

A Sequence of Weakly Monotonic Automata with Increasing Level

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Abstract

In this paper, we exhibit a sequence of strongly connected aperiodic weakly monotonic automata, whose level increases with the number of states.

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1 Introduction

Forty five years ago, Černý [1] presented a family of synchronizing automata with n states whose shortest reset words have length $(n - 1)^2$ and conjectured that for every automaton with n states, if there is a synchronizing word, then there is one with at most $(n - 1)^2$ letters. Several advances have been made towards the proof of this conjecture, but the general case remains open. For more information on the history of the Černý Conjecture and the particular cases that have been solved, see for example [3].

The class of aperiodic automata has received some attention recently regarding its synchronizing properties. Particularly, in [2] Trahtman established the Černý conjecture for this class of automata, proving that every synchronizing aperiodic automaton with n states has a reset word whose length is at most $n(n-1)/2$. Later in [4], Volkov introduced the concept of weakly monotonic automaton and proved that every aperiodic automaton is weakly monotonic. Then he established that every synchronizing strongly connected weakly monotonic automaton has a reset word whose length is not greater than $\lfloor n(n-1)/6 \rfloor$. This way, Trahtman's bound was improved for the case of strongly connected aperiodic automata.

In this paper, we show that given $l \in \mathbb{N}$, there is a synchronizing strongly connected aperiodic automaton that is weakly monotonic of level l . The reason to search for strongly connected and aperiodic automata is the fact that in his paper Volkov suspects that the bound $\lfloor n(n-1)/6 \rfloor$ can be further improved for this particular case. In private communication, Volkov asked the second author whether there is a bound on the level of weak monotonicity of strongly connected aperiodic automata, in the hope that such a property would entail fast synchronization.

2 Definitions

Given a complete deterministic finite automaton $\mathcal{A} = (S, \Sigma, \delta)$, a binary relation $\rho \subset S \times S$ in \mathcal{A} is *stable* if for every $\sigma \in \Sigma$ and every $p, q \in S$, $(p, q) \in \rho$ implies $(\delta(p, \sigma), \delta(q, \sigma)) \in \rho$. The equivalence closure of a binary relation ρ , denoted by $\text{Eq}(\rho)$, is the smallest equivalence relation that contains ρ . Of course, if ρ is stable so is $\text{Eq}(\rho)$. A *congruence* over \mathcal{A} is a stable equivalence relation $\pi \subset S \times S$.

Given a congruence π , denote by $[p]_\pi$ the π -class that contains the state $p \in S$. One defines the *quotient automaton* \mathcal{A}/π as the automaton $(S/\pi, \Sigma, \delta_\pi)$, where $S/\pi = \{[p]_\pi : p \in S\}$ and the transition function δ_π is such that for every π -class $[p]_\pi$ and every $\sigma \in \Sigma$, $\delta_\pi([p]_\pi, \sigma) = [\delta(p, \sigma)]_\pi$.

The complete deterministic finite automaton $\mathcal{A} = (S, \Sigma, \delta)$ is said to be *weakly monotonic of level l* , see [4], if there is a strictly increasing chain of stable binary relations $\rho^0 \subset \rho^1 \subset \dots \subset \rho^l$ on \mathcal{A} such that:

- ρ^0 is the equality relation $\{(s, s) : s \in S\}$;
- for each $i \in \{1, 2, \dots, l\}$, $\pi^{i-1} = \text{Eq}(\rho^{i-1}) \subset \rho^i$ and ρ^i/π^{i-1} is a partial order on S/π^{i-1} ;
- $\pi^l = \text{Eq}(\rho^l)$ is the universal relation on S .

A semigroup A is said to be *aperiodic* if all its subgroups are trivial, which is equivalent to the property that for every $a \in A$, there is some $m \in \mathbb{N}$ such

that $a^m = a^{m+1}$. An automaton $\mathcal{A} = (S, \Sigma, \delta)$ whose transition semigroup is aperiodic is called *aperiodic* as well.

We say that the automaton $\mathcal{A} = (S, \Sigma, \delta)$ is *strongly connected* if its underlying digraph G is strongly connected, that is, for all $p, q \in S$ there is a directed path in G going from p to q .

3 A Sequence Of Automata

For each positive integer n , consider the complete deterministic finite automaton $\mathcal{A}_n = (S_n, \Sigma_n, \delta_n)$, with set of states $S_n = \{0, 1, \dots, n-1\}$, alphabet $\Sigma_n = \{\sigma_0, \sigma_1, \dots, \sigma_{n-1}\}$ and transition function δ_n such that:

- for each $i \in 0, 1, \dots, n-2$ and each $s \in S$,

$$\delta_n(s, \sigma_i) = \begin{cases} i & \text{if } s < n-i-1 \text{ or } s = n-i-1 \geq \lceil \frac{n}{2} \rceil; \\ i+1 & \text{if } s \geq n-i \text{ or } s = n-i-1 < \lceil \frac{n}{2} \rceil \end{cases};$$

- for every $s \in S$,

$$\delta_n(s, \sigma_{n-1}) = \begin{cases} n-s-1 & \text{if } s < \lceil \frac{n}{2} \rceil \\ n-s & \text{if } s \geq \lceil \frac{n}{2} \rceil \end{cases}.$$

Note that

$$\text{for every } n > 1 \text{ and every } \sigma \in \Sigma_n, \delta_n\left(\lceil \frac{n}{2} \rceil, \sigma\right) = \delta_n\left(\lceil \frac{n}{2} \rceil - 1, \sigma\right). \quad (1)$$

To represent the automaton \mathcal{A}_n we use a $n \times n$ matrix over \mathbb{N} such that for $i, j \in \{0, 1, \dots, n-1\}$ the entry (i, j) is $\delta_n(i, \sigma_j)$.

Example 3.1. The matrices that represent the automata $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ and \mathcal{A}_5 are, respectively:

$$(0), \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 2 & 3 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 3 \\ 0 & 1 & 3 & 4 & 2 \\ 0 & 1 & 3 & 4 & 2 \\ 0 & 2 & 3 & 4 & 1 \end{pmatrix}.$$

Lemma 3.2. *Let ρ be a stable and transitive binary relation on \mathcal{A}_n . Suppose that $(s, t) \in \rho$ with $s \neq t$ and that there is a nonempty set I_t^s of consecutive elements of $\{0, 1, \dots, n-2\}$ such that for each $i \in I_t^s$, $\delta_n(s, \sigma_i) = i$ and $\delta_n(t, \sigma_i) = i+1$. Suppose also that for $p = \min I_t^s$ and $q = \max I_t^s + 1$, $\delta_n(s, \sigma_{n-1}) = q$ and $\delta_n(t, \sigma_{n-1}) = p$. Then the relation ρ cannot be antisymmetric.*

Proof. We know that for every $i \in I_t^s$, $\delta_n(s, \sigma_i) = i$ and $\delta_n(t, \sigma_i) = i + 1$, therefore

$$\begin{aligned} \delta_n(s, \sigma_p) &= p \text{ and } \delta_n(t, \sigma_p) = p + 1, \\ \delta_n(s, \sigma_{p+1}) &= p + 1 \text{ and } \delta_n(t, \sigma_{p+1}) = p + 2, \\ &\vdots \\ \delta_n(s, \sigma_{q-1}) &= q - 1 \text{ and } \delta_n(t, \sigma_{q-1}) = q. \end{aligned}$$

Since ρ is stable and $(s, t) \in \rho$, we have $(p, p+1), (p+1, p+2), \dots, (q-1, q) \in \rho$. Using the transitivity of ρ , we conclude that $(p, q) \in \rho$. But we also have $(q, p) \in \rho$, because $\delta_n(s, \sigma_{n-1}) = q$ and $\delta_n(t, \sigma_{n-1}) = p$ and ρ is stable. Thus, since $p \neq q$ because I_t^s is nonempty, ρ cannot be antisymmetric. \square

Lemma 3.3. *Let π_n^0 be the equality relation on \mathcal{A}_n . The only stable partial orders on \mathcal{A}_n are π_n^0 , $\rho_n^1 = \pi_n^0 \cup \{(\lceil n/2 \rceil - 1, \lceil n/2 \rceil)\}$ and $\bar{\rho}_n^1 = \pi_n^0 \cup \{(\lceil n/2 \rceil, \lceil n/2 \rceil - 1)\}$.*

Proof. Let ρ be a stable and transitive binary relation on \mathcal{A}_n and suppose that $(s, t) \in \rho$ with $s \neq t$ and $\{s, t\} \neq \{\lceil n/2 \rceil - 1, \lceil n/2 \rceil\}$. We can assume without loss of generality that $s < t$ for the usual order on \mathbb{N} , otherwise it would be enough to consider the reverse order of ρ . Since $\{s, t\} \neq \{\lceil n/2 \rceil - 1, \lceil n/2 \rceil\}$, we have the following possibilities:

1. $s < t < \lceil n/2 \rceil$;
2. $\lceil n/2 \rceil < s < t$;
3. $s < \lceil n/2 \rceil < t$;
4. $s = \lceil n/2 \rceil < t$;
5. $s < \lceil n/2 \rceil = t$.

In case 1, note that the set $I_t^s = \{i \in \{0, 1, \dots, n-2\} : n-t-1 \leq i < n-s-1\}$ is nonempty. For each $i \in I_t^s$, we have $s < n-i-1 \leq t$, hence $\delta_n(s, \sigma_i) = i$ and $\delta_n(t, \sigma_i) = i+1$. We also have $\delta_n(s, \sigma_{n-1}) = n-s-1$ and $\delta_n(t, \sigma_{n-1}) = n-t-1$. This way we are in the conditions of Lemma 3.2 and ρ cannot be antisymmetric, which means that it is not a partial order.

In case 2, we put $I_t^s = \{i \in \{0, 1, \dots, n-2\} : n-t \leq i < n-s\}$, which is again a nonempty set. For each $i \in I_t^s$, we have $s < n-i \leq t$, hence $\delta_n(s, \sigma_i) = i$ and $\delta_n(t, \sigma_i) = i+1$. We also have $\delta_n(s, \sigma_{n-1}) = n-s$ and $\delta_n(t, \sigma_{n-1}) = n-t$. This way we are in the conditions of Lemma 3.2 and ρ cannot be antisymmetric, which means that it is not a partial order.

In case 3, $s < t-1$ and so the set $I_t^s = \{i \in \{0, 1, \dots, n-2\} : n-t \leq i < n-s-1\}$ is nonempty. For each $i \in I_t^s$, we have $s < n-i-1 < t$, hence

$\delta_n(s, \sigma_i) = i$ and $\delta_n(t, \sigma_i) = i + 1$. We also have $\delta_n(s, \sigma_{n-1}) = n - s - 1$ and $\delta_n(t, \sigma_{n-1}) = n - t$. This way we are in the conditions of Lemma 3.2 and ρ cannot be antisymmetric, which means that it is not a partial order.

In case 4, we have $(\lceil n/2 \rceil, t) \in \rho$, with $t > \lceil n/2 \rceil$. Since $\delta_n(\lceil n/2 \rceil, \sigma_{n-t}) = n - t$ and $\delta_n(t, \sigma_{n-t}) = n - t + 1$, so that $(n - t, n - t + 1) \in \rho$, we fall again in case 1 unless n is even and $t = n/2 + 1$, in which case we have $(n - t, n - t + 1) = (n/2 - 1, n/2) \in \rho$ and also $(n/2, n/2 + 1) \in \rho$. But $\delta_n(n/2, \sigma_{n-1}) = n - n/2 = n/2$ and $\delta_n(n/2 + 1, \sigma_{n-1}) = n - (n/2 + 1) = n/2 - 1$, so that $(n/2, n/2 - 1) \in \rho$, since ρ is stable. This proves that ρ cannot be antisymmetric because both $(n/2, n/2 - 1)$ and $(n/2 - 1, n/2)$ belong to ρ .

In case 5, since $\{s, t\} \neq \{\lceil n/2 \rceil - 1, \lceil n/2 \rceil\}$, we must have $s < \lceil n/2 \rceil - 1$ and $t = \lceil n/2 \rceil$, with $(s, t) \in \rho$. Since $\delta_n(s, \sigma_{n-\lceil n/2 \rceil}) = \lceil n/2 \rceil$ and $\delta_n(t, \sigma_{n-\lceil n/2 \rceil}) = \lceil n/2 \rceil + 1$, we deduce that $(\lceil n/2 \rceil, \lceil n/2 \rceil + 1) \in \rho$, which falls in case 5, that we have already treated.

To finish the proof, it is enough to verify that ρ_n^1 is a stable partial order on \mathcal{A}_n , since $\bar{\rho}_n^1$ is the reverse order of ρ_n^1 . In view of (1), we deduce that ρ_n^1 is stable and it is trivial to check that it is reflexive, transitive and antisymmetric. \square

Given a positive integer n , consider the automaton $\mathcal{B}_n = (S_n, \Sigma_n \uplus \{\tau\}, \bar{\delta}_n)$, where $\bar{\delta}_n$ is such that for every $s \in S_n$,

$$\bar{\delta}_n|_{S_n \times \Sigma_n} = \delta_n \text{ and } \bar{\delta}_n(s, \tau) = \lceil n/2 \rceil - 1.$$

Lemma 3.4. *Consider the stable equivalence relation*

$$\pi_n^1 = \pi_n^0 \cup \{(\lceil n/2 \rceil - 1, \lceil n/2 \rceil), (\lceil n/2 \rceil, \lceil n/2 \rceil - 1)\}$$

on \mathcal{A}_n . Then, for every $n \geq 2$,

$$\mathcal{A}_n / \pi_n^1 \simeq \mathcal{B}_{n-1}.$$

Proof. Consider the functions

$$\begin{aligned} \phi : S_n / \pi_n^1 &\longrightarrow S_{n-1} & \psi : \Sigma_n &\longrightarrow \Sigma_{n-1} \cup \{\tau\} \\ [s] &\longmapsto \begin{cases} s & \text{if } s < \lceil \frac{n}{2} \rceil \\ s - 1 & \text{if } s \geq \lceil \frac{n}{2} \rceil \end{cases} & \sigma_i &\longmapsto \begin{cases} \sigma_i & \text{if } i < \lceil \frac{n}{2} \rceil - 1 \\ \tau & \text{if } i = \lceil \frac{n}{2} \rceil - 1 \\ \sigma_{i-1} & \text{if } i > \lceil \frac{n}{2} \rceil - 1 \end{cases} \end{aligned}$$

Note that ϕ is well defined, because the only nontrivial class is $\{\lceil \frac{n}{2} \rceil - 1, \lceil \frac{n}{2} \rceil\}$. For the same reason, ϕ is bijective and it is obvious that ψ is also a bijection. Hence, to finish the proof all we need to check is that the pair (ϕ, ψ) defines a morphism between the automata \mathcal{A}_n / π_n^1 and \mathcal{B}_{n-1} , that is, for every $[s] \in S_n / \pi_n^1$ and every $\sigma_i \in \Sigma_n$,

$$\bar{\delta}_{n-1}(\phi([s]), \psi(\sigma_i)) = \phi(\delta_n([s], \sigma_i)). \quad (2)$$

We have the following possibilities:

1. $i = \lceil n/2 \rceil - 1$;
2. $i < \lceil n/2 \rceil - 1$ and $s \leq \lceil n/2 \rceil$;
3. $i < \lceil n/2 \rceil - 1$ and $s > \lceil n/2 \rceil$;
4. $i > \lceil n/2 \rceil - 1$ and $s \leq \lceil n/2 \rceil$;
5. $i > \lceil n/2 \rceil - 1$ and $s > \lceil n/2 \rceil$.

In case 1, $\psi(\sigma_i) = \tau$ and $\bar{\delta}_{n-1}(t, \tau) = \lceil n/2 \rceil - 1$, for all $t \in S_{n-1}$, hence

$$\bar{\delta}_{n-1}(\phi([s]), \psi(\sigma_i)) = \left\lceil \frac{n}{2} \right\rceil - 1, \text{ for every } [s] \in S_n/\pi_n^1.$$

On the other hand,

$$\delta_n([s], \sigma_i) = \begin{cases} [i] & \text{if } s < n - i - 1 \text{ or } s = n - i - 1 \geq \lceil n/2 \rceil \\ [i + 1] & \text{if } s \geq n - i \text{ or } s = n - i - 1 < \lceil n/2 \rceil \end{cases}.$$

Since $[i] = \{i, i + 1\} = [i + 1]$ and $\phi([i]) = \lceil n/2 \rceil - 1$, we have

$$\phi(\delta_n([s], \sigma_i)) = \left\lceil \frac{n}{2} \right\rceil - 1, \text{ for every } [s] \in S_n/\pi_n^1.$$

Therefore, the equality (2) holds in this case.

In case 2, $\psi(\sigma_i) = \sigma_i$ and $\phi([s]) = s$, for $s < \lceil n/2 \rceil$. For $s = \lceil n/2 \rceil$, $[s] = \lceil n/2 \rceil - 1$, so it is enough to consider $s < \lceil n/2 \rceil$. It follows that

$$\begin{aligned} \bar{\delta}_{n-1}(\phi([s]), \psi(\sigma_i)) &= \bar{\delta}_{n-1}(s, \sigma_i) = \delta_{n-1}(s, \sigma_i) = \\ &= \begin{cases} i & \text{if } s < n - i - 2 \text{ or } s = n - i - 2 \geq \lceil \frac{n-1}{2} \rceil \\ i + 1 & \text{if } s \geq n - i - 1 \text{ or } s = n - i - 2 < \lceil \frac{n-1}{2} \rceil \end{cases}. \end{aligned}$$

But $i < \lceil n/2 \rceil - 1$ implies $n - i - 2 \geq \lceil (n - 1)/2 \rceil$, therefore the condition $s = n - i - 2 < \lceil (n - 1)/2 \rceil$ is impossible and

$$\bar{\delta}_{n-1}(\phi([s]), \psi(\sigma_i)) = \begin{cases} i & \text{if } s < n - i - 1 \\ i + 1 & \text{if } s \geq n - i - 1 \end{cases}.$$

On the other hand,

$$\phi(\delta_n([s], \sigma_i)) = \begin{cases} \phi([i]) & \text{if } s < n - i - 1 \\ \phi([i + 1]) & \text{if } s \geq n - i - 1 \end{cases} = \begin{cases} i & \text{if } s < n - i - 1 \\ i + 1 & \text{if } s \geq n - i - 1 \end{cases},$$

because

$$i < \left\lceil \frac{n}{2} \right\rceil - 1 \text{ implies } \left(\phi([i]) = i \text{ and } \phi([i + 1]) = i + 1 \right). \quad (3)$$

Thus, the equality (2) holds in this case.

In case 3, $\psi(\sigma_i) = \sigma_i$, $\phi([s]) = s - 1$ and $s - 1 \geq \lceil n/2 \rceil \geq \lceil (n - 1)/2 \rceil$. Hence

$$\begin{aligned} \bar{\delta}_{n-1}(\phi([s]), \psi(\sigma_i)) &= \bar{\delta}_{n-1}(s - 1, \sigma_i) = \delta_{n-1}(s - 1, \sigma_i) = \\ &= \begin{cases} i & \text{if } s - 1 < n - i - 1 \\ i + 1 & \text{if } s - 1 \geq n - i - 1 \end{cases} = \begin{cases} i & \text{if } s < n - i \\ i + 1 & \text{if } s \geq n - i \end{cases}. \end{aligned}$$

On the other hand, in view of (3), we have

$$\phi(\delta_n([s], \sigma_i)) = \begin{cases} \phi([i]) & \text{if } s < n - i \\ \phi([i + 1]) & \text{if } s \geq n - i \end{cases} = \begin{cases} i & \text{if } s < n - i \\ i + 1 & \text{if } s \geq n - i \end{cases}.$$

Thus, the equality (2) holds in this case.

In case 4, $\psi(\sigma_i) = \sigma_{i-1}$ and $\phi([s]) = s$, for $s < \lceil n/2 \rceil$. For $s = \lceil n/2 \rceil$, $[s] = \lceil \lceil n/2 \rceil - 1 \rceil$, so it is enough to consider $s < \lceil n/2 \rceil$. It follows that

$$\begin{aligned} \bar{\delta}_{n-1}(\phi([s]), \psi(\sigma_i)) &= \bar{\delta}_{n-1}(s, \sigma_{i-1}) = \delta_{n-1}(s, \sigma_{i-1}) = \\ &= \begin{cases} i - 1 & \text{if } s < n - i - 1 \text{ or } s = n - i - 1 \geq \lceil \frac{n-1}{2} \rceil \\ i & \text{if } s \geq n - i \text{ or } s = n - i - 1 < \lceil \frac{n-1}{2} \rceil \end{cases}. \end{aligned}$$

But

$$i > \left\lceil \frac{n}{2} \right\rceil - 1 \text{ implies } n - i - 1 < \left\lceil \frac{n-1}{2} \right\rceil \leq \left\lceil \frac{n}{2} \right\rceil, \quad (4)$$

therefore the condition $s = n - i - 1 \geq \lceil (n - 1)/2 \rceil$ is impossible and

$$\bar{\delta}_{n-1}(\phi([s]), \psi(\sigma_i)) = \begin{cases} i - 1 & \text{if } s < n - i - 1 \\ i & \text{if } s \geq n - i - 1 \end{cases}.$$

On the other hand,

$$\phi(\delta_n([s], \sigma_i)) = \begin{cases} \phi([i]) & \text{if } s < n - i - 1 \\ \phi([i + 1]) & \text{if } s \geq n - i - 1 \end{cases} = \begin{cases} i - 1 & \text{if } s < n - i - 1 \\ i & \text{if } s \geq n - i - 1 \end{cases},$$

because

$$i \geq \left\lceil \frac{n}{2} \right\rceil \text{ implies } \left(\phi([i]) = i - 1 \text{ and } \phi([i + 1]) = i \right). \quad (5)$$

Thus, the equality (2) holds in this case.

In case 5, $\psi(\sigma_i) = \sigma_{i-1}$, $\phi([s]) = s - 1$ and $s - 1 \geq \lceil n/2 \rceil \geq \lceil (n - 1)/2 \rceil$. Hence

$$\bar{\delta}_{n-1}(\phi([s]), \psi(\sigma_i)) = \bar{\delta}_{n-1}(s - 1, \sigma_{i-1}) = \delta_{n-1}(s - 1, \sigma_{i-1}) =$$

$$= \begin{cases} i-1 & \text{if } s-1 \leq n-1-(i-1) \\ i & \text{if } s-1 > n-1-(i-1) \end{cases} = \begin{cases} i-1 & \text{if } s \leq n-i+1 \\ i & \text{if } s > n-i+1 \end{cases}.$$

On the other hand, in view of (5), we have

$$\phi(\delta_n([s], \sigma_i)) = \begin{cases} \phi([i]) & \text{if } s < n-i \\ \phi([i+1]) & \text{if } s \geq n-i \end{cases} = \begin{cases} i-1 & \text{if } s < n-i \\ i & \text{if } s \geq n-i \end{cases}.$$

Thus, equality (2) holds in this case as well, concluding our proof. \square

Lemma 3.5. *Let $\mathcal{A} = (S, \Sigma, \delta)$ be a complete deterministic finite automaton and consider the automaton $\mathcal{B} = (S, \Sigma \uplus \{\tau\}, \bar{\delta})$, where $\bar{\delta}|_{S \times \Sigma} = \delta$ and there is $\bar{s} \in S$ such that, for all $s \in S$, $\bar{\delta}(s, \tau) = \bar{s}$. Then for every $l \in \mathbb{N}$, \mathcal{A} is weakly monotonic of level l if and only if \mathcal{B} is weakly monotonic of level l .*

Proof. Suppose that \mathcal{A} is weakly monotonic of level l for some natural number l . Then there is a strictly increasing chain of stable binary relations $\rho^0 \subset \rho^1 \subset \dots \subset \rho^l$ on \mathcal{A} such that:

1. ρ^0 is the equality relation $\{(s, s) : s \in S\}$;
2. for each $i \in \{1, 2, \dots, l\}$, $\pi^{i-1} = \text{Eq}_{\mathcal{A}}(\rho^{i-1}) \subset \rho^i$ and ρ^i/π^{i-1} is a partial order on S/π^{i-1} ;
3. $\pi^l = \text{Eq}_{\mathcal{A}}(\rho^l)$ is the universal relation on S .

It is clear that $\rho^0 \subset \rho^1 \subset \dots \subset \rho^l$ is a strictly increasing chain of stable binary relations on \mathcal{B} , since the state set is the same in both automata, the only letter present in \mathcal{B} that is not in \mathcal{A} acts as a constant, and all these relations contain the equality relation. Trivially, condition 1 holds in \mathcal{B} . Condition 2 is also verified, because $\text{Eq}_{\mathcal{A}}(\rho) = \text{Eq}_{\mathcal{B}}(\rho)$ for any binary relation on these automata. Finally, condition 3 holds for the same reason. Which means that \mathcal{B} is weakly monotonic of level l . The converse is proved in the same way. \square

Theorem 3.6. *For every positive integer n , the automaton \mathcal{A}_n is strongly connected, aperiodic and weakly monotonic of level $n-1$.*

Proof. It is clear that \mathcal{A}_n is strongly connected for every positive integer n , because given $s \in S_n$, $\delta_n(s, \sigma_0) = 0$, $\delta_n(0, \sigma_s) = s$.

Let T_n be the transformation monoid of the automaton \mathcal{A}_n and consider the associated function $\zeta_n : \Sigma_n^* \rightarrow T_n$. To see that \mathcal{A}_n is aperiodic, we will use induction on n . The automaton \mathcal{A}_1 is clearly aperiodic. Moreover, for every idempotent $e \in T_1$ and every word $w \in \zeta_1^{-1}(e)$, we have $|\delta_1(S_1, w)| = 1$, that is w is a synchronizing word. Suppose that, for some integer $n \geq 2$, \mathcal{A}_{n-1} is aperiodic and, for every idempotent $\bar{e} \in T_{n-1}$ and every word $\bar{w} \in \zeta_{n-1}^{-1}(\bar{e})$, we have $|\delta_{n-1}(S_{n-1}, \bar{w})| = 1$.

Consider the functions

$$\begin{aligned} \xi : S_n &\longrightarrow S_{n-1} & \theta : \Sigma_n &\longrightarrow \Sigma_{n-1} \\ s &\longmapsto \begin{cases} s & \text{if } s < \lceil \frac{n}{2} \rceil \\ s - 1 & \text{if } s \geq \lceil \frac{n}{2} \rceil \end{cases} & \sigma_i &\longmapsto \begin{cases} \sigma_i & \text{if } i < \lceil \frac{n}{2} \rceil - 1 \\ \sigma_0 \sigma_{\lceil n/2 \rceil - 1} & \text{if } i = \lceil \frac{n}{2} \rceil - 1 \\ \sigma_{i-1} & \text{if } i > \lceil \frac{n}{2} \rceil - 1 \end{cases} \end{aligned}$$

In the proof of Lemma 3.4 we presented an isomorphism $\mathcal{A}_n/\pi_n^1 \longrightarrow \mathcal{B}_{n-1}$ that was obtained from functions $\phi : S_n/\pi_n^1 \longrightarrow S_{n-1}$ and $\psi : \Sigma_n \longrightarrow \Sigma_{n-1} \cup \{\tau\}$, where π_n^1 is the kernel of the mapping ξ . Thus the pair (ξ, θ) is obtained by composing the quotient morphism $\mathcal{A}_n \longrightarrow \mathcal{A}_n/\pi_n^1$ with the isomorphism $(\phi, \psi) : \mathcal{A}_n/\pi_n^1 \longrightarrow \mathcal{B}_{n-1}$ and finally with the morphism $\mathcal{B}_{n-1} \longrightarrow \mathcal{A}_{n-1}$ that fixes the states and the letters σ_i and maps the letter τ to $\sigma_0 \sigma_{\lceil n/2 \rceil - 1}$. Hence (ξ, θ) is a morphism of automata. Now, using θ , we can define a morphism $\Theta : T_n \longrightarrow T_{n-1}$, by putting the image of an element of T_n written as a product of letters in Σ_n , to be the value in T_{n-1} of the product of the images of those letters by θ . All we need to check is that Θ is well defined, that is, if α and β represent the same element of T_n , then $\Theta(\alpha)$ and $\Theta(\beta)$ represent the same element of T_{n-1} . But if α and β represent the same element of T_n , then $\delta_n(s, \alpha) = \delta_n(s, \beta)$, for every $s \in S_n$. This implies that $\delta_{n-1}(\xi(s), \theta(\alpha)) = \delta_{n-1}(\xi(s), \theta(\beta))$, for every $s \in S_n$. Or equivalently $\delta_{n-1}(t, \theta(\alpha)) = \delta_{n-1}(t, \theta(\beta))$, for every $t \in S_{n-1}$, because ξ is surjective. But this means that $\Theta(\alpha)$ and $\Theta(\beta)$ represent the same element of T_{n-1} .

Now, consider an idempotent $e \in T_n$, since Θ is a monoid morphism, we know that $\Theta(e)$ is an idempotent of T_{n-1} . Thus, for $w \in \zeta_n^{-1}(e)$, we have $|\xi(\delta_n(S_n), w)| = |\delta_{n-1}(\xi(S_n), \theta(w))| = 1$, according to the induction hypothesis, because $\theta(w) \in \zeta_{n-1}^{-1}(\Theta(e))$. If $|\delta_n(S_n, w)| \neq 1$, then $\delta_n(S_n, w) = \{\lceil n/2 \rceil - 1, \lceil n/2 \rceil\}$, because these are the only two distinct elements in S_n that have the same image under the function ξ . Since e is an idempotent, for all $s \in S_n$ we have $\delta_n(\delta_n(s, w), w) = \delta_n(s, w^2) = \delta_n(s, w)$, that is, w fixes all the elements in $\delta_n(S_n, w)$. But we know that for every $\sigma \in \Sigma$, $\delta_n(\lceil n/2 \rceil - 1, \sigma) = \delta_n(\lceil n/2 \rceil, \sigma)$, therefore there is no $w \in \zeta_n^{-1}(e)$ such that $\delta_n(S_n, w) = \{\lceil n/2 \rceil - 1, \lceil n/2 \rceil\}$. Which means that $|\delta_n(S_n, w)| = 1$.

We proved that, for every word w in Σ_n^* such that $\zeta_n(w)$ is an idempotent of T_n , w is a synchronizing word. Now suppose that $x \in T_n$ and $m > 1$ are such that $x^m = x$. Then x^{m-1} is an idempotent, thus every word $w \in \zeta_n^{-1}(x^{m-1})$ is a synchronizing word in the automaton \mathcal{A}_n . Consider the word $v \in \zeta_n^{-1}(x)$. Then wv is still a synchronizing word and it belongs to $\zeta_n^{-1}(x^{m-1}x) = \zeta_n^{-1}(x^m) = \zeta_n^{-1}(x)$, therefore x is an idempotent because for each $s \in S_n$, $\delta_n(s, (wv)^2) = \delta_n(s, wv)$. But this means that if $x^m = x$ and $m > 1$, then $x^2 = x$, that is, T_n is aperiodic, which means that \mathcal{A}_n is aperiodic.

We will also use induction on n to prove that \mathcal{A}_n is weakly monotonic of

level $n - 1$. Clearly \mathcal{A}_1 is weakly monotonic of level 0. Suppose that for some integer $n \geq 2$, \mathcal{A}_{n-1} is weakly monotonic of level $n - 2$.

According to Lemma 3.3, the only stable partial orders on \mathcal{A}_n are the equality relation, $\rho_n^1 = \pi_n^0 \cup \{(\lceil n/2 \rceil - 1, \lceil n/2 \rceil)\}$ and $\bar{\rho}_n^1 = \pi_n^0 \cup \{(\lceil n/2 \rceil, \lceil n/2 \rceil - 1)\}$. Now using Lemmas 3.4 and 3.5, we know that \mathcal{A}_n/π_n^1 and \mathcal{A}_{n-1} have the same level, where $\pi_n^1 = \text{Eq}(\rho_n^1) = \text{Eq}(\bar{\rho}_n^1)$. But if the level of \mathcal{A}_n/π_n^1 is $n - 2$ and π_n^1 is the only stable equivalence relation obtained from a stable partial order in \mathcal{A}_n that is not the equality relation, then \mathcal{A}_n is weakly monotonic of level $n - 2 + 1 = n - 1$.

We have established that for every positive integer n , \mathcal{A}_n is a strongly connected aperiodic weakly monotonic automaton of level $n - 1$. \square

We found a family of strongly connected aperiodic automata whose level of weak monotonicity increases with the number of states. Of course, higher levels do not correspond necessarily to bigger reset words. Indeed, adding a letter that acts as a constant function has no effect on the level of weak monotonicity of an automaton, while it makes synchronization quite trivial.

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