 1, \ldots, m \)) and \( \beta_j \) (\( j = 1, \ldots, n \)) do not depend on \( k \) and \( t \),

\[
C^{(x,y)} = \sum_{j=1}^{n} \sum_{i=1}^{m} \left( \beta_j \alpha_i \right) C^{(a_i,b_j)}.
\]

Thus the theorem is proved. \( \square \)

On the set of all positive integers consider the following binary relation: \( k \sim_{\mathcal{R}_p} n \) if and only if \( k \) is representable by \( n \) over the field \( F \) under the right regular representation \( \mathcal{R}_p \).
Corollary 3.4 If \( k \sim_{R_{\mathcal{F}}} n \) and \( t \sim_{R_{\mathcal{F}}} m \) for some positive integers \( k, t, n, m \) then \( kt \sim_{R_{\mathcal{F}}} nm \).

Proof. Assume

\[
k \sim_{R_{\mathcal{F}}} n \quad \text{and} \quad t \sim_{R_{\mathcal{F}}} m
\]

for some positive integers \( k, t, n, m \). Then there are left reductive semigroups \( S_1 \) and \( S_2 \) such that

\[
|S_1| = n \quad \text{and} \quad |S_2| = m
\]

and

\[
dim A(R_{\mathcal{F}}(S_1)) = k \quad \text{and} \quad dim A(R_{\mathcal{F}}(S_2)) = t.
\]

By Theorem 3.3,

\[
dim A(R_{\mathcal{F}}(S_1 \times S_2)) = kt.
\]

Thus \( kt \sim_{\mathcal{F}} nm \). \( \Box \)

Theorem 3.5 Let \( \mathcal{F} \) be a field and \( S_1, S_2 \) be arbitrary finite left reductive semigroups. Then

\[
A(R_{\mathcal{F}}(S_1)) \bigotimes A(R_{\mathcal{F}}(S_2)) \cong_{\text{Alg}} A(R_{\mathcal{F}}(S_1 \times S_2)),
\]

where \( \bigotimes \) denotes the tensor product and \( \cong_{\text{Alg}} \) denotes the algebra isomorphism.

Proof. We use the notations of the proof of Theorem 3.3. Consider the tensor product

\[
A(R_{\mathcal{F}}(S_1)) \bigotimes A(R_{\mathcal{F}}(S_2))
\]

of the vector spaces \( A(R_{\mathcal{F}}(S_1)) \) and \( A(R_{\mathcal{F}}(S_2)) \). The tensors

\[
A^{(a_i)} \otimes B^{(b_j)} \quad (i = 1, \ldots, m; j = 1, \ldots n)
\]

form a basis of \( A(R_{\mathcal{F}}(S_1)) \bigotimes A(R_{\mathcal{F}}(S_2)) \) and the product between them is

\[
(A^{(a_i)} \otimes B^{(b_j)})(A^{(a_k)} \otimes B^{(b_l)}) = (A^{(a_ia_k)} \otimes B^{(b jb_l)}).
\]

By the proof of Theorem 3.3,

\[
\{ C^{(a_i,b_j)} : i = 1, \ldots m; j = 1, \ldots n \}
\]
is a basis of the algebra \( \mathcal{A}(\mathcal{R}_F(S_1 \times S_2)) \). The product between the elements of this basis is the following:

\[
C^{(a_i, b_j)} C^{(a_k, b_t)} = C^{(a_i a_k, b_j b_t)}.
\]

As

\[
dim(\mathcal{A}(\mathcal{R}_F(S_1))) \otimes \mathcal{A}(\mathcal{R}_F(S_2)) = dim(\mathcal{A}(\mathcal{R}_F(S_1 \times S_2)))
\]

by Theorem 3.3, the mapping

\[
\phi : (A^{(a_i)} \otimes B^{(b_j)}) \mapsto C^{(a_i, b_j)} \quad i = 1, \ldots m; j = 1, \ldots n
\]

is an isomorphism of the vector space \( \mathcal{A}(\mathcal{R}_F(S_1)) \otimes \mathcal{A}(\mathcal{R}_F(S_2)) \) onto the vector space \( \mathcal{A}(\mathcal{R}_F(S_1 \times S_2)) \). As

\[
\phi((A^{(a_i)} \otimes B^{(b_j)})(A^{(a_k)} \otimes B^{(b_t)})) = \phi((A^{(a_i, a_k)} \otimes B^{(b_j, b_t)})) =
\]

\[
= C^{(a_i a_k, b_j b_t)} = C^{(a_i, b_j)}(a_k, b_t) = C^{(a_i, b_j)} C^{(a_k, b_t)} =
\]

\[
= \phi((A^{(a_i)} \otimes B^{(b_j)})) \phi((A^{(a_k)} \otimes B^{(b_t)})),
\]

\( \phi \) is an algebra isomorphism of the tensor product \( \mathcal{A}(\mathcal{R}_F(S_1)) \otimes \mathcal{A}(\mathcal{R}_F(S_2)) \) onto the algebra \( \mathcal{A}(\mathcal{R}_F(S_1 \times S_2)) \). 

A congruence \( \sigma \) on a semigroup \( S \) is called a semilattice congruence if the factor semigroup \( Y = S/\sigma \) is a semilattice (a commutative semigroup in which every element is idempotent). If \( \sigma \) is a semilattice congruence of a semigroup \( S \) then the \( \sigma \)-classes \( S_\alpha \ (\alpha \in Y) \) of \( S \) are subsemigroups of \( S \). We say that a semigroup \( S \) is a semilattice of subsemigroups \( S_\alpha \ (\alpha \in Y) \) of \( S \) if there is a semilattice congruence \( \sigma \) on \( S \) such that \( S/\sigma \) is isomorphic to \( Y \) and the \( \sigma \)-classes of \( S \) are the subsemigroups \( S_\alpha \ (\alpha \in Y) \).

Theorem 3.6 Let \( S \) be a finite semigroup which is a semilattice of two left reductive subsemigroups \( A \) and \( B \) of \( S \). Then

\[
dim \mathcal{A}(\mathcal{R}_F(S)) \geq dim \mathcal{A}(\mathcal{R}_F(A)) + dim \mathcal{A}(\mathcal{R}_F(B)).
\]

Proof. It is clear that one of \( A \) and \( B \), for example, \( A \) is an ideal of \( S \). If \( c, d \in S \) be arbitrary elements such that \( xc = xd \) holds for all \( x \in S \) then \( c^2 = cd = d^2 \) and so both of \( c \) and \( d \) are in either \( A \) or \( B \). As \( A \) and \( B \) are left reductive, we get \( c = d \). Thus \( S \) is left reductive and so the right regular representation of \( S \) is faithful. Let \( A = \{a_1, \ldots, a_n\} \) and \( B = \{b_1, \ldots, b_m\} \). Let

\[
A^{(a_i)} \ (i = 1, \ldots n) \quad \text{and} \quad B^{(b_j)} \ (j = 1, \ldots, m)
\]
denote the right matrices of the elements \( a_i \in A \) and \( b_j \in B \) corresponding to the above ordering of \( A \) and \( B \), respectively.

Consider the following ordering of \( S \):

\[
S = \{ a_1, \ldots, a_n, b_1, \ldots, b_m \}.
\]

The right matrices \( C^{(s)} \) of the elements \( s \) of \( S \) corresponding to the above ordering of \( S \) are matrices of blocks

\[
C^{(s)}_{k,t} \ (k, t \in \{1, 2\})
\]

such that the type of \( C^{(s)}_{1,1} \) is \( n \times n \) and the type of \( C^{(s)}_{2,2} \) is \( m \times m \). Moreover, \( C^{(a_i)}_{1,1} = A^{(a_i)} \), \( C^{(a_i)}_{2,2} = 0_{m \times m} \) for every \( a_i \in A \), and \( C^{(b_j)}_{2,2} = B^{(b_j)} \) for every \( b_j \in B \). Assume

\[
dim A(\mathcal{R}_B(A)) = k \quad \text{and} \quad dim A(\mathcal{R}_B(B)) = t.
\]

We can suppose that \( A^{(a_i)} \) \((i = 1, \ldots, k)\) and \( B^{(b_j)} \) \((j = 1, \ldots, t)\) are the basis of \( A(\mathcal{R}_B(A)) \) and \( A(\mathcal{R}_B(B)) \), respectively. We show that the system of matrices \( C^{(a_i)} \) and \( C^{(b_j)} \) \((i = 1, \ldots, k; j = 1, \ldots, t)\) is linearly independent. Assume

\[
\Sigma_{i=1}^k \alpha_i C^{(a_i)} + \Sigma_{j=1}^t \beta_j C^{(b_j)} = 0_{(n+m) \times (n+m)}
\]

Then

\[
\Sigma_{i=1}^k \alpha_i C^{(a_i)}_{2,2} + \Sigma_{j=1}^t \beta_j C^{(b_j)}_{2,2} = 0_{m \times m}
\]

and so

\[
\Sigma_{j=1}^t \beta_j B^{(b_j)} = 0_{m \times m},
\]

because \( C^{(a_i)}_{2,2} = 0_{m \times m} \) and \( C^{(b_j)}_{2,2} = B^{(b_j)} \) for every \( a_i \in A \) and \( b_j \in B \). As the matrices \( B^{(b_j)} \) \((j = 1, \ldots, t)\) are linearly independent, we get \( \beta_j = 0 \) for every \( j = 1, \ldots, t \). Then

\[
\Sigma_{i=1}^k \alpha_i C^{(a_i)} = 0_{(n+m) \times (n+m)}
\]

and so

\[
0_{n \times n} = \Sigma_{i=1}^k \alpha_i C^{(a_i)}_{1,1} = \Sigma_{i=1}^k \alpha_i A^{(a_i)}.
\]

As the matrices \( A^{(a_i)} \) \((i = 1, \ldots, k)\) are linearly independent, we get \( \alpha_i = 0 \) for every \( i = 1, \ldots, k \). Thus the matrices

\[
C^{(a_1)}, \ldots, C^{(a_k)}, C^{(b_1)}, \ldots, C^{(b_t)}
\]
are linearly independent. From this it follows that
\[
dim \mathcal{A}(\mathcal{R}_Y(S)) \geq \dim \mathcal{A}(\mathcal{R}_Y(A)) + \dim \mathcal{A}(\mathcal{R}_Y(B)).
\]

Let a semigroup \( S \) be a semilattice \( Y \) of semigroups \( S_\alpha, \alpha \in Y \). Assume that, for every \( \alpha, \beta \in Y \) with \( \alpha \geq \beta \), there is a homomorphism \( f_{\alpha,\beta} \) of \( S_\alpha \) into \( S_\beta \) such that the following are satisfied.

1. For each \( \alpha \in Y \), \( f_{\alpha,\alpha} \) is the identity mapping of \( S_\alpha \).
2. If \( \alpha \geq \beta \geq \gamma \) then \( f_{\alpha,\beta}f_{\beta,\gamma} = f_{\alpha,\gamma} \).
3. If \( a \in S_\alpha \) and \( b \in S_\beta \) then \( ab = (a)f_{\alpha,\beta}(b)f_{\beta,\alpha} \).

In such a case \( S \) is called a strong semilattice \( Y \) of semigroups \( S_\alpha \) \( (\alpha \in Y) \).

**Theorem 3.7** Let \( S \) be a finite semigroup which is a strong semilattice of two left reductive subsemigroups \( A \) and \( B \) of \( S \) with \( AB \subseteq A \). If \( \dim \mathcal{A}(\mathcal{R}_Y(B)) = |B| \) then
\[
\dim \mathcal{A}(\mathcal{R}_Y(S)) = \dim \mathcal{A}(\mathcal{R}_Y(A)) + \dim \mathcal{A}(\mathcal{R}_Y(B)).
\]

**Proof.** We use the notations of the proof of Theorem 3.6. As \( S \) is a strong semilattice of subsemigroups \( A = \{a_1, \ldots, a_n\} \) and \( B = \{b_1, \ldots, b_m\} \) such that \( A \) is an ideal of \( S \), there is a homomorphism \( \varphi \) of \( B \) into \( A \) such that \( b_ja_i = \varphi(b_j)a_i \) for every \( b_j \in B \) and \( a_i \in A \). This homomorphism induces a mapping \( \varphi^* \) of \( \{1, \ldots, m\} \) into \( \{1, \ldots, n\} \) with the following way: \( \varphi^*(j) = i \) if and only if \( \varphi(b_j) = a_i \). From this it follows that the \( j^{th} \) row of the matrix \( C_{2,1}^{(a_i)} \) \( (j = 1, \ldots, m) \) equals the \( (\varphi^*(j))^{th} \) row of the right matrix \( A^{(a_i)} \) for every \( a_i \in A \). Thus if a linear combination \( \sum_{i=1}^{k} \beta_i A^{(a_i)} \) equals a right matrix \( A^{(a)} \) \( (a \in A) \) then \( \sum_{i=1}^{k} \beta_i C_{2,1}^{(a_i)} \) equals the matrix \( C_{2,1}^{(a)} \). As \( \dim \mathcal{A}(\mathcal{R}_Y(B)) = |B| = m \), Theorem 3.6 implies that the matrices
\[
C^{(a_1)}, \ldots, C^{(a_k)}, C^{(b_1)}, \ldots, C^{(b_m)}
\]
are linearly independent. We show that they form a basis of the subalgebra \( \mathcal{A}(\mathcal{R}_Y(S)) \) of the matrix algebra \( \mathbb{F}_{(n+m)\times(n+m)} \). It is sufficient to show that every matrix \( C^{(a_j)} \) \( (j = k + 1, \ldots, n) \) can be expressed as a linearly combination of the matrices \( C^{(a_1)}, \ldots, C^{(a_k)}, C^{(b_1)}, \ldots, C^{(b_m)} \). Let \( C^{(a)}, a \in \{a_{j+1}, \ldots, a_n\} \) be an arbitrary matrix. Then
\[
A^{(a)} = \sum_{i=1}^{k} \beta_i A^{(a_i)}
\]
for some $\beta_i \in \mathbb{F}$. By the above result, this equation implies
\[
C_{2,1}^{(a)} = \sum_{i=1}^{k} \beta_i C_{2,1}^{(a_i)}
\]
and so
\[
C^{(a)} = \sum_{i=1}^{k} \beta_i C^{(a_i)}.
\]
Thus the matrices
\[
C^{(a_1)}, \ldots, C^{(a_k)}, C^{(b_1)}, \ldots, C^{(b_m)}
\]
form a basis of the subalgebra $\mathcal{A}(\mathcal{R}_Y(S))$. Hence
\[
\dim \mathcal{A}(\mathcal{R}_Y(S)) = \dim \mathcal{A}(\mathcal{R}_Y(A)) + \dim \mathcal{A}(\mathcal{R}_Y(B)).
\]
\[\square\]

**Theorem 3.8** If a finite semigroup $S$ is a semilattice $Y$ of left reductive semigroups $S_\alpha$ ($\alpha \in Y$) then
\[
\dim \mathcal{A}(\mathcal{R}_Y(S)) \geq \sum_{\alpha \in Y} \dim \mathcal{A}(\mathcal{R}_Y(S_\alpha)).
\]

**Proof.** The assertion will be proved by induction on $n = |Y|$. If $n = 1$ then the assertion is obvious. If $n = 2$ then the assertion follows from Theorem 3.6. Let $n \geq 3$. Assume that the assertion is true for all semilattices of order less then $n$. Let $Y$ be a semilattice such that $|Y| = n$. Let $S$ be a semigroup which is a semilattice $Y$ of left reductive semigroups $S_\alpha$, $\alpha \in Y$. As $Y$ is a semilattice and $|Y| \geq 3$, there are elements $\alpha, \beta \in Y$ such that
\[
\alpha \beta \neq \beta.
\]
Let $I_\beta$ denote the ideal of $Y$ generated by $\beta$. It is known that
\[
I_\beta = \{ \xi \in Y : \xi \beta = \xi \}.
\]
As
\[
\beta, \alpha \beta \in I_\beta,
\]
$\alpha \beta \neq \beta$ implies
\[
|I_\beta| \geq 2.
\]
First consider the case when $I_\beta \neq Y$. Then $|Y \setminus I_\beta| \leq n - 2$. As $I_\beta$ is a subsemigroup of $Y$, the union $A_\beta$ of subsemigroups $S_\xi$ ($\xi \in I_\beta$) form a subsemigroup of $S$. As $I_\beta \subset Y$, we get

$$dim A(R_\Phi(A_\beta)) \geq \sum_{\xi \in I_\beta} dim A(R_\Phi(S_\xi))$$

by induction. As $I_\beta$ is an ideal of $Y$, the semigroup $S$ is a semilattice of the semigroups $S_\eta$ ($\eta \in Y \setminus I_\beta$) and the subsemigroup $A_\beta$. As $|Y \setminus I_\beta| + 1 \leq n - 1$, we get

$$dim A(R_\Phi(S)) \geq dim A(R_\Phi(A_\beta)) + \sum_{\eta \in Y \setminus I_\beta} dim A(R_\Phi(S_\eta))$$

by induction. This and the above

$$dim A(R_\Phi(A_\beta)) \geq \sum_{\xi \in I_\beta} dim A(R_\Phi(S_\xi))$$

together imply

$$dim A(R_\Phi(S)) \geq \sum_{\alpha \in Y} dim A(R_\Phi(S_\alpha)).$$

In the next consider the case when $I_\beta = Y$. It means that $\beta$ is the identity element of $Y$. In this case $\xi \eta \neq \beta$ for every $\beta \not\in \{\xi, \eta\}$. Indeed, if there were elements $\xi, \eta \in Y$ with $\xi \neq \beta$ and $\eta \neq \beta$ such that $\eta \xi = \beta$ then, for every $\alpha \in Y$, we would have $\alpha \eta \xi = \alpha \beta = \alpha$ and so $\alpha \xi = \alpha$. It would imply that $\xi$ is an identity element of $Y$ which would contradicts $\xi \neq \beta$.

Thus $X = Y \setminus \{\beta\}$ is a subsemilattice of $Y$. Let $S^*$ denote the subsemigroup of $S$ which is a semilattice $X$ of semigroups $S_\tau$, $\tau \in X$. Then $S$ is a semilattice of $S^*$ and $S_\beta$ and so

$$dim A(R_\Phi(S)) \geq dim A(R_\Phi(S^*)) + dim A(R_\Phi(S_\beta))$$

by Theorem 3.6. As $|X| = |Y| - 1$,

$$dim A(R_\Phi(S^*)) = \sum_{\tau \in X} dim A(R_\Phi(S_\tau))$$

by induction. Consequently

$$dim A(R_\Phi(S)) \geq \sum_{\alpha \in Y} dim A(R_\Phi(S_\alpha)).$$
Theorem 3.9 Let $S$ be a semigroup which is a semilattice $Y$ of left reductive finite semigroups $S_\alpha$ ($\alpha \in Y$) such that $\dim A\mathcal{R}_F(S_\alpha) = |S_\alpha|$ for every $\alpha \in Y$. Then $\dim A\mathcal{R}_F(S) = |S|$.

Proof. Applying Theorem 3.8 and the assumptions of this theorem, we get

$$\sum_{\alpha \in Y} |S_\alpha| = |S| \geq \dim A\mathcal{R}_F(S) \geq \sum_{\alpha \in Y} \dim A\mathcal{R}_F(S_\alpha) = \sum_{\alpha \in Y} |S_\alpha|$$

and so $\dim A\mathcal{R}_F(S) = |S|$.

Theorem 3.10 If a finite semigroup $S$ is a semilattice $Y$ of monoids $S_\alpha$ ($\alpha \in Y$) $\alpha \in Y$ then $\dim A\mathcal{R}_F(S) = |S|$.

Proof. It is easy to see that every monoid $M$ is left reductive and $\dim A\mathcal{R}_F(M) = |M|$. Thus our assertion follows from Theorem 3.9.

Theorem 3.11 If $S$ is a finite Clifford semigroup then $\dim A\mathcal{R}_F(S) = |S|$.

Proof. It is known that a semigroup is a Clifford semigroup if and only if it is a semilattice of groups (see Theorem 2.1 of [3]). Thus our assertion follows from Theorem 3.10.

Theorem 3.12 If $S$ is a finite semilattice then $\dim A\mathcal{R}_F(S) = |S|$.

Proof. As a semilattice is a semilattice of one-element monoids, our assertion follows from Theorem 3.10.

Corollary 3.13 If $k \sim_{\mathcal{R}_F} n$ then, for every positive integer $t$, $k + t \sim_{\mathcal{R}_F} n + t$.

Proof. Assume $k \sim_{\mathcal{R}_F} n$ for some positive integers $k$ and $n$. Then there is an $n$-element semigroup $A$ such that $\dim A\mathcal{R}_F(A) = k$. Let $t$ be a positive integer and $B$ a $t$-element semilattice. As $A$ is a finite semigroup, it has an idempotent element $e$. Let $\varphi$ denote the mapping of $B$ into $A$ such that $\varphi(b) = e$ for every $b \in B$. It is easy to see that $\varphi$ is a homomorphism. On the set $S = A \cup B$ define the following multiplication. Let the new multiplication on $A$ and $B$ is the old multiplication, respectively. For arbitrary $a \in A$ and $b \in B$, let $ab = a\varphi(b) = ae$ and $ba = \varphi(b)a = ea$. Then $S$ is a strong semilattice of $A$ and $B$ with $AB \subseteq A$. By Theorem 3.12, $\dim A\mathcal{R}_F(B) = |B| = t$ and so Theorem 3.6 implies $\dim A\mathcal{R}_F(S) = k + t$. Thus $k + t \sim_{\mathcal{R}_F} n + t$. 

\qed
References


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