**International Journal of Algebra, Vol. 6, 2012, no. 9, 447 - 455**

# Coefficient Modules and Rees Polynomials of Arbitrary Modules

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**Abstract**

Let $(R, \mathfrak{m})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$, $G = \oplus_{n\geq 0} G_n$ a standard graded algebra with $D := \dim(\mathrm{Proj}(G))$ and $I$ a $R$-submodule of $G_1$. We use the degree of Rees polynomial of $I$ to show that coefficient modules exist in general case for standard graded algebras and for $R$-submodules $E$ of $R^p$ over a Noetherian local ring $R$.

**Mathematic Subject Classification:** 13H15 (primary), 13D40, 14C17, 13C15

**Keywords:** coefficient modules, Rees polynomials

## 1 Introduction

Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring of Krull dimension $d$. Let $I$ be an $\mathfrak{m}$-primary ideal of $R$. From a classical result of Samuel in [9] the length of $R/I^n$ is a polynomial in $n$ of degree $d$, for all large $n$. This polynomial is called the Hilbert-Samuel polynomial of $I$ and can be written in the form:

$$\sum_{i=0}^{d}(-1)^i \mathrm{e}_i(I,R)\left(\begin{array}{c} n+d-i-1 \\ d-i \end{array}\right),$$

where $\mathrm{e}_i(I,R)$ are the $i$-th Hilbert-Samuel coefficient of $I$, for $i = 1, \ldots, d$. The integer $\mathrm{e}_0(I,R)$ is the classical Hilbert-Samuel multiplicity of $I$.

Recall that an ideal $J \subset I$ is called a reduction of $I$ if $JI^n = I^{n+1}$ for some positive integer $n$. The Rees multiplicity theorem asserts that if $R$ is quasi-unmixed local ring then $J \subset I$ is a reduction of $I$ if and only if $\mathrm{e}_0(I,R) = \mathrm{e}_0(J,R)$ (more details see [7]).

Let $J \subset I$ ideals of $R$ and $\ell(\frac{I}{J}) < \infty$. Amao [1] showed that the numerical function $H(\frac{I}{J}, n) := \ell(\frac{I^n}{J^n})$ is a polynomial function of degree at most $d$. We call $H(\frac{I}{J}, n)$ the Rees function of the pair $(J, I)$ and the corresponding polynomial $P(\frac{I}{J}, n)$, the Rees polynomial of the pair $(J, I)$. The Rees multiplicity theorem has been generalized for ideal that are not $\mathfrak{m}$-primary by a number authors. In [7] Rees showed that $(R, \mathfrak{m})$ is quasi-unmixed then $J \subset I$ is reduction of $I$ if and only if $\deg(P(\frac{I}{J}, n)) < d$. It it natural to ask for limiters to the degree of $\deg(P(\frac{I}{J}, n)) < d$. In the case $I$ and $J$ are $\mathfrak{m}$-primary it is easy to see that $\deg(P(\frac{I}{J}, n)) < d - k$ if and only if $\mathrm{e}_i(I,R) = \mathrm{e}_i(J,R)$ for $i = 0, \dots, k$ for all $k = 0, \dots, d$.

Shah proved in [8] the existence of unique largest ideals $I^R_{\{k\}}$, for $0 \leq k \leq d$ between the ideals $I$ and $\overline{I}$ such that $I^R_{\{k\}}$ preserves the first $k+1$ Hilbert-Samuel coefficients of $I$, i.e., there exists ideals $I^R_{\{k\}}$ which contain the ideal $I$ such that $I \subset I^R_{\{d\}} \subset \dots \subset I^R_{\{1\}} \subset I^R_{\{0\}} = \overline{I}$ and $\mathrm{e}_i(I,R) = \mathrm{e}_i(I^R_{\{k\}}, R)$, for $i = 0, \dots, k$. The ideal $I^R_{\{k\}}$ is called $k$-th coefficient ideal of $I$. Shah also determined a colon ideal structure for each coefficient ideal $I^R_{\{k\}}$ of $I$ as follows, $I^R_{\{k\}} = (I^{n_0+1} : x_1, \dots, x_k)$, for $k = 1, \dots, d$, some fixed $n_0 \geq 1$ and some fixed minimal reduction $\{x_1, \dots, x_d\}$ of $I^{n_0}$. Herzog-Puthenpurakal-Verma, proved in [4], that coefficient ideals exist even in the general case. Computation of coefficient ideals seems to be a difficult problem at present.

On the other hand, Liu in [6] partially extended the work of Shah (*loc.cit.*) for finitely generated torsion free modules $E$ over a Noetherian integral domain $R$, contained in a free $R$-module $F$. Suppose that $F/E$ has finite length as an $R$-module, then Liu defined the coefficients modules of $E$ in $F$ and also proved their existence and uniqueness when $R$ is formally equidimensional. Thus, she partially extended the work of Shah (*loc.cit.*) but she didn't give the colon structure of this modules which in the ideal case is one of the main results in the work of Shah (*loc.cit.*).

In [3], they extend the aforementioned results about the existence of Shah and Liu for standard graded algebras and as consequence is proved that coefficient modules exist, for $R$-submodules $E$ of $R^p$ over a Noetherian local ring $R$. In this work is extended the result of Herzog-Puthenpurakal-Verma (*loc.cit.*) and showed that coefficient modules exists for standard graded algebras and for $R$-submodules $E$ of $R^p$ over a Noetherian local ring $R$.

# 2 Coefficient modules for graded algebras

Let $(R, \mathfrak{m})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$, $G = \oplus_{n\geq 0} G_n$ a standard graded algebra with $D := \dim(\mathrm{Proj}(G))$ and $I$ a $R$-submodule of $G_1$, Then Kleiman-Thorup proved in [5, Theorem 5.7] that the length function $\ell_R(G_n/I^n)$ is a polynomial $P_{(I,G)}(n)$ of degree, at most, $D$, for all large $n$. This polynomial is called Buchsbaum-Rim polynomial of $(I, G)$. We write $P_{(I,G)}(n)$ in the form:

$$P_G(I, n) = \sum_{0\leq i\leq D} (-1)^i \mathrm{e}_i(I, G) \begin{pmatrix} n + D - 1 - i \\ D - i \end{pmatrix} \tag{1}$$

call $\mathrm{e}_i(I, G)$ the $i$-th **Buchsbaum-Rim coefficient** of $(I, G)$, $i = 0, \ldots, D$, and the integer $\mathrm{e}_0(I, G)$ is the Buchsbaum-Rim multiplicity of $(I, G)$. This multiplicity was first introduced and studied by Buchsbaum-Rim in [2] but in this context it was first studied in [7] and [5].

We say that an $R$-submodule $J$ of $G_1$ ia a reduction of $I$ if there exist an integer $n$ such that $JI^n = I^{n+1}$. In this way, the *integral closure* of $I$ in $G_1$ is the largest submodule $\overline{I}$ of $G_1$ having $I$ as a reduction (see [10, page 416]). Assume that the local ring $(R, \mathfrak{m})$ is formally equidimensional, then by a result of [5, Proposition 6.3] and [7, Theorem 4.11] the integral closure $\overline{I}$ of $I$ in $G_1$ is the largest submodule of $G_1$ which contains $I$ and have the same Buchsbaum-Rim multiplicity as $I$. The integral closure $\overline{I}$ of $I$ is well known to be an intersection of certain valuation modules (see [10, page 418, Remark 8.9]). In [3] prove the existence of a unique chain of modules between $I$ and $\overline{I}$, called the coefficient modules of $I$ and we find their colon structure. An important result was proved by Kleiman-Thorup in [5] and states that considering an $R$-algebra $G = \oplus_{n\geq 0} G_n$ formally equidimensional with $D := \dim(\mathrm{Proj}(G))$, $I \subseteq K$ two $R$-submodules of $G_1$. Then $\deg\left(P_G(\frac{K}{I}, n)\right) < D$ if and only if $I$ is a reduction of $K$. This result generalize the Amao's theorem for ideals and for modules, see [1].

# 3 Existence of coefficient modules

Let $\mathcal{M} = \mathfrak{m} \oplus G_+$ the maximal ideal of $G$, where $G_+ = \oplus_{n\geq 1} G_n$ is the maximal homogenous ideal of $G$. Let $I$ a $R$-submodule of $G_1$ and we denoted by $\mathcal{I}$ the ideal of $G$ generated by the $R$-submodule $I$. If $K$ is another $R$-submodule of $G_1$ we say that $I$ is a reduction of $K$ if $\mathcal{I}$ is a reduction of $\mathcal{K}$, where $\mathcal{K}$ is the homogenous ideal generated by $K$. Observe that $\mathcal{I}$ is an homogenous ideal and we define the integral closure of $I$ as the linear part of the integral closure of $\mathcal{I}$ as an ideal of $G$, i.e., $\overline{I} := [\overline{\mathcal{I}}]_1$.

If $R$ is a local ring with maximal ideal $\mathfrak{m}$, we define the maximal ideal of $G$ as $\mathcal{M} = \mathfrak{m} \bigoplus G_+$, where $G_+ = \oplus_{n>0} G_n$.

The fiber cone of $I$ and the fiber cone of $\mathcal{I}$ are, respectively,

$$\mathcal{F}(I) := \bigoplus_{n\geq 0} \frac{\mathcal{R}_n(I)}{\mathfrak{m}\mathcal{R}_n(I)} \text{ and } \mathcal{F}(\mathcal{I}) := \bigoplus_{n\geq 0} \frac{\mathcal{R}_n(\mathcal{I})}{\mathcal{M}\mathcal{R}_n(\mathcal{I})}.$$

The analytic spread of $I$, $s(I)$, and the analytic spread of $\mathcal{I}$, $s(\mathcal{I})$, are defined by the Krull dimension of the quotient rings given below:

$$s(I) := \dim(\mathcal{F}(I)) \text{ and } s(\mathcal{I}) := \dim(\mathcal{F}(\mathcal{I})).$$

In [4], they defined the saturation and the relative integral closure associated with an ideal $\mathfrak{a}$ of a Noetherian local ring $(R, \mathfrak{m})$ in the following way:

$$\mathfrak{a}^{\text{sat}} = \bigcup_{n\geq 1} (\mathfrak{a} : \mathfrak{m}^n) = \{x \in R \mid x\mathfrak{m}^n \subseteq \mathfrak{a}, \text{ for some } n \geq 0\}$$

is the the saturation of $\mathfrak{a}$ and the relative integral closure of $\mathfrak{a}$ is defined by

$$q(\mathfrak{a}) = \overline{\mathfrak{a}} \cap \mathfrak{a}^{\text{sat}}.$$

Note that as $R$ is a Noetherian ring the saturation and the relative integral closure of an ideal $\mathfrak{a}$ are ideals which contains $\mathfrak{a}$.

**Definition 3.1 (Relative integral closure of a module).** *Let $I$ be a $R$-submodule of $G_1$. We defined the saturation of $I$ by $I^{sat} = [\mathcal{I}^{sat}]_1$ where*

$$\mathcal{I}^{sat} = \bigcup_{n\geq 1} (\mathcal{I} : \mathcal{M}^n) = \{x \in G \mid x\mathcal{M}^n \subseteq \mathcal{I}, \textit{ for some } n \geq 0\}$$

*and the relative integral closure of $I$ by $q(I) = \overline{I} \cap I^{sat}$.*

In this case, we have

$$I^{\text{sat}} = \left[\bigcup_{n\geq 1} (\mathcal{I} : \mathcal{M}^n)\right]_1 = [\{x \in G \mid x\mathcal{M}^n \subseteq \mathcal{I}, \text{ for some } n \geq 0\}]_1$$

and

$$q(I) = \overline{I} \cap I^{\text{sat}} = [\overline{\mathcal{I}}]_1 \cap [\mathcal{I}^{\text{sat}}]_1 = [\overline{\mathcal{I}} \cap \mathcal{I}^{\text{sat}}]_1.$$

We can easily see that $q(I) = [q(\mathcal{I})]_1$.

**Proposition 3.2.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $G = \oplus_{n\geq 0} G_n$ a standard graded algebra and $I$ a $R$-submodule of $G_1$. Then*

1. *$I$ is a reduction of $q(I)$ and $\ell_R\left(\frac{q(I)}{I}\right) < \infty$.*

*2. If $I$ is a reduction of $K$ with $\ell_R\left(\frac{\mathcal{K}}{\mathcal{I}}\right) < \infty$ then $K \subseteq q(I)$.*

*3. If $J \subseteq I$ then $q(J) \subseteq q(I)$.*

*4. $q(q(I)) = q(I)$.*

*Proof.* We have just to remember that all this results are valid for ideals, see [4].

1. We know that $\mathcal{I}$ is a reduction of $q(\mathcal{I})$, then $[\mathcal{I}]_1$ is a reduction of $[q(\mathcal{I})]_1$, but we know that $[q(\mathcal{I})]_1 = q(I)$.

2. If $I$ is a reduction of $J$ then $\mathcal{I}$ is a reduction of $\mathcal{J}$, by [4, Proposition 4.3, pg. 635] since $\ell_R\left(\frac{\mathcal{J}}{\mathcal{I}}\right) < \infty$ we have $\mathcal{J} \subseteq q(\mathcal{I})$ thus $J = [\mathcal{J}]_1 \subseteq [q(\mathcal{I})]_1 = q(I)$.

3. If $K \subseteq I$ then $\mathcal{K} \subseteq \mathcal{I}$ and $q(\mathcal{K}) \subseteq q(\mathcal{I})$.

4. If $q(I)$ is a reduction of $K$ with $\ell_R\left(\frac{\mathcal{K}}{q(\mathcal{I})}\right) < \infty$ then note that $K$ is also a reduction of $I$, since $I$ is a reduction of $q(I)$ and also $\ell_R\left(\frac{\mathcal{K}}{\mathcal{I}}\right) < \infty$. By *2.* we have $K \subseteq q(I)$ thus $K = I$. Take $K = q(q(I))$ and we get the desired result.

□

We know that if $(R, \mathfrak{m})$ is a Noetherian local ring, $M = \oplus_{n\geq 0} M_n$ is a graded $R$-module finitely generated and $\ell_R(M_n) < \infty$, for all $n \geq 0$, there exists an $r > 0$ such that $\mathfrak{m}^r M_n = 0$, for all $n > 0$. Thus $\mathfrak{m}^r \subseteq \operatorname{ann}(M)$.

**Theorem 3.3 (Theorem of existence for arbitrary modules).** *Let $(R, \mathfrak{m})$ be a local ring with infinite residual field $\frac{R}{\mathfrak{m}}$, $G = \oplus_{n\geq 0} G_n$ a formally equidimensional standard graded algebra and $I$ a $R$-submodule of $G_1$ with analytic spread $s := s(I)$. Then, for $k = 1, \ldots, s$, there exist unique largest $R$-submodules $I_{\{k\}}$ of $G_1$ with $\ell_R\left(\frac{I_{\{k\}}G}{\mathcal{I}}\right) < \infty$ and*

$$I \subseteq I_{\{s\}} \subseteq \ldots \subseteq I_{\{1\}} \subseteq I_{\{0\}} = q(I)$$

*such that $\deg\left(p_{\frac{I_{\{k\}}}{I}}\right) < s - k$, for $k = 1, \ldots, s$.*

*Proof.* Fix $0 \leq k \leq s$. Consider the set of homogenous $R$-submodules

$$V_{\{k\}} = \left\{ L \subseteq G_1 |\ I \subseteq L,\ \ell_R\left(\frac{LG}{\mathcal{I}}\right) < \infty,\ \deg\left(p_{\frac{L}{I}}(n)\right) < s - k \right\}$$

If $L \in V_{\{k\}}$, then $\deg\left(p_{\frac{L}{I}}\right) < D - 1$ and $\ell_R\left(\frac{LG}{\mathcal{I}}\right) < \infty$. Hence by Kleiman-Thorup's theorem [5, Proposition 6.3, pg. 202], $I$ is a reduction of $L$ and by item 2. of Proposition 3.2, $L \subseteq q(I)$. Since $I \in V_{\{k\}}$ and $G_1$ is a Noetherian $R$-module, $V_{\{k\}}$ has a maximal element, say $J$.

We show that $J$ is unique. Let $L \in V_{\{k\}}$ and $x \in L$. Since $I \subseteq (I, x) \subseteq L$,

$$\ell_R\left(\frac{(I,x)^n}{I^n}\right) < \ell_R\left(\frac{L^n}{I^n}\right) < \infty,$$

for all large $n$. Hence $\deg\left(p_{\frac{(I,x)}{I}}\right) < \deg\left(p_{\frac{L}{I}}\right) < s-k$. By Kleiman-Thorup's theorem [5, Proposition 6.3, pg. 202], $I$ is a reduction of $(I,x)$, i.e., $I(I,x)^r = (I,x)^{r+1}$ for some $r$. Then $x^{r+1} \in I(I,x)^{r+1} \subseteq J(J,x)^r$. Hence $J(J,x)^r = (J,x)^{r+1}$. It follows that, for all $n \geq r$, $J^{n-r}(J,x)^r = (J,x)^n$.

Since $\deg\left(p_{\frac{(I,x)}{I}}\right) < \infty$, we have $\ell_R\left(\frac{(I,x)}{I}\right) < \infty$. If we consider $M = \oplus_{n\geq 0}\frac{(J,x)^n}{I^n}$ there exist some $t > 0$ such that $(x, I)\,\mathfrak{m}^t \subseteq I \subseteq J$ then $x\mathfrak{m}^t \subseteq I \subseteq J$ and $I\mathfrak{m}^t \subseteq I \subseteq J$ hence $I\mathfrak{m}^t \subseteq I \subseteq (J,x)$ then $\ell_R\left(\frac{(J,x)}{I}\right) < \infty$.

We have:

$$\begin{aligned}
\ell_R\left(\frac{(J,x)^n}{I^n}\right) &= \ell_R\left(\frac{(J,x)^t J^{n-t}}{I^n}\right)\\
&\leq \sum_{i=1}^{t} \ell_R\left(\frac{J^{n-i}x^i + J^n}{I^n}\right)\\
&\leq \sum_{i=1}^{t}\left[\ell_R\left(\frac{J^{n-i}x^i + I^n}{J^n}\right) + \ell_R\left(\frac{J^n}{I^n}\right)\right]\\
&\leq \sum_{i=1}^{t}\left[\ell_R\left(\frac{J^{n-i}}{I^{n-i}}\right) + \ell_R\left(\frac{(J,x)^n}{I^n}\right) + \ell_R\left(\frac{J^n}{I^n}\right)\right].
\end{aligned}$$

Hence $\deg\left(p_{\frac{(J,x)}{I}}\right) < s-k$. Therefore $(J,x) \subseteq V_{\{k\}}$. By maximality of $J$, we conclude that $x \in J$. Hence $L \subseteq J$ and therefore $J$ is unique and will be denoted by $I_{\{k\}}$. □

**Definition 3.4.** *In the same conditions of Theorem 3.3, for $k = 0, \ldots, s$, we defined the $R$-submodule $I_{\{k\}}$ of $G_1$ called the $k$-th coefficient module of $I$ in $G$.*

Observe that in Theorem 3.3 we obtain $\ell_R\left(\frac{I_{\{k\}}G}{\mathcal{I}}\right) < \infty$, for $k = 0, \ldots, s$, then we also get $\ell_R\left(\frac{I_{\{k\}}}{I}\right) < \infty$, for $k = 0, \ldots, s$.

In the next results we obtain the relation between the coefficient modules and a boundary for the degree of the Rees polynomial. The proof is similiar to [3].

**Corollary 3.5.** *In the same conditions of Theorem 3.3, let $I \subseteq K$ be two $R$-submodules of $G_1$ with $\ell_R\left(\frac{K}{I}\right) < \infty$, then we have*

$$I \subseteq K \subseteq I_{\{k\}} \subseteq \overline{I} \Longleftrightarrow I \subseteq K \text{ and } deg\left(p_{\frac{K}{I}}\right) < s - k.$$

**Corollary 3.6.** *In the same conditions of Theorem 3.3, let $I \subseteq K$ be two $R$-submodules of $G_1$ with $\ell_R\left(\frac{K}{I}\right) < \infty$, then we have*

$$deg\left(p_{\frac{K}{I}}\right) = s - k \Longleftrightarrow K \subseteq I_{\{k-1\}} \text{ and } K \nsubseteq I_{\{k\}}$$

*for $k = 1, \dots, s$.*

**Corollary 3.7.** *In the same conditions of Theorem 3.3, we have*

$$I_{\{k\}} = \left(I_{\{k\}}\right)_{\{k\}}, \text{ for } k = 0, \dots, s.$$

# 4 Application

In this section we give some application considering a pair of modules $E \subseteq F \subseteq R^p$. Let $(R, \mathfrak{m})$ be a Noetherian local ring of Krull dimension $d$ and $E$ be $R$-submodule of the free $R$-module $R^p$, the symmetric algebra $G := \mathrm{Sym}(R^p) = \oplus_n S_n(R^p)$ of $R^p$ is a polynomial ring $R[T_1, \dots, T_p]$. If $h = (h_1, \dots, h_p) \in R^p$ we define the element $w(h) = h_1T_1 + \ldots + h_pT_p \in S_1(R^p) =: G_1$. We denote by $\mathcal{R}(E) := \oplus_{n\geq 0}\mathcal{R}_n(E)$ the subalgebra of $G$ generated in degree one by $\{w(h) : h \in E\}$ and call it the Rees algebra of $E$. We know that $\mathcal{R}(E)$ has dimension $d + p$.

If $E \subseteq F$ are two $R$-submodules finitely generated of $R^p$ then we obtain two $R$-subalgebras, $\mathcal{R}(E)$ and $\mathcal{R}(F)$ such that

$$\mathcal{R}(E) \subseteq \mathcal{R}(F) \subseteq R[T_1, \dots, T_p].$$

We say that $E$ is a reduction of $F$ if exists a $n_0 > 0$ such that

$$\mathcal{R}_{n+1}(F) = \mathcal{R}_1(E)\mathcal{R}_n(F),$$

for all $n \geq n_0$. If $E$ is the smallest $R$-submodule that satisfies this condition of reduction of $F$ then we say that $E$ is a minimal reduction of $F$.

The fiber cone of a $R$-module $E$ in $R^p$ is defined by

$$\mathcal{F}(E) := \bigoplus_{n\geq 0} \frac{\mathcal{R}_n(E)}{\mathfrak{m}\mathcal{R}_n(E)}.$$

The analytic spread of $E$ is defined by $s = s(E) := \dim(\mathcal{F}(E))$.

We consider $G = \mathcal{R}(F) = \oplus_{n\geq 0}\mathcal{R}_n(F)$ a graded $R$-subalgebra of $S(R^p)$ and $I = \mathcal{R}_1(E)$ a $R$-submodule of $\mathcal{R}_1(F)$ we define the function

$$\begin{array}{rccc} h_{\frac{F}{E}}: & \mathbb{N} & \longrightarrow & \mathbb{N} \\ & n & \longmapsto & \ell_R\left(\frac{\mathcal{R}_n(F)}{\mathcal{R}_n(E)}\right). \end{array}$$

this function $h_{\frac{F}{E}}$ becomes a polynomial $p_{\frac{E}{F}}(n)$ in $n$ of degree at most $d+p-1$, for all large $n$ called generalized Buchsbaum-Rim polynomial of the pair $(E, F)$.

By definition we obtain

$$q(\mathcal{R}_1(E)) = \overline{\mathcal{R}_1(E)} \cap [\mathcal{R}_1(E)]^{\text{sat}},$$

then similarly we define the relative integral closure of $E$ as a $R$-submodule of $R^p$ in the following way:

$$q(E) = \overline{E} \cap E^{\text{sat}}$$

where $E^{\text{sat}} = \{x \in F; w(x) \in [\mathcal{R}_1(E)]^{\text{sat}}\}$.

In the context above, when we consider $G = \mathcal{R}(F)$, $I = \mathcal{R}_1(E)$ we have the same conditions of graded algebras. Then we achieved as application the same results in case of a pair of modules $E \subseteq F \subseteq R^p$.

**Theorem 4.1.** *Let $(R, \mathfrak{m})$ be a $d$-dimensional formally equidimensional local Noetherian ring with infinite residual field $\frac{R}{\mathfrak{m}}$ and $E \subseteq F$ be $R$-submodules finitely generated of the free $R$-module $R^p$. Then, for $k = 1, \dots, s$, there exists unique largest $R$-submodules $E_{\{k\}}$ of $F$ with $\ell_R\left(\frac{E_{\{k\}}}{E}\right) < \infty$ and*

$$E \subseteq E_{\{s\}} \subseteq \ldots \subseteq E_{\{1\}} \subseteq E_{\{0\}} = q(E)$$

*such that deg $\left(p_{\frac{E_{\{k\}}}{E}}\right) < s - k$, for $k = 1, \dots, s$.*

The $R$-submodule $E_{\{k\}}$ of $F$ is called $k$-th coefficient module of the pair $(E, F)$, for $k = 0, \dots, s$.

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**Received: August, 2011**