Multiplication Graded Modules

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Abstract

Let $G$ be a multiplicative group and $R$ be a $G$-graded commutative ring and $M$ a $G$-graded $R$-module. Various properties of multiplicative ideals in a graded ring are discussed and we extend this to graded modules over graded rings. We have also discussed the set of $P$-primary ideals and modules of $R$ when $P$ is a graded multiplication prime ideals and modules.

Keywords: graded rings, graded modules, graded multiplication Modules.

1. Introduction

Let $G$ be a group. A ring $R$ is called $G$-graded ring if there exist a family $\{R_g\}_{g \in G}$ of additive subgroup of $R$ such that $R = \bigoplus_{g \in G} R_g$ that $R_g R_h \subset R_{gh}$ for each $g, h \in G$. A $R$- module $M$ is called $R$-graded Module over $G$ if $M = \bigoplus_{g \in G} M_g$ and $R_g M_h \subset M_{gh}$ for all $g, h \in G$. Thus each $M_g$ submodule of $M$ is $R=R_g$ - module. An element of a graded ring $R$ is called homogeneous if it belongs to $\bigcup_{g \in G} M_g$. If an element $m \in M$ is belongs to $\bigcup_{g \in G} M_g$, then $m$ is called homogeneous element and the set of all homogeneous elements of $M$ is denoted by $\hat{H}(M)$ (for a ring $R$ is denoted by $\hat{H}(R)$). A graded submodule $N$ of a graded $R$-Module $M$ ($R$ is a graded ring) is a submodule such that $N = \bigoplus_{g \in G} (M_g \cap N) = \bigoplus_{g \in G} N_g$. Equivalently, $N$ is graded in $M$ if and only if
\(N\) has a homogeneous set of generators. If \(R = \bigoplus_{g \in G} R_g\) and \(R' = \bigoplus_{g \in G} R'_g\) are two graded rings, then the mapping \(\phi : R \to R'\) with \(\phi(1_R) = 1_{R'}\) is called graded homomorphism if \(\phi(R_g) \subseteq R'_g\), for all \(g \in G\).

If \(M = \bigoplus_{g \in G} M_g\) and \(M' = \bigoplus_{g \in G} M'_g\) are two graded \(R\)-modules (\(R\) is a graded ring), the mapping \(\lambda : M \to M'\) is called graded homomorphism if \(\lambda(M_g) \subseteq M'_g\), for all \(g \in G\). A graded ideal \(P\) of a graded ring \(R\) is called gr-prime if whenever \(x, y \in H(R)\) with \(xy \in P\) then \(x \in P\) or \(y \in P\). And a graded submodule of a graded module \(M\) over a graded ring \(R\) is called gr-prime if \(\lambda(M) \subseteq M'\), for all \(g \in G\). A graded ideal \(P\) of a graded ring \(R\) is called gr-maximal if it is maximal in the lattice of graded ideals of \(R\). (similarly we have for \(R\)-modules). A graded ring \(R\) is called a gr-local ring if it has unique gr-maximal ideal. Let \(R\) be a graded ring and let \(S \subseteq H(R)\) be a multiplicatively closed subset of \(R\). Then the ring of fraction \(S^{-1}R\) is a graded ring which is called a gr-ring of fractions. Indeed, \(S^{-1}R = \bigoplus_{g \in G} (S^{-1}R)_g\)

where \((S^{-1}R)_g = \left\{ \frac{r}{s} \mid r, s \in R, s \in S, g = \frac{\deg(r)}{\deg(s)} \right\}\). And \(S^{-1}M = \bigoplus_{g \in G} (S^{-1}M)_g\)

where \((S^{-1}M)_g = \left\{ \frac{m}{s} \mid m \in M, s \in S, g = \frac{\deg(M)}{\deg(s)} \right\}\). Consider the ring gr-homomorphism \(\pi : R \to S^{-1}R\) defined by \(\pi(r) = \frac{r}{1}\). And \(\pi : M \to S^{-1}M\) is called gr-homomorphism if \(\pi(m) = \frac{m}{1}\). Let \(P\) be any gr-prime ideal of a graded ring \(R\) and consider the multiplicatively closed subset \(S = H(R) - P\). We denote the graded ring of fraction \(S^{-1}R\) of \(R\) by \(R^g_P\) and we call it the gr-localization of \(R\).

This is a gr-local with the unique gr-maximal ideal \(S^{-1}P\) which will be denoted by \(PP^g\). Let \(I\) be a graded ideal in a graded ring \(R\). The graded radical of \(I\) (\(\text{gr-rad}(I)\)) is defined the set of all \(x \in R\) such that for each \(g \in G\), there exists \(n_g > 0\) such that \(x^n_g \in I\). A graded radical submodule \(N\) of a graded \(R\)-module \(M\) (\(R\) is a graded ring) is the intersections of graded prime submodules of \(M\) such that containing \(N\) as a submodule. A submodule \(N\) of on \(R\)-module \(M\) is called multiplication if \(N = IM\), for some gr-ideal \(I\) of \(R\). If each submodule of \(M\) is gr-multiplication, \(M\) is called gr-multiplication \(R\)-module.
In this paper, we study some properties of gr-multiplication submodules in a graded multiplication R-module $M$, when $M$ is gr-module over gr-ring $R$. And give a characterization of finitely generated gr-multiplication submodules of a gr-multiplication $M$ over a gr-ring $R$.

**Definition 1** Let $R$ be a graded ring over the group $G$ and $M$ an $R$-graded module. A graded submodule $N$ of $M$ is called graded multiplication. If $K < N$ then there is a gr-ideal of $R$ such that $K = NI$.

**Definition 2** A graded $R$-module $M$ is called gr-multiplication module if every gr-submodule of $M$ is gr-multiplication.

**Definition 3** A graded ideal $Q$ of a graded ring $R$ is called gr-primary if $Q \neq R$ and whenever, $a, b \in H(R)$ with $ab \in Q$, then $a \in Q$ or $b^0 \in Q$ . If $Q$ is gr-primary ideal of $R$ and gr-rad $(Q) = P$, we say that $Q$ is gr-p-primary.

**Definition 4** An gr-submodule $N$ of graded $R$-module $M$ is called gr-primary, if $a, b \in H(R)$, $b \in H(M)$ and $ab \in N$, then $b \in N$ or $a^n M \subset N$ for some integer $n \geq 0$.

Recall that if $N, K$ are two gr-submodules of a graded $R$-module $M$, then $(N : K) = \left\{ r \in R \middle| rK \subset N \right\}$ is a graded ideal of $R$.

**Lemma 1** Let $I$ be a graded ideal in a graded ring $R$ then $I$ is multiplication if $I \cap J = I(J : I)$ for gr-ideal $J \subset I$.

**Proof.** Suppose that $J \subset I$ for some gr-ideal $J$ of $R$. Then $J = I \cap J = I(J : I)$. Hence $J$ is gr-multiplication ideal of $I$.

Conversely, let $I$ be a graded multiplication ideal in $R$, let $J$ be any graded ideal of $R$. Then $I \cap J \subset I$, so there is a graded ideal $K$ of $R$ such that $I \cap J = IK$. Therefore $K \subset (I \cap J : I) \subset (J : I)$, and then $I \cap J = IK \subset (J : I)$. On the other hand, clearly $I(J : I) \subset I \cap J$. Hence $J = I \cap J = (J : I)I$.

**Proposition 1** Let $M$ be a graded $R$-module ($R$ is a graded ring). Then $M$ is gr-multiplication if for every gr-submodule $N$ of $M$, $N = [N : M]M$.

**Proof.** Let $M$ be gr-multiplication $R$-module, and $N$ a gr-submodule of $M$, then there is an gr-ideal $I$ of $R$ such that $N = IM$, as $IM \subset N$ we have $I \subset [N : M]$ and $N = IM \subset M[N : M]$. Since $[N : M]M \subset N$, so $N = [N : M]M$. Conversely it is clearly. Recall that Graded $R$-module $M$ is called graded cyclic if $M = Rx$, for some $x \in H(M)$. 

Theorem 1 Let $M$ be a gr-multiplication Module over a graded local ring $R$. Then $M$ is gr-multiplication if $M$ is a graded cyclic $R$-module. 

Proof. If $M = \langle m \rangle$ for some $m \in H(M)$ then clearly $M$ is gr-multiplication $R$-module. Conversely, Let $M = \langle m_\alpha \mid \alpha \in A \rangle$ where each $M_\alpha$ is a homogeneous element ($m_\alpha \in H(M)$). Since $M$ is gr-multiplication we have $Rm_\alpha = [M_\alpha : M]M = \sum_{\alpha \in A} Rm_\alpha = \sum_{\alpha \in A} [m_\alpha : M]M = M\left(\sum_{\alpha \in A} [m_\alpha : M]\right)$. 

If $\sum_{\alpha \in A} [m_\alpha : M] = R$, then $[m_{\alpha_0} : M] = R$. Since otherwise if $\forall \alpha [m_\alpha : M] \neq R$, then $[m_\alpha : M] \subset J$, where $J$ is the only maximal ideal of $R$, and hence $\sum_{\alpha \in A} [m_\alpha : M] = R \subset J$ that is a contradiction, so $[m_{\alpha_0} : M] = R$ for some $\alpha_0 \in A$ therefore $\langle m_\alpha \rangle = [m_\alpha : M]M = M$. Hence $M$ is gr-principal. If $\sum_{\alpha \in A} [m_\alpha : M] \neq R$, then $\sum_{\alpha \in A} [m_\alpha : M] \subset J$, and then $M = \sum_{\alpha \in A} [m_\alpha : M]M \subset J_m \subset M$ therefore $J_m = M$, hence $M = \langle 0 \rangle$.

Proposition 2. If $M$ is gr-multiplication $R$-module where $R$ is a graded ring, and $S \subset H(R)$ is a multiplicatively closed subset of $R$. Then $S^{-1}M$ is a gr-multiplication $S^{-1}R$-module. 

Proof. Let $K$ be a graded $S^{-1}R$-submodule of $S^{-1}M$. Then $K = S^{-1}N$ for some graded submodule of $N$ of $M$. Now since $M$ is gr-multiplication $R$-module, then $N = [N : M]M$ so $S^{-1}N = (S^{-1}[N : M])(S^{-1}M)$ Hence $S^{-1}M$ is a gr-multiplication $S^{-1}R$-module.

Definition 5 A graded submodule $N$ of graded $R$-module $M$ is locally gr-principal if $N \cdot R_p^g$ is gr-principal for every gr-prime ideal $P$ of $R$.

Proposition 3 Let $R$ be a gr-local ring with graded maximal ideal $J$ and $M$ a graded $R$-module such that $M = \langle m_1, m_2, \ldots, m_k \rangle$, where $m_i \in H(M)$ for every $1 \leq i \leq k$, then $M = \langle m_i \rangle$ for some $1 \leq i \leq k$.

Proof. Suppose that $M = \langle a \rangle$ for some $a \in H(M)$ and $M = \langle a_1, a_2, \ldots, a_k \rangle$, then $a = \sum_{i=1}^{k} r_i a_i$ and each $a_i = s_i a$, so $a = \sum_{i=1}^{k} r_i s_i a$
and \( a(1 - \sum_{i=1}^{k} r_i x_i) = 0 \) if \( 1 - \sum_{i=1}^{k} r_i s_i \) is a unit in \( R \) then \( a = 0 \) since \( a_i = s_i a_i \), \( a_i = 0 \), for all \( 1 \leq i \leq k \). and \( M = \langle 0 \rangle = \langle a_i \rangle \), for all \( i = 1, 2, \ldots, k \). If \( 1 - \sum_{i=1}^{k} r_i s_i \) is not a unit, then \( \sum_{i=1}^{k} r_i s_i \notin J \) and so \( \sum_{i=1}^{k} r_i s_i \) is a unit. Therefore, there is an \( i \in \{1, 2, \ldots, k\} \) such that \( r_i s_j \) is a unit. Otherwise since each \( r_i s_j \) is not unit then \( r_i x_i \in M \), for all \( i = 1, 2, \ldots, k \), hence \( \sum_{i=1}^{k} r_i s_i \in M \). That is a contradiction. So \( r_i s_j \) is a unit for some \( i \), then \( s_j \) is a unit. Hence \( a = a s_i s_j^{-1} = a_j^{-1} s_j \in \langle a_i \rangle \) Then \( M = \langle a_i \rangle \).

**Theorem 2** Let \( M = \langle m_1, m_2, \ldots, m_k \rangle \) be a finitely generated graded \( R \)-module over a graded ring \( R \). Then the following are equivalent:

1. \( M \) is gr-multiplication.
2. \( M \) is locally gr-principal.
3. \( \sum_{i=1}^{k} [m_i : M_i] = R \), where \( M_i = \langle a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k \rangle \)

**Proof.**

1. \( \rightarrow \) (2) By Theorem 1.
2. \( \rightarrow \) (3) Let \( M \) be a locally gr-principal. Then for graded prime ideal \( P \) of \( R \), we have by Proposition 3 \( MR_p^g = \langle m_1, m_2, \ldots, m_k \rangle = \langle m_i \rangle > R_p^g \), for some \( i \in \{1, 2, \ldots, k\} \). Hence for any gr-prime ideal \( P \) of \( R[(m_i)R_p^g : M_i R_p^g] = R_p^g \), where \( M_i = \langle a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k \rangle \) and then \( \sum_{i=1}^{k} [(m_i) : M_i] R_p^g = \sum_{i=1}^{k} [(m_i)R_p^g : M_i R_p^g] = R_p^g \). Since \( M_i \) is finitely generated for each \( i \), there for \( \sum_{i=1}^{k} [(m_i) : M_i] = R \).

3. \( \rightarrow \) (2) Suppose that \( \sum_{i=1}^{k} [(m_i) : M_i] = R \). Then for any gr-prime \( P \) of \( R \) we have \( \sum_{i=1}^{k} [(m_i)R_p^g : M_i R_p^g] = \sum_{i=1}^{k} [(m_i) : M_i] R_p^g = \sum_{i=1}^{k} [(m_i) : M_i] R_p^g = R_p^g \). Therefore, there is \( i \in \{1, 2, \ldots, k\} \) such that \( (m_i)R_p^g : M_i R_p^g = R_p^g \) and then
\[ \text{MR}_p^g \subset (a_i)R_p^g = \langle \frac{a_i}{1} \rangle. \] It follows that \[ \text{MR}_p^g = \langle \frac{a_i}{1} \rangle \] for each gr-prime ideal \( P \) of \( R \). Hence \( M \) is locally gr-principal. If \( M \) is a graded module over the graded ring \( R \) we define the \( \theta^g(M) = \sum_{x \in (M)} [(x): M] \). It is clear that \( \theta^g(M) \) is a graded ideal of \( R \).

**Proposition 4** Let \( M \) be a graded multiplication module over a graded ring \( R \). Then

1. \[ M = M \theta^g(M) \]
2. \[ N = N \theta^g(M) \] for any graded submodule \( N \) of \( M \).

**Proof.** (1) Let \( x \in M \) as \( M \) is graded multiplication \( R \)-module, then \( <x> = [(x): M]M \) since

\[ M = \sum_{x \in M} <x> = \sum_{x \in M} [(x): M]M = M \sum_{x \in M} [(x): M] = M \theta^g(M). \]

(2) suppose that \( N \) is a graded submodule of \( M \). Then \( N = [N : M]M \), where \( [N : M] \) is a graded ideal of \( R \). Hence

\[ N = [N : M]M = [N : M] \theta^g(M)M = N \theta^g(M). \]

**Proposition 5** Let \( N \) and \( K \) be graded submodules of graded multiplication \( R \)-module \( M \) and \( S \subset H(R) \) be a multiplicatively closed subset of \( R \). Then

1. \( \theta^g(N) \theta^g(K) \subset \theta^g(NK) \)
2. \( S^{-1}(\theta^g(N)) \subset \theta^g(S^{-1}(N)) \)

**Proof.** (1) If \( M \) is a multiplication \( R \)-module and \( N = lM \) and \( K = JM \) we defined \( NK = lM \). If \( x \in M \) and \( y \in K \), then \( xy = \sum_{i=1}^{n} r_i m_i \), where \( r_i \in lJ \), for all \( i = 1,2,\cdots,n \) and \( n \geq 1 \). See [2].

Let \( a \in N \cap H(M) \) and \( b \in K \cap H(M) \). It is enough to prove that

\[ [(a):N][(b):K] \subset [(ab):NK] \]

where \( x_i \in [(a):N] \) and \( y_i \in [(b):K] \), for \( i = 1,2,\cdots,n \). Then \( x_i N \subset (a) \) and \( y_i K \subset (b) \), for \( i = 1,2,\cdots,n \). Hence, \( x_i y_i NK \subset (ab) \) and then

\[ x_i y_i \in [(ab):NK]. \] Therefore \( \sum_{i=1}^{n} x_i y_i \in [(ab):NK]. \)
Recall that a graded module $M$ over graded ring $R$ is called gr-finitely generated if $M$ is generated by a finite set of homogeneous elements.

**Theorem 3** Let $M$ be a graded $R$-module where $R$ is a graded ring. Then $M$ is gr-finitely generated and locally gr-principal if $\theta^g(M) = R$.

**Proof.** Let $J$ be a gr-maximal ideal in $R$. Then $MR^g_j = (x)R^g_j$ for some $x \in H(M)$. Hence, $R^g_j = [(x)R^g_j : MR^g_j] = [(x) : M]R^g_j$ since $M$ is gr-finitely generated. Therefore $R^g_j = \theta^g(M)R^g_j$ and they by local property $\theta^g(M) = R$.

Conversely, suppose $\theta^g(M) = R$. Then there exist, $m_1, m_2, \ldots, m_k \in H(N)$ such that $R = \theta^g(M) = [(m_1) : M] + [(m_2) : M] + \cdots + [(m_k) : M]$. Thus $M = \theta^g(M)M = M[(m_1) : M] + M[(m_2) : M] + \cdots + M[(m_k) : M] \subseteq [m_1] + [m_2] + \cdots + [m_k] \subseteq M$ so $M = (m_1, m_2, \ldots, m_k)$ is gr-finitely generated. Let $J$ be a gr-maximal ideal of $R$. Since $\theta^g(M) = R$, there is $x \in H(M)$, with $[(x) : M] \subseteq J$. Therefore, there exists $r \in R - J$ with $rM \subseteq (x)$ and then $rMR^g_j = r > R^g_j : MR^g_j = MR^g_j \subseteq (x)R^g_j$. Hence $MR^g_j = (x)R^g_j$, for any gr-maximal ideal $J$ of $R$ and so $M$ is locally gr-principal.

**References**


Received: November, 2012