International Journal of Algebra, Vol. 7, 2013, no. 3, 115 - 129
HIKARI Ltd, www.m-hikari.com

On Faithful Representations of Finite [1] Semigroups S of Degree $|S|$ over the Fields

Attila Nagy

Department of Algebra, Mathematical Institute
Budapest University of Technology and Economics
1521 Budapest, Pf. 91, Hungary
nagyat@math.bme.hu

Abstract

By a representation of a semigroup S of degree n over a field $\mathbb{F}$ we mean a homomorphism γ of S into the multiplicative semigroup of the algebra $M_n(\mathbb{F})$ of all $n \times n$ matrices with entries in $\mathbb{F}$. A representation is called faithful if it is injective. In this paper we focus our attention to the dimension of the subalgebra of $M_n(\mathbb{F})$ generated by $\gamma(S)$, where S is an n-element semigroup and γ is a faithful representation of S of degree n over a field $\mathbb{F}$. In Section 2 we deal with the case when S and γ are arbitrary; in Section 3 we focus our attention to the case when S is left reductive and γ is the right regular representation of S.

Mathematics Subject Classification: 20M30

Keywords: semigroup, representation of semigroups, full matrix algebra

1 Introduction

The representation of semigroups by matrices is a central problem in the theory of semigroups. The literature of this topic is very rich, but here we refer to only the books [1], [6] and the survey [4].

Let S be a semigroup and $\mathbb{F}$ a field. By a representation of S of degree n over $\mathbb{F}$ we mean a homomorphism γ of S into the multiplicative semigroup of

[1]Research supported by the Hungarian NFSR grant No. K77476

the algebra $M_n(\mathbb{F})$ of all $n \times n$ matrices with entries in $\mathbb{F}$. If γ is injective then the representation is said to be faithful.

In this paper we focus our attention to representations of finite semigroups S of degree $|S|$. We prove theorems about the dimension of the subalgebra of $M_n(\mathbb{F})$ generated by $\gamma(S)$, where S is an n-element semigroup and γ is a faithful representation of S of degree n. We also present some results on couples (k, n) of positive integers k and n with $k \leq n$ which satisfy, for a fixed field $\mathbb{F}$, the following condition: there is an n-element semigroup S and a faithful representation γ of S of degree n over $\mathbb{F}$ such that the dimension of the subalgebra of $M_n(\mathbb{F})$ generated by $\gamma(S)$ equals k. This is equivalent to the condition that the dimension of the kernel of the extension γ^* of γ to the semigroup algebra $\mathbb{F}[S]$ is $n - k$ (see [1]).

In Section 2, we deal with the general case: the considered finite semigroups S are arbitrary and the representations are their arbitrary faithful representation of degree $|S|$.

In Section 3 we consider a special case: the semigroups S are the finite left reductive semigroups and the representations are their right regular representation.

For notations and notions not defined here, we refer to [1], [3], [5], [6] and [7].

2 The case of arbitrary representations

Definition 2.1 *Let k and n be positive integers. We say that k is representable by n (or n represents k) over a field $\mathbb{F}$ if $k \leq n$ and there is an n-element semigroup S and a faithful representation γ of S of degree n over $\mathbb{F}$ such that the dimension of the subalgebra $\mathcal{A}(\gamma(S))$ of the matrix algebra $M_n(\mathbb{F})$ generated by $\gamma(S)$ is k.*

It is clear that k is representable by n if and only if there is an n-element semigroup of the multiplicative semigroup of the matrix algebra $M_n(\mathbb{F})$ such that the dimension of the subalgebra of $M_n(\mathbb{F})$ generated by S is k.

Theorem 2.2 *Let n be a positive integer. Then every positive integer k with $\frac{n}{2} \leq k \leq n$ is representable by n over every field $\mathbb{F}$ with $char(\mathbb{F}) \neq 2$.*

Proof. Let $\mathbb{F}$ be a field with $char(\mathbb{F}) \neq 2$. Let n and k be positive integers with $\frac{n}{2} \leq k \leq n$. Denote $\mathbf{E}_i$ $(i = 1, \ldots, k)$ the matrix of $M_n(\mathbb{F})$ defined by the following way: $\mathbf{E}_i$ is a diagonal matrix, in which the first i upper elements in the diagonal equal the identity element of the field $\mathbb{F}$ and the other elements are the zero of $\mathbb{F}$. It is easy to see that

$$\mathbf{E}_i\mathbf{E}_j = \mathbf{E}_{min\{i,j\}}$$

for every $i, j \in \{1, \ldots, k\}$. Let $\mathcal{A}$ denote the subalgebra of the algebra $M_n(\mathbb{F})$ generated by the matrices

$$\mathbf{E}_1, \ldots, \mathbf{E}_k.$$

As the matrices $\mathbf{E}_1, \ldots, \mathbf{E}_k$ are linearly independent over $\mathbb{F}$,

$$dim(\mathcal{A}) = k.$$

Since $\frac{n}{2} \leq k$, that is, $n - k \leq k$ then the matrices

$$-\mathbf{E}_1, \ldots, -\mathbf{E}_{n-k}$$

are in $\mathcal{A}$. As $char(\mathbb{F}) \neq 2$, the matrices

$$\mathbf{E}_1, \ldots, \mathbf{E}_k, -\mathbf{E}_1, \ldots, -\mathbf{E}_{n-k}$$

are pairwise distinct and

$$S = \{\mathbf{E}_1, \ldots, \mathbf{E}_k, -\mathbf{E}_1, \ldots, -\mathbf{E}_{n-k}\}$$

is an n-element subset of $M_n(\mathbb{F})$. As

$$(\pm\mathbf{E}_i)(\pm\mathbf{E}_j) \in \{\mathbf{E}_{min\{i,j\}}, -\mathbf{E}_{min\{i,j\}}\}$$

for every $i, j \in \{1, \ldots, k\}$,

$$S = \{\mathbf{E}_1, \ldots, \mathbf{E}_k, -\mathbf{E}_1, \ldots, -\mathbf{E}_{n-k}\}$$

is an n-element subsemigroup of the multiplicative semigroup of the algebra $M_n(\mathbb{F})$ such that S generates the subalgebra $\mathcal{A}$ of $M_n(\mathbb{F})$. Since $dim(\mathcal{A}) = k$ then k is representable by n over $\mathbb{F}$. □

Problem 1. Is Theorem 2.2 true for arbitrary field?

Theorem 2.3 *If k is a positive integer which is representable by a positive integer n over a finite field $\mathbb{F}$ then $log_{|\mathbb{F}|} n \leq k$.*

Proof. Let $\mathbb{F}$ be a finite field and k a positive integer which is representable by a positive integer n. Then there is an n-element semigroup S in the multiplicative semigroup of the full matrix algebra $M_n(\mathbb{F})$ such that the dimension of the subalgebra $\mathcal{A}$ of $M_n(\mathbb{F})$ generated by S is k. Then $n = |S| \leq |\mathcal{A}| = |\mathbb{F}|^k$. Thus $log_{|\mathbb{F}|} n \leq k$. □

Let n be a positive integer and $\mathbb{F}$ a finite field with $char(\mathbb{F}) \neq 2$. By Theorem 2.2, the integers belonging to the interval $[\frac{n}{2}, n]$ are representable by n over $\mathbb{F}$. By Theorem 2.3, the positive integers k with $k < log_{|\mathbb{F}|} n$ are not representable by n. What can we say about the positive integers belonging to the interval $[log_{|\mathbb{F}|}n, \frac{n}{2}]$.

Problem 2. Let n be a positive integer and $\mathbb{F}$ a finite field with the condition $char(\mathbb{F}) \neq 2$. Is every positive integer k belonging to the interval $[log_{|\mathbb{F}|}n, \frac{n}{2}]$ representable by n?

If the answer was yes, then a positive integer k would be representable by a positive integer n over a finite field $\mathbb{F}$ with $char(\mathbb{F}) \neq 2$ if and only if k would be in the interval $[log_{|\mathbb{F}|}n, n]$.

Problem 3. Is it true that, for a fixed positive integer n and an arbitrary field $\mathbb{F}$, there is a positive integer $k_0(n, \mathbb{F}) \leq n$ depending on $\mathbb{F}$ and n such that a positive integer k is representable by n over $\mathbb{F}$ if and only if k belongs to the interval $[k_0(n, \mathbb{F}), n]$?

3 The case of the right regular representation

Let S be a finite semigroup and $\mathbb{F}$ a field. By an S-matrix over $\mathbb{F}$ we mean a single valued mapping A of the descartes product $S \times S$ into $\mathbb{F}$. If we fix an ordering of the elements of S, for example, $S = \{s_1, \ldots, s_n\}$, then an S-matrix A can be written in the usual form: the element of A being in the i^{th} row and the j^{th} column equals $A((s_i, s_j))$. In most of our proofs we will consider the semigroups S with a fixed ordering, and the S-matrices will be written in the usual form detailed above.

Let e and 0 denote the identity element and the zero element of a field $\mathbb{F}$, respectively. For an arbitrary element s of a finite semigroup $S = \{s_1, \ldots, s_n\}$, consider the S-matrix

$$\mathbf{R}^{(s)} = [r_{i,j}^{(s)}]_{n \times n},$$

where

$$r_{i,j}^{(s)} = \begin{cases} e & \text{if } s_i s = s_j, \\ 0 & \text{otherwise.} \end{cases}$$

This matrix will be called the right matrix of s over $\mathbb{F}$.

It is known (see, for example, Exercise 4(b) of §3.5 of [1]) that if S is a finite n-element semigroup then

$$\mathcal{R}_{\mathbb{F}} : \ s \mapsto \mathbf{R}^{(s)}$$

is a representations of S of degree n over $\mathbb{F}$. This representation (which is called the right regular representation of S) is faithful if and only if S is left reductive, that is, for every $a, b \in S$, the assumption "$xa = xb$ for all $x \in S$" implies $a = b$.

For an arbitrary n-element semigroup S and an arbitrary field $\mathbb{F}$, let $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S))$ denote the subalgebra of the matrix algebra $M_n(\mathbb{F})$ generated by $\mathcal{R}_{\mathbb{F}}(S)$.

Definition 3.1 *Let k and n be positive integers. We say that k is representable by n (or n represents k) over a field $\mathbb{F}$ under the right regular representation $\mathcal{R}_{\mathbb{F}}$ if $k \leq n$ and there is an n-element left reductive semigroup S such that the dimension of the subalgebra $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S))$ of the matrix algebra $M_n(\mathbb{F})$ generated by $\mathcal{R}_{\mathbb{F}}(S)$ is k.*

Theorem 3.2 *If a positive integer $n \leq 4$ represents a positive integer k under the right regular representation $\mathcal{R}_{\mathbb{F}}(S)$ then $k = n$.*

Proof. In [2], we can find the Cayley-table of all nonisomorphic and nonanti-isomorphic semigroups containing n elements for $2 \leq n \leq 5$. It is a matter of checking to see that the dimension of the subalgebra of $M_n(\mathbb{F})$ generated by $\mathcal{R}_{\mathbb{F}}(S)$ equals $|S|$ for every left reductive semigroup S with $|S| \leq 4$. □

The next example shows that Theorem 3.2 is not true in case $n \geq 5$.

Example 2. Let $S = \{1, 2, 3, 4, 5\}$ be a semigroup defined by the following Cayley table:

	1	2	3	4	5
1	2	2	1	1	2
2	2	2	2	2	2
3	2	2	3	3	2
4	2	2	4	4	2
5	1	2	1	2	5

(see the Cayley table in the 7^{th} row and the 10^{th} column on page 167 of [2]).

As the columns of the table are pairwise distinct, S is left reductive. It is a matter of checking to see that, for every field $\mathbb{F}$,

$$\mathbf{R}^{(4)} = -\mathbf{R}^{(1)} + \mathbf{R}^{(2)} + \mathbf{R}^{(3)} + 0\mathbf{R}^{(5)}$$

and the matrices

$$\mathbf{R}^{(1)}, \mathbf{R}^{(2)}, \mathbf{R}^{(3)}, \mathbf{R}^{(5)}$$

are linearly independent over $\mathbb{F}$. Thus $dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S)) = 4$ and so 4 is representable by 5 over every field $\mathbb{F}$ under the right regular representation $\mathcal{R}_{\mathbb{F}}$.

Theorem 3.3 *Let $\mathbb{F}$ be a field and S_1, S_2 arbitrary left reductive finite semigroups. Then*

$$dim[\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1))]dim[\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_2))] = dim[\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1 \times S_2))].$$

Proof. Let $S_1 = \{a_i : i = 1, \dots, |S_1|\}$ and $S_2 = \{b_j : j = 1, \dots, |S_2|\}$ be arbitrary finite semigroups and $\mathbb{F}$ an arbitrary field. Consider the right regular representations of S_1 and S_2, respectively. Let $\mathbf{A}^{(a_i)}$ and $\mathbf{B}^{(b_j)}$ denote the right matrices of the elements $a_i \in S_1$ and $b_j \in S_2$ (corresponding to the above orderings of S_1 and S_2), respectively. Assume

$$dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1)) = m \quad \text{and} \quad dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_2)) = n.$$

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ denote a bases of $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1))$ and $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_2))$, respectively. We can suppose that $\mathcal{B}_1 = \{\mathbf{A}^{(a_1)}, \dots, \mathbf{A}^{(a_m)}\}$ and $\mathcal{B}_2 = \{\mathbf{B}^{(b_1)}, \dots, \mathbf{B}^{(b_n)}\}$.

It is clear that the direct product $S_1 \times S_2$ is also left reductive. Thus the right regular representation of $S_1 \times S_2$ is faithful. Consider the following ordering of the elements of $S_1 \times S_2$:

$$S_1 \times S_2 = \{(a_1, b_1); \dots; (a_1, b_{|S_2|}); \dots; (a_{|S_1|}, b_1); \dots; (a_{|S_1|}, b_{|S_2|})\}.$$

It is a matter of checking to see that the right matrix $\mathbf{C}^{(a_i,b_j)}$ of the element $(a_i, b_j) \in S_1 \times S_2$ (corresponding to the above ordering of $S_1 \times S_2$)is a matrix of blocks $\mathbf{C}_{k,t}^{(a_i,b_j)}$ $(k, t \in \{1, \dots, |S_1|\})$ such that

$$\mathbf{C}_{k,t}^{(i,j)} = a_{k,t}^{(a_i)}\mathbf{B}^{(b_j)},$$

where $a_{k,t}^{(a_i)}$ $(k, t = 1, \dots, |S_1|)$ are the elements of the right matrix $\mathbf{A}^{(a_i)}$. We show that the right matrices $\mathbf{C}^{(a_i,b_j)}$ $(i = 1, \dots m; j = 1, \dots n)$ form a basis of $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1 \times S_2))$.

To show that the matrices $\mathbf{C}^{(a_i,b_j)}$ $(i = 1, \dots m; j = 1, \dots n)$ are linearly independent (over $\mathbb{F}$), assume

$$\Sigma_{j=1}^{n}\Sigma_{i=1}^{m}\gamma_{j,i}\mathbf{C}^{(a_i,b_j)} = \mathbf{0}_{mn \times mn}$$

for some $\gamma_{j,i} \in \mathbb{F}$. Then, for every $k, t \in \{1, \dots |S_1|\}$,

$$\Sigma_{j=1}^{n}\Sigma_{i=1}^{m}\gamma_{j,i}\mathbf{C}_{k,t}^{(a_i,b_j)} = \mathbf{0}_{n \times n},$$

that is,

$$\Sigma_{j=1}^{n}\Sigma_{i=1}^{m}\gamma_{j,i}a_{k,t}^{(a_i)}\mathbf{B}^{(b_j)} = \mathbf{0}_{n \times n}.$$

Then

$$\Sigma_{j=1}^{n}(\Sigma_{i=1}^{m}\gamma_{j,i}a_{k,t}^{(a_i)})\mathbf{B}^{(b_j)} = \mathbf{0}_{n \times n}$$

from which we obtain that, for every $j = 1, \ldots, n$ (and every $k, t = 1, \ldots, |S_1|$),

$$\Sigma_{i=1}^{m} \gamma_{j,i} a_{k,t}^{(a_i)} = 0,$$

because the matrices $\mathbf{B}^{(b_1)}, \ldots, \mathbf{B}^{(b_n)}$ are linearly independent. As the coefficients $\gamma_{j,i}$ do not depend on k and t, we have

$$\Sigma_{i=1}^{m} \gamma_{j,i} \mathbf{A}^{(a_i)} = \mathbf{0}_{m \times m}$$

for every $j = 1, \ldots, n$. As the matrices $\mathbf{A}^{(a_1)}, \ldots, \mathbf{A}^{(a_m)}$ are linearly independent, we get $\gamma_{j,i} = 0$ for every $j = 1, \ldots, n$ and $i = 1, \ldots, m$.

In the next, we show that the matrices $\mathbf{C}^{(a_i,b_j)}$ $(i = 1, \ldots m; j = 1, \ldots n)$ generate $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1 \times S_2))$. Let $(x, y) \in S_1 \times S_2$ be arbitrary. As $\mathcal{B}_2$ is a basis of $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_2))$, there are $\beta_j \in F$ $(j = 1, \ldots, n)$ such that

$$\mathbf{B}^{(y)} = \Sigma_{j=1}^{n} \beta_j \mathbf{B}^{(b_j)}.$$

Then, for every $k, t \in \{1, \ldots, |S_1|\}$,

$$a_{k,t}^{(x)} \mathbf{B}^{(y)} = \Sigma_{j=1}^{n} \beta_j a_{k,t}^{(x)} \mathbf{B}^{(b_j)}.$$

As $\mathcal{B}_1$ is a basis of $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1))$, there are $\alpha_i \in F$ $(i = 1, \ldots, m)$ such that

$$\mathbf{A}^{(x)} = \Sigma_{i=1}^{m} \alpha_i \mathbf{A}^{(a_i)},$$

that is,

$$a_{k,t}^{(x)} = \Sigma_{i=1}^{m} \alpha_i a_{k,t}^{(a_i)}$$

for every $k, t = 1, \ldots, |S_1|$. Then

$$a_{k,t}^{(x)} \mathbf{B}^{(y)} = \Sigma_{j=1}^{n} \beta_j (\Sigma_{i=1}^{m} \alpha_i a_{k,t}^{(a_i)}) \mathbf{B}^{(b_j)} =$$

$$\Sigma_{j=1}^{n} \Sigma_{i=1}^{m} (\beta_j \alpha_i)(a_{k,t}^{(a_i)} \mathbf{B}^{(b_j)})$$

and so

$$\mathbf{C}_{k,t}^{(x,y)} = \Sigma_{j=1}^{n} \Sigma_{i=1}^{m} (\beta_j \alpha_i) \mathbf{C}_{k,t}^{(a_i,b_j)}$$

for every $k, t = 1, \ldots, |S_1|$. As the coefficients α_i $(i = 1, \ldots, m)$ and β_j $(j = 1, \ldots, n)$ do not depend on k and t,

$$\mathbf{C}^{(x,y)} = \Sigma_{j=1}^{n} \Sigma_{i=1}^{m} (\beta_j \alpha_i) \mathbf{C}^{(a_i,b_j)}.$$

Thus the theorem is proved. □

On the set of all positive integers consider the following binary relation: $k \sim_{\mathcal{R}_{\mathbb{F}}} n$ if and only if k is representable by n over the field $\mathbb{F}$ under the right regular representation $\mathcal{R}_{\mathbb{F}}$.

Corollary 3.4 *If $k \sim_{\mathcal{R}_{\mathbb{F}}} n$ and $t \sim_{\mathcal{R}_{\mathbb{F}}} m$ for some positive integers k, t, n, m then $kt \sim_{\mathcal{R}_{\mathbb{F}}} nm$.*

Proof. Assume

$$k \sim_{\mathcal{R}_{\mathbb{F}}} n \quad \text{and} \quad t \sim_{\mathcal{R}_{\mathbb{F}}} m$$

for some positive integers k, t, n, m. Then there are left reductive semigroups S_1 and S_2 such that

$$|S_1| = n \quad \text{and} \quad |S_2| = m$$

and

$$dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1)) = k \quad \text{and} \quad dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_2)) = t.$$

By Theorem 3.3,

$$dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1 \times S_2)) = kt.$$

Thus $kt \sim_{\mathbb{F}} nm$. □

Theorem 3.5 *Let $\mathbb{F}$ be a field and S_1, S_2 be arbitrary finite left reductive semigroups. Then*

$$\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1)) \bigotimes \mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_2)) \cong_{Alg} \mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1 \times S_2)),$$

where $\bigotimes$ denotes the tensor product and $\cong_{Alg}$ denotes the algebra isomorphism.

Proof. We use the notations of the proof of Theorem 3.3. Consider the tensor product

$$\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1)) \bigotimes \mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_2))$$

of the vector spaces $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1))$ and $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_2))$. The tensors

$$\mathbf{A}^{(a_i)} \otimes \mathbf{B}^{(b_j)} \quad (i = 1, \ldots, m; j = 1, \ldots n)$$

form a basis of $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1)) \bigotimes \mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_2))$ and the product between them is

$$(\mathbf{A}^{(a_i)} \otimes \mathbf{B}^{(b_j)})(\mathbf{A}^{(a_k)} \otimes \mathbf{B}^{(b_t)}) = (\mathbf{A}^{(a_i a_k)} \otimes \mathbf{B}^{(b_j b_t)}).$$

By the proof of Theorem 3.3,

$$\{\mathbf{C}^{(a_i, b_j)} : \ i = 1, \ldots m; j = 1, \ldots, n\}$$

is a basis of the algebra $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1 \times S_2))$. The product between the elements of this basis is the following:

$$\mathbf{C}^{(a_i,b_j)}\mathbf{C}^{(a_k,b_t)} = \mathbf{C}^{(a_ia_k,b_jb_t)}.$$

As

$$dim(\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1)) \bigotimes \mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_2)) = dim(\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1 \times S_2)))$$

by Theorem 3.3, the mapping

$$\phi \ : \ (\mathbf{A}^{(a_i)} \otimes \mathbf{B}^{(b_j)}) \ \mapsto \ \mathbf{C}^{(a_i,b_j)} \quad i = 1, \dots m; j = 1, \dots n$$

is an isomorphism of the vector space $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1)) \bigotimes \mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_2))$ onto the vector space $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1 \times S_2))$. As

$$\phi((\mathbf{A}^{(a_i)} \otimes \mathbf{B}^{(b_j)})(\mathbf{A}^{(a_k)} \otimes \mathbf{B}^{(b_t)})) = \phi((\mathbf{A}^{(a_i,a_k)} \otimes \mathbf{B}^{(b_j,b_t)})) =$$

$$= \mathbf{C}^{(a_ia_k,b_jb_t)} = \mathbf{C}^{(a_i,b_j)(a_k,b_t)} = \mathbf{C}^{(a_i,b_j)}\mathbf{C}^{(a_k,b_t)} =$$

$$= \phi((\mathbf{A}^{(a_i)} \otimes \mathbf{B}^{(b_j)}))\phi((\mathbf{A}^{(a_k)} \otimes \mathbf{B}^{(b_t)})),$$

ϕ is an algebra isomorphism of the tensor product $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1)) \bigotimes \mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_2))$ onto the algebra $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_1 \times S_2))$. □

A congruence σ on a semigroup S is called a *semilattice congruence* if the factor semigroup $Y = S/\sigma$ is a semilattice (a commutative semigroup in which every element is idempotent). If σ is a semilattice congruence of a semigroup S then the σ-classes S_α ($\alpha \in Y$) of S are subsemigroups of S. We say that a semigroup S is a *semilattice Y of subsemigroups S_α* ($\alpha \in Y$) of S if there is a semilattice congruence σ on S such that S/σ is isomorphic to Y and the σ-classes of S are the subsemigroups S_α ($\alpha \in Y$).

Theorem 3.6 *Let S be a finite semigroup which is a semilattice of two left reductive subsemigroups A and B of S. Then*

$$dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S)) \geq dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(A)) + dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(B)).$$

Proof. It is clear that one of A and B, for example, A is an ideal of S. If $c, d \in S$ be arbitrary elements such that $xc = xd$ holds for all $x \in S$ then $c^2 = cd = d^2$ and so both of c and d are in either A or B. As A and B are left reductive, we get $c = d$. Thus S is left reductive and so the right regular representation of S is faithful. Let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_m\}$. Let

$$\mathbf{A}^{(a_i)} \ (i = 1, \dots n) \quad \text{and} \quad \mathbf{B}^{(b_j)} \ (j = 1, \dots, m)$$

denote the right matrices of the elements $a_i \in A$ and $b_j \in B$ corresponding to the above ordering of A and B, respectively.

Consider the followin ordering of S:

$$S = \{a_1, \ldots, a_n, b_1, \ldots b_m\}.$$

The right matrices $\mathbf{C}^{(s)}$ of the elements s of S corresponding to the above ordering of S are matrices of blocks

$$\mathbf{C}_{k,t}^{(s)} \ (k, t \in \{1, 2\})$$

such that the type of $\mathbf{C}_{1,1}^{(s)}$ is $n \times n$ and the type of $\mathbf{C}_{2,2}^{(s)}$ is $m \times m$. Moreover, $\mathbf{C}_{1,1}^{(a_i)} = \mathbf{A}^{(a_i)}$, $\mathbf{C}_{2,2}^{(a_i)} = \mathbf{0}_{m\times m}$ for every $a_i \in A$, and $\mathbf{C}_{2,2}^{(b_j)} = \mathbf{B}^{(b_j)}$ for every $b_j \in B$. Assume

$$dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(A)) = k \quad \text{and} \quad dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(B)) = t.$$

We can suppose that $\mathbf{A}^{(a_i)}$ $(i = 1, \ldots, k)$ and $\mathbf{B}^{(b_i)}$ $(j = 1, \ldots, t)$ are the basis of $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(A)$ and $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(B)$, respectively. We show that the system of matrices $\mathbf{C}^{(a_i)}$ and $\mathbf{C}^{(b_j)}$ $(i = 1, \ldots k; j = 1, \ldots t)$ is linearly independent. Assume

$$\Sigma_{i=1}^{k} \alpha_i \mathbf{C}^{(a_i)} + \Sigma_{j=1}^{t} \beta_j \mathbf{C}^{(b_j)} = \mathbf{0}_{(n+m)\times(n+m)}.$$

Then

$$\Sigma_{i=1}^{k} \alpha_i \mathbf{C}_{2,2}^{(a_i)} + \Sigma_{j=1}^{t} \beta_j \mathbf{C}_{2,2}^{(b_j)} = \mathbf{0}_{m\times m}$$

and so

$$\Sigma_{j=1}^{t} \beta_j \mathbf{B}^{(b_j)} = \mathbf{0}_{m\times m},$$

because $\mathbf{C}_{2,2}^{(a_i)} = \mathbf{0}_{m\times m}$ and $\mathbf{C}_{2,2}^{(b_j)} = \mathbf{B}^{(b_j)}$ for every $a_i \in A$ and $b_j \in B$. As the matrices $\mathbf{B}^{(b_j)}$ $(j = 1, \ldots t)$ are linearly independent, we get $\beta_j = 0$ for every $j = 1, \ldots, t$. Then

$$\Sigma_{i=1}^{k} \alpha_i \mathbf{C}^{(a_i)} = \mathbf{0}_{(n+m)\times(n+m)}$$

and so

$$\mathbf{0}_{n\times n} = \Sigma_{i=1}^{k} \alpha_i \mathbf{C}_{1,1}^{(a_i)} = \Sigma_{i=1}^{k} \alpha_i \mathbf{A}^{(a_i)}.$$

As the matrices $\mathbf{A}^{(a_i)}$ $(i = 1, \ldots k)$ are linearly independent, we get $\alpha_i = 0$ for every $i = 1, \ldots, k$. Thus the matrices

$$\mathbf{C}^{(a_1)}, \ldots, \mathbf{C}^{(a_k)}, \mathbf{C}^{(b_1)}, \ldots, \mathbf{C}^{(b_t)}$$

are linearly independent. From this it follows that

$$dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S)) \geq dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(A)) + dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(B)).$$

□

Let a semigroup S be a semilattice Y of semigroups S_α, $\alpha \in Y$. Assume that, for every $\alpha, \beta \in Y$ with $\alpha \geq \beta$, there is a homomorphism $(\ )f_{\alpha,\beta}$ of S_α into S_β such that the following are satisfied.

(1) For each $\alpha \in Y$, $f_{\alpha,\alpha}$ is the identity mapping of S_α.

(2) If $\alpha \geq \beta \geq \gamma$ then $f_{\alpha,\beta} f_{\beta,\gamma} = f_{\alpha,\gamma}$.

(3) If $a \in S_\alpha$ and $b \in S_\beta$ then $ab = (a)f_{\alpha,\alpha\beta}(b)f_{\beta,\alpha\beta}$.

In such a case S is called a *strong semilattice* Y of semigroups S_α $(\alpha \in Y)$.

Theorem 3.7 *Let S be a finite semigroup which is a strong semilattice of two left reductive subsemigroups A and B of S with $AB \subseteq A$. If $dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(B)) = |B|$ then*

$$dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S)) = dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(A)) + dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(B)).$$

Proof. We use the notations of the proof of Theorem 3.6. As S is a strong semilattice of subsemigroups $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ such that A is an ideal of S, there is a homomorphism φ of B into A such that $b_j a_i = \varphi(b_j) a_i$ for every $b_j \in B$ and $a_i \in A$. This homomorphism induces a mapping φ^* of $\{1, \ldots, m\}$ into $\{1, \ldots, n\}$ with the following way: $\varphi^*(j) = i$ if and only if $\varphi(b_j) = a_i$. From this it follows that the j^{th} row of the matrix $\mathbf{C}_{2,1}^{(a_i)}$ $(j = 1, \ldots m)$ equals the $(\varphi^*(j))^{th}$ row of the right matrix $\mathbf{A}^{(a_i)}$ for every $a_i \in A$. Thus if a linear combination $\sum_{i=1}^{k} \beta_i \mathbf{A}^{(a_i)}$ equal a right matrix $\mathbf{A}^{(a)}$ $(a \in A)$ then $\sum_{i=1}^{k} \beta_i \mathbf{C}_{2,1}^{(a_i)}$ equals the matrix $\mathbf{C}_{2,1}^{(a)}$. As $dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(B)) = |B| = m$, Theorem 3.6 implies that the matrices

$$\mathbf{C}^{(a_1)}, \ldots, \mathbf{C}^{(a_k)}, \mathbf{C}^{(b_1)}, \ldots, \mathbf{C}^{(b_m)}$$

are linearly independent. We show that they form a basis of the subalgebra $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S))$ of the matrix algebra $\mathbb{F}_{(n+m)\times(n+m)}$. It is sufficient to show that every matrix $\mathbf{C}^{(a_j)}$ $(j = k+1, \ldots, n)$ can be expressed as a linearly combination of the matrices $\mathbf{C}^{(a_1)}, \ldots, \mathbf{C}^{(a_k)}, \mathbf{C}^{(b_1)}, \ldots, \mathbf{C}^{(b_m)}$. Let $\mathbf{C}^{(a)}$, $a \in \{a_{j+1}, \ldots, a_n\}$ be an arbitrary matrix. Then

$$\mathbf{A}^{(a)} = \sum_{i=1}^{k} \beta_i \mathbf{A}^{(a_i)}$$

for some $\beta_i \in \mathbb{F}$. By the above result, this equation implies

$$\mathbf{C}_{2,1}^{(a)} = \sum_{i=1}^{k} \beta_i \mathbf{C}_{2,1}^{(a_i)}$$

and so

$$\mathbf{C}^{(a)} = \sum_{i=1}^{k} \beta_i \mathbf{C}^{(a_i)}.$$

Thus the matrices

$$\mathbf{C}^{(a_1)}, \ldots, \mathbf{C}^{(a_k)}, \mathbf{C}^{(b_1)}, \ldots, \mathbf{C}^{(b_m)}$$

form a basis of the subalgebra $\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S))$. Hence

$$dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S)) = dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(A)) + dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(B)).$$

□

Theorem 3.8 *If a finite semigroup S is a semilattice Y of left reductive semigroups S_α ($\alpha \in Y$) then*

$$dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S)) \geq \sum_{\alpha \in Y} dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_\alpha)).$$

Proof. The assertion will be proved by induction on $n = |Y|$. If $n = 1$ then the assertion is obvious. If $n = 2$ then the assertion follows from Theorem 3.6. Let $n \geq 3$. Assume that the assertion is true for all semilattice of order less then n. Let Y be a semilattice such that $|Y| = n$. Let S be a semigroup which is a semilattice Y of left reductive semigroups S_α, $\alpha \in Y$. As Y is a semilattice and $|Y| \geq 3$, there are elements $\alpha, \beta \in Y$ such that

$$\alpha\beta \neq \beta.$$

Let I_β denote the ideal of Y generated by β. It is known that

$$I_\beta = \{\xi \in Y : \ \xi\beta = \xi\}.$$

As

$$\beta, \alpha\beta \in I_\beta,$$

$\alpha\beta \neq \beta$ implies

$$|I_\beta| \geq 2.$$

First consider the case when $I_\beta \neq Y$. Then $|Y \setminus I_\beta| \leq n-2$. As I_β is a subsemigroup of Y, the union A_β of subsemigroups S_ξ ($\xi \in I_\beta$) form a subsemigroup of S. As $I_\beta \subset Y$, we get

$$dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(A_\beta)) \geq \sum_{\xi \in I_\beta} dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_\xi))$$

by induction. As I_β is an ideal of Y, the semigroup S is a semilattice of the semigroups S_η ($\eta \in Y \setminus I_\beta$) and the subsemigroup A_β. As $|Y \setminus I_\beta| + 1 \leq n-1$, we get

$$dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S)) \geq dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(A_\beta)) + \sum_{\eta \in Y \setminus I_\beta} dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_\eta))$$

by induction. This and the above

$$dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(A_\beta)) \geq \sum_{\xi \in I_\beta} dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_\xi))$$

together imply

$$dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S)) \geq \sum_{\alpha \in Y} dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_\alpha)).$$

In the next consider the case when $I_\beta = Y$. It means that β is the identity element of Y. In this case $\xi\eta \neq \beta$ for every $\beta \notin \{\xi, \eta\}$. Indeed, if there were elements $\xi, \eta \in Y$ with $\xi \neq \beta$ and $\eta \neq \beta$ such that $\eta\xi = \beta$ then, for every $\alpha \in Y$, we would have $\alpha\eta\xi = \alpha\beta = \alpha$ and so $\alpha\xi = \alpha$. It would imply that ξ is an identity element of Y which would contradicts $\xi \neq \beta$.

Thus $X = Y \setminus \{\beta\}$ is a subsemilattice of Y. Let S^* denote the subsemigroup of S which is a semilattice X of semigroups S_τ, $\tau \in X$. Then S is a semilattice of S^* and S_β and so

$$dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S)) \geq dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S^*)) + dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_\beta))$$

by Theorem 3.6. As $|X| = |Y| - 1$,

$$dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S^*)) = \sum_{\tau \in X} dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_\tau))$$

by induction. Consequently

$$dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S)) \geq \sum_{\alpha \in Y} dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_\alpha)).$$

□

Theorem 3.9 *Let S be a semigroup which is a semilattice Y of left reductive finite semigroups S_α ($\alpha \in Y$) such that $dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_\alpha)) = |S_\alpha|$ for every $\alpha \in Y$. Then $dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S)) = |S|$.*

Proof. Applying Theorem 3.8 and the assumptions of this theorem, we get

$$\sum_{\alpha \in Y} |S_\alpha| = |S| \geq dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S)) \geq \sum_{\alpha \in Y} dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S_\alpha)) = \sum_{\alpha \in Y} |S_\alpha|$$

and so $dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S)) = |S|$. □

Theorem 3.10 *If a finite semigroup S is a semilattice Y of monoids S_α ($\alpha \in Y$) $\alpha \in Y$ then $dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S)) = |S|$.*

Proof. It is easy to see that every monoid M is left reductive and $dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(M)) = |M|$. Thus our assertion follows from Theorem 3.9. □

Theorem 3.11 *If S is a finite Clifford semigroup then $dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S)) = |S|$.*

Proof. It is known that a semigroup is a Clifford semigroup if and only if it is a semilattice of groups (see Theorem 2.1 of [3]). Thus our assertion follows from Theorem 3.10. □

Theorem 3.12 *If S is a finite semilattice then $dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S)) = |S|$.*

Proof. As a semilattice is a semilattice of one-element monoids, our assertion follows from Theorem 3.10. □

Corollary 3.13 *If $k \sim_{\mathcal{R}_{\mathbb{F}}} n$ then, for every positive integer t, $k+t \sim_{\mathcal{R}_{\mathbb{F}}} n+t$.*

Proof. Assume $k \sim_{\mathcal{R}_{\mathbb{F}}} n$ for some positive integers k and n. Then there is an n-element semigroup A such that $dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(A)) = k$. Let t be a positive integer and B a t-element semilattice. As A is a finite semigroup, it has an idempotent element e. Let φ denote the mapping of B into A such that $\varphi(b) = e$ for every $b \in B$. It is easy to see that φ is a homomorphism. On the set $S = A \cup B$ define the following multiplication. Let the new multiplication on A and B is the old multiplication, respectively. For arbitrary $a \in A$ and $b \in B$, let $ab = a\varphi(b) = ae$ and $ba = \varphi(b)a = ea$. Then S is a strong semilattice of A and B with $AB \subseteq A$. By Theorem 3.12, $dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(B)) = |B| = t$ and so Theorem 3.6 implies $dim\mathcal{A}(\mathcal{R}_{\mathbb{F}}(S)) = k + t$. Thus $k + t \sim_{\mathcal{R}_{\mathbb{F}}} n + t$. □

References

[1] Clifford, A.H. and G.B. Preston, *The Algebraic Theory of Semigroups*, Amer. Math. Soc., Providence, R.I., I(1961), II(1967)

[2] Bogdanovič, S., Semigroups with a system of subsemigroups, Novi Sad, 1985

[3] Howie, J.M., *An Introduction to Semigroup Theory*, Academic Press, London/New York/San Francisco, 1976

[4] McAlister, D.B., *Representations of semigroups by linear transformations, I,II*, Semigroup Forum 2(1971), 189-263; 2(1971), 283-320

[5] Nagy, A., *Special Classes of Semigroups*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2001

[6] Okniński, J., *Semigroup Algebras*, Monographs and Textbooks in Pure and Applied Mathematics, 138, Marcel Dekker, Inc., New York, 1991

[7] Petrich, M., *Lectures in Semigroups*, Akademie-Verlag Berlin, 1977

Received: November, 2012