Semi Hopfian and Semi Co-Hopfian Modules over Generalized Power Series Rings

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Abstract

Let \((S, \leq)\) be a strictly totally ordered monoid, \(R\) be a commutative ring and \(M\) be an \(R\)-module. We show the following results: (1) If \((S, \leq)\) satisfies the condition that \(0 \leq s\) for all \(s \in S\), then the module \([M_S, \leq]\) of generalized power series is a semi Hopfian \([R_S, \leq]\)-module if and only if \(M\) is a semi Hopfian \(R\)-module; (2) If \((S, \leq)\) is artinian, then the generalized inverse polynomial module \([M_S, \leq]\) is a semi co-Hopfian \([R_S, \leq]\)-module if and only if \(M\) is a semi co-Hopfian \(R\)-module.

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1 Introduction

Let \(R\) be a ring and \((S, \leq)\) be a strictly totally ordered monoid. Assume that \([R_S, \leq]\) is the ring of generalized power series with coefficients in \(R\) and exponents in \(S\). The generalized inverse polynomial modules \([M^S, \leq]\) and the generalized power series modules \([M^S, \leq]\) are two important classes of modules over \([R_S, \leq]\), which have been studied by many authors, see for example [3, 4, 5, 6, 7, 8, 9, 16, 17]. The motivation for the paper comes from the following results. Recall that a module \(R\) is called Hopfian (resp. co-Hopfian) if any surjective (resp. injective) endomorphism of \(R\) is an isomorphism. Note that any noetherian module is Hopfian, and any artinian module is co-Hopfian. Let \(R\) be an associative ring not necessarily containing an identity element, it was shown in[16, Theorem 4.6] that if \(M\) is a noetherian left \(R\)-module possessing property (F), \((S, \leq)\) is narrow, \(S\) is cancellative, torsion-free and
$S = \langle s_1, \ldots, s_n \rangle + G(S)$ for some finite set $\{s_1, \ldots, s_n\}$ of elements in $S \setminus G(S)$, then $[[M^{S, \leq}]]$ is a noetherian left $[[R^{S, \leq}]]$-module, and that for a nonzero $R$-module $M$ possessing property (F), if $[[M^{S, \leq}]]$ is noetherian, then so is $M$; if $S$ is cancellative then there exist a finite number $s_1, \ldots, s_n$ of elements in $S \setminus G(S)$ satisfying $S = \langle s_1, \ldots, s_n \rangle + G(S)$; if $0 \leq s$ for all $s \in S$, then $(s, \leq)$ is narrow ([16, Proposition 2.1, Theorem 2.2]), where $G(S)$ denotes the largest subgroup of $S$ and $\langle s_1, \ldots, s_n \rangle$ the submonoid of $S$ generated by $s_1, \ldots, s_n$. It was shown in [3, Theorem 4] that if $(S, \leq)$ is a strictly totally ordered monoid which is finitely generated and satisfies the condition that $0 \leq s$ for any $s \in S$, then $[[M^{S, \leq}]]$ is a Hopfian left $[[R^{S, \leq}]]$-module if and only if $M$ is a Hopfian left $R$-module. Let $R$ be an associated ring not necessarily with identity, $M$ a left $R$-module having the property (F), and $(S, \leq)$ a strictly totally ordered monoid which is also artinian, it was proven in [17, Theorem 4] that if $S$ is a finitely generated monoid then $[M^{S, \leq}]$ is an artinian left $[[R^{S, \leq}]]$-module if and only if $M$ is an artinian left $R$-module, and in [8, Theorem 2.2], it was shown that $[M^{S, \leq}]$ is a co-Hopfian left $[[R^{S, \leq}]]$-module if and only if $M$ is a co-Hopfian left $R$-module. As a generalization of the notions Hopficity and co-Hopficity, semi Hopfian modules and semi co-Hopfian modules were introduced in [1] and a new characterization of artinian rings was obtained by using these concepts. Let $R$ be a commutative ring and $M$ an $R$-module, following [1], $M$ is said to be semi Hopfian (resp. semi co-Hopfian) if for any $r \in R$, the endomorphism of $M$ induced by multiplication by $r$ is an isomorphism, provided it is surjective (resp. injective). Clearly, over a commutative ring $R$, any Hopfian (resp. co-Hopfian) $R$-module is semi-Hopfian (resp. semi co-Hopfian). In [2], it was shown that $M$ is semi Hopfian $R$-module if and only if $M[1, \ldots, X_n]$ is semi Hopfian $R[1, \ldots, X_n]$-module if and only if $M[[X_1, \ldots, X_n]]$ is semi Hopfian $R[[X_1, \ldots, X_n]]$-module (Theorem 3.4), and that $M$ is semi co-Hopfian $R$-module if and only if $M[X_1^{-1}, \ldots, X_n^{-1}]$ is semi co-Hopfian $R[X_1^{-1}, \ldots, X_n^{-1}]$-module if and only if $M[1, \ldots, X_n^{-1}]$ is semi co-Hopfian $R[[X_1, \ldots, X_n]]$-module (Theorem 3.5), where $X_1, \ldots, X_n$ are $n$ commuting indeterminates over $R$. Motivated by these facts, in this paper, the semi Hopficity of generalized power series modules and the semi co-Hopficity of generalized inverse polynomial modules will be investigated, respectively. To be precise, let $(S, \leq)$ be a strictly totally ordered monoid, $R$ a commutative ring with identity and $M$ an $R$-module, we will prove that if $(S, \leq)$ satisfies the condition that $0 \leq s$ for all $s \in S$, then the module $[[M^{S, \leq}]]$ of generalized power series is a semi Hopfian $[[R^{S, \leq}]]$-module if and only if $M$ is a semi Hopfian $R$-module, and that if $(S, \leq)$ is artinian, then the generalized inverse polynomial module $[M^{S, \leq}]$ is a semi co-Hopfian $[[R^{S, \leq}]]$-module if and only if $M$ is a semi co-Hopfian $R$-module.
2 Definitions and Notations

In this Section, we recall some definitions and notations, any concept and notation not defined here can be found in [3, 7, 14, 15, 16].

Let \((S, \leq)\) be an ordered set. Recall that \((S, \leq)\) is artinian if every strictly decreasing sequence of elements of \(S\) is finite, and that \((S, \leq)\) is narrow if every subset of pairwise order-incomparable elements of \(S\) is finite. Let \(S\) be a commutative monoid. Unless stated otherwise, the operation of \(S\) shall be denoted additively, and the neutral element by 0. The following definition is due to P. Ribenboim [14, 15].

**Definition 2.1** Let \((S, \leq)\) be a strictly ordered monoid (that is, \((S, \leq)\) is an ordered monoid satisfying the condition that, if \(s, s', t \in S\) and \(s < s'\), then \(s + t < s' + t\), and \(R\) a ring. Let \([[R^{S, \leq}]\]) be the set of all maps \(f : S \rightarrow R\) such that \(\text{supp}(f) = \{s \in S \mid f(s) \neq 0\}\) is artinian and narrow. For every \(s \in S\) and \(f, g \in [[R^{S, \leq}]\], let \(X_s(f, g) = \{(u, v) \in S \times S \mid u + v = s, f(u) \neq 0, g(v) \neq 0\}\). It follows from [14, 4.1] that \(X_s(f, g)\) is finite. This fact allows to define the operation of convolution:

\[(fg)(s) = \sum_{(u,v) \in X_s(f,g)} f(u)g(v).\]

With this operation, and pointwise addition, \([[R^{S, \leq}]\]) becomes a ring, which is called the ring of generalized power series with coefficients in \(R\) and exponents in \(S\).

Examples and basic properties of rings of generalized power series are given in [14, 15].

**Definition 2.2** Let \(M\) be a left \(R\)-module and \((S, \leq)\) a strictly ordered monoid. We denote by \([[M^{S, \leq}]\]) the set of all maps \(\phi : S \rightarrow M\) such that \(\text{supp}(\phi)\) is artinian and narrow, where \(\text{supp}(\phi) = \{s \in S \mid \phi(s) \neq 0\}\). For each \(s \in S, f \in [[R^{S, \leq}]\] and \(\phi \in [[M^{S, \leq}]\], let \(X_s(f, \phi) = \{(u, v) \in S \times S \mid u + v = s, f(u) \neq 0, \phi(v) \neq 0\}\). Then by [3, Lemma 1], \(X_s(f, \phi)\) is finite. This allows to define the scalar multiplication as follows:

\[(f\phi)(s) = \sum_{(u,v) \in X_s(f,\phi)} f(u)\phi(v).\]

With this operation and pointwise addition, \([[M^{S, \leq}]\]) becomes a left \([[R^{S, \leq}]\])-module, which is called the module of generalized power series with coefficients in \(M\) and exponents in \(S\).

Similarly, if \(M\) is a right \(R\)-module, then \([[M^{S, \leq}]\]) is a right \([[R^{S, \leq}]\])-module. Examples and some results of modules of generalized power series are given in [3, 4, 6, 9, 16].
Definition 2.3 Let $M$ be a left $R$-module, we let $[M^{S,\leq}]$ be the set of all maps $\phi : S \to M$ such that the set $\text{supp}(\phi) = \{s \in S | \phi(s) \neq 0\}$ is finite. Now $[M^{S,\leq}]$ can be turned into a left $[[R^{S,\leq}]]$-module under some additional conditions. The addition in $[M^{S,\leq}]$ is componentwise and the scalar multiplication is defined as follows:

$$(f\phi)(s) = \sum_{t \in S} f(t)\phi(s + t), \quad \text{for every } s \in S,$$

where $f \in [[R^{S,\leq}]]$, and $\phi \in [M^{S,\leq}]$. Since the set $\text{supp}(\phi)$ is finite, this multiplication is well-defined. If $(S, \leq)$ is a strictly totally ordered monoid which is also artinian, then, from [7], $[M^{S,\leq}]$ becomes a left $[[R^{S,\leq}]]$-module, which we call the generalized inverse polynomial module.

Similarly, if $M$ is a right $R$-module, then $[M^{S,\leq}]$ is a right $[[R^{S,\leq}]]$-module. For example, if $S = \mathbb{N} \cup \{0\}$ and $\leq$ is the usual order, then $[M^{\mathbb{N} \cup \{0\},\leq}] \cong M[x^{-1}]$, the usual left $R[[x]]$-module introduced in [10] and [11], which is called the Macaulay-Northcott module in [12] and [13]. Further examples of modules of generalized inverse polynomials are given in [7].

Next, we explain some notations and facts involved. To any $r \in R$, we define $c_r \in [[R^{S,\leq}]]$ by

$$c_r(x) = \begin{cases} r, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0. \end{cases}$$

To any $m \in M$ and any $s \in S$, we associate the map $d^s_m \in [[M^{S,\leq}]]$ via

$$d^s_m(x) = \begin{cases} m, & \text{if } x = s, \\ 0, & \text{if } x \neq s. \end{cases}$$

For any $m \in M$ and any $s \in S$, we define $\phi_{s,m} \in [M^{S,\leq}]$ via

$$\phi_{s,m}(x) = \begin{cases} m, & \text{if } x = s, \\ 0, & \text{if } x \neq s. \end{cases}$$

Throughout this paper, $R$ denotes a commutative ring with identity, all modules are unitary and $(S, \leq)$ is a strictly totally ordered monoid. In this situation, for every $0 \neq \phi \in [[M^{S,\leq}]]$ (resp. $0 \neq f \in [[R^{S,\leq}]]$), $\text{supp}(\phi)$ (resp. $\text{supp}(f)$) has a minimal element, we denote it by $\pi(\phi)$ (resp. $\pi(f)$). If $(S, \leq)$ satisfies the condition that $0 \leq s$ for all $s \in S$, then $(f\phi)(0) = f(0)\phi(0)$ for any $\phi \in [[M^{S,\leq}]]$ and any $f \in [[R^{S,\leq}]]$. If $(S, \leq)$ is also artinian, then by [7], $0 \leq s$ for any $s \in S$, and for any $0 \neq \varphi \in [M^{S,\leq}]$, $\text{supp}(\varphi)$ has a maximal element, we denote by $\sigma(\varphi)$. 
3 Main results and proofs

Definition 3.1 Let $R$ be a commutative ring. An $R$-module $M$ is said to be semi Hopfian (resp. semi co-Hopfian) if for any $r \in R$, the endomorphism of $M$ induced by multiplication by $r$ is an isomorphism, provided it is surjective (resp. injective).

In [2, Theorem 3.4], it was shown that $M$ is semi Hopfian $R$-module if and only if $M[X_1, \ldots, X_n]$ is semi Hopfian $R[X_1, \ldots, X_n]$-module if and only if $M[[X_1, \ldots, X_n]]$ is semi Hopfian $R[[X_1, \ldots, X_n]]$-module. Here we have:

Theorem 3.2 Let $(S, \leq)$ be a strictly totally ordered monoid and satisfies the condition that $0 \leq s$ for any $s \in S$, $R$ a commutative ring and $M$ an $R$-module. Then $[[M^{S, \leq}]]$ is a semi Hopfian $[[R^{S, \leq}]]$-module if and only if $M$ is a semi Hopfian $R$-module.

Proof $\Rightarrow$ Let $\alpha : M \rightarrow M$ defined by $\alpha(m) = rm$ for some fixed $r \in R$ be a surjective $R$-homomorphism. Define $\beta : [[M^{S, \leq}]] \rightarrow [[M^{S, \leq}]]$ via $\beta(\varphi) = c_r \varphi$ for any $\varphi \in [[M^{S, \leq}]]$. Then it is easy to see that $\beta$ is an $[[R^{S, \leq}]]$-homomorphism. For any $\psi \in [[M^{S, \leq}]]$ and any $s \in S$, there exists an element $m_s \in M$ such that $\alpha(m_s) = \psi(s)$ since $\alpha$ is surjective. Define $\varphi : S \rightarrow M$ via

$$\varphi(s) = \begin{cases} m_s, & \text{if } s \in \text{supp}(\psi), \\ 0, & \text{if } s \notin \text{supp}(\psi). \end{cases}$$

Clearly, supp($\varphi$)=supp($\psi$), and so $\varphi \in [[M^{S, \leq}]]$. For any $s \in S$,

$$\beta(\varphi)(s) = (c_r \varphi)(s) = r \varphi(s) = \begin{cases} rm_s, & s \in \text{supp}(\psi), \\ 0, & s \notin \text{supp}(\psi), \end{cases} = \psi(s).$$

This means that $\beta(\varphi) = \psi$, and thus $\beta$ is a surjective $[[R^{S, \leq}]]$-homomorphism. Hence $\beta$ is an isomorphism since $[[M^{S, \leq}]]$ is a semi Hopfian $[[R^{S, \leq}]]$-module.

Let $m \in M$ be such that $rm = 0$. Then $\beta(d_m^0) = c_r d_m^0 = 0$. Thus $d_m^0 = 0$ and so $m = 0$ since $\beta$ is an isomorphism. This means that $\alpha$ is an isomorphism.

$\Leftarrow$ Let $\alpha : [[M^{S, \leq}]] \rightarrow [[M^{S, \leq}]]$ defined by $\alpha(\varphi) = f \varphi$ for some fixed $f \in [[R^{S, \leq}]]$ be a surjective $[[R^{S, \leq}]]$-homomorphism. Define $\beta : M \rightarrow M$ via $\beta(m) = f(0)m$ for any $m \in M$. Then it is easy to see that $\beta$ is an $R$-homomorphism. For any $m \in M$, there exists an element $\varphi \in [[M^{S, \leq}]]$ such that $\alpha(\varphi) = d_m^0$ since $\alpha$ is surjective. Then $m = d_m^0(0) = \alpha(\varphi)(0) = (f \varphi)(0) = f(0) \varphi(0) = \beta(\varphi(0))$ since $0 \leq s$ for any $s \in S$. This implies that $\beta$ is a surjective $R$-homomorphism, which must be an isomorphism since $M$ is semi-Hopfian.

Let $\varphi \in [[M^{S, \leq}]]$ be such that $f \varphi = \alpha(\varphi) = 0$. Then $\beta(\varphi(0)) = f(0) \varphi(0) = (f \varphi)(0) = 0$, and so $\varphi(0) = 0$ since $\beta$ is an isomorphism. Now, suppose that
u ∈ S and for any v ∈ S with v < u, φ(v) = 0. We will show that φ(u) = 0. Note that \( π(φ - d^u_{φ(u)}) > u \). So \( π(f(φ - d^u_{φ(u)})) \geq π(f + π(φ - d^u_{φ(u)}) > π(f) + u \geq u \). Thus

\[
\beta(φ(u)) = f(0)φ(u) = (fd^u_{φ(u)})(u) = (f - d^u_{φ(u)})(u) + (f(φ - d^u_{φ(u)}))(u) = (fφ)(u) = α(φ)(u) = 0.
\]

Hence \( φ(u) = 0 \) since \( β \) is an isomorphism. Therefore \( φ = 0 \) and thus \( α \) is an isomorphism.

\[\square\]

**Corollary 3.3** Let \((S, \leq)\) be a strictly totally ordered monoid and satisfies the condition that \(0 \leq s\) for any \(s \in S\), \(R\) be a commutative ring and \(M, N\) be \(R\)-modules. Then the following are equivalent:

1. Any \([R^{S,≤}]\)-epimorphism \([M^{S,≤}] \rightarrow [N^{S,≤}]\) induced by multiplication by some \(f \in [R^{S,≤}]\) is an isomorphism.
2. Any \(R\)-epimorphism \(M \rightarrow N\) induced by multiplication by some \(r \in R\) is an isomorphism.

**Proof** It is similar to the proof of Theorem 3.2.

\[\square\]

In [2, Theorem 3.5], it was shown that \(M\) is semi co-Hopfian \(R\)-module if and only if \(M[X_1^{-1}, \ldots, X_n^{-1}]\) is semi co-Hopfian \(R[X_1, \ldots, X_n]\)-module if and only if \(M[X_1^{-1}, \ldots, X_n^{-1}]\) is semi co-Hopfian \(R[[X_1, \ldots, X_n]]\)-module. Here we have:

**Theorem 3.4** Let \((S, ≤)\) be a strictly totally ordered monoid which is also artinian, \(R\) a commutative ring and \(M\) an \(R\)-module. Then \([M^{S,≤}]\) is a semi co-Hopfian \([R^{S,≤}]\)-module if and only if \(M\) is a semi co-Hopfian \(R\)-module.

**Proof** \(\implies\) Let \(α : M \rightarrow M\) defined by \(α(m) = rm\) for some fixed \(r \in R\) be an injective \(R\)-homomorphism. Define \(β : [M^{S,≤}] \rightarrow [M^{S,≤}]\) via \(β(φ) = crφ\) for any \(φ \in [M^{S,≤}]\). Then it is easy to see that \(β\) is an \([R^{S,≤}]\)-homomorphism. If \(β(φ) = 0\) where \(φ \in [M^{S,≤}]\). Then, for any \(s \in S\),

\[
0 = β(φ)(s) = (c_rφ)(s) = \sum_{x \in S} c_r(x)φ(x + s) = rφ(s) = α(φ(s)).
\]

Thus \(φ(s) = 0\) since \(α\) is injective. Hence \(φ = 0\). This means that \(β\) is injective which must be an isomorphism since \([M^{S,≤}]\) is semi co-Hopfian. Now, for any \(m \in M\), there exists a \(φ \in [M^{S,≤}]\) such that \(c_rφ = β(φ) = φ_{0,m}\). Thus

\[
m = φ_{0,m}(0) = (c_rφ)(0) = \sum_{x \in S} c_r(x)φ(x) = rφ(0) = α(φ(0)).
\]

This means that \(α\) is surjective.
\( \leftarrow \) Let \( \alpha : [M^{S, \leq}] \rightarrow [M^{S, \leq}] \) defined by \( \alpha(\varphi) = f \varphi \) for some fixed \( f \in \llbracket [R^{S, \leq}] \rrbracket \) be an injective \( \llbracket [R^{S, \leq}] \rrbracket \)-homomorphism. Define \( \beta : M \rightarrow M \)

via \( \beta(m) = f(0)m \) for any \( m \in M \). Then it is easy to see that \( \beta \) is an \( R \)-homomorphism. If \( \beta(m) = 0 \), then for any \( s \in S \),

\[
\alpha(\phi_{0, m})(s) = (f \phi_{0, m})(s) = \sum_{x \in S} f(x) \phi_{0, m}(x + s) = \begin{cases} f(0)m = \beta(m), & s = 0, \\ 0, & s > 0, \end{cases}
\]

which implies that \( \alpha(\phi_{0, m}) = 0 \), and so \( \phi_{0, m} = 0 \) since \( \alpha \) is injective. Thus \( m = 0 \). This means that \( \beta \) is injective which must be an isomorphism since \( M \) is semi co-Hopfian. Now we show that \( \alpha \) is surjective.

Let \( \varphi \in [M^{S, \leq}] \) with \( \sigma(\varphi) = s \). If \( s = 0 \), set \( m = \beta^{-1}(\varphi(0)) \). Then, for any \( t \in S \),

\[
\alpha(\phi_{0, m})(t) = \begin{cases} f(0)m = \beta(m) = \beta(\beta^{-1}(\varphi(0))) = \varphi(0), & t = 0, \\ 0, & t > 0, \end{cases} = \varphi(t),
\]

which means that \( \alpha(\phi_{0, m}) = \varphi \).

Now, suppose that \( 0 < s \). Assume that for any \( \psi \in [M^{S, \leq}] \) with \( \sigma(\psi) < s \), there exists \( \psi' \in [M^{S, \leq}] \) such that \( \alpha(\psi') = \psi \). Since \( 0 \neq \varphi(s) \in M \), there exists an \( m \in M \) such that \( \varphi(s) = \beta(m) = f(0)m \). For any \( s \leq t \in S \), from

\[
\alpha(\phi_{s, m})(t) = (f \phi_{s, m})(t) = \sum_{x \in S} f(x) \phi_{s, m}(x + t) = \begin{cases} f(0)m = \varphi(s), & t = s, \\ 0, & s < t, \end{cases} = \varphi(t)
\]

it follows that \( \sigma(\varphi - \alpha(\phi_{s, m})) < s \). By the hypothesis, there exists \( \varphi' \in [M^{S, \leq}] \) such that \( \varphi - \alpha(\phi_{s, m}) = \alpha(\varphi') \). Thus \( \varphi = \alpha(\varphi' + \phi_{s, m}) \).

Therefore, by the transfinite induction, we have shown that \( \alpha \) is surjective.

\( \square \)

**Corollary 3.5** Let \((S, \leq)\) be a strictly totally ordered monoid which is also artinian, \( R \) be a commutative ring and \( M, N \) be \( R \)-modules. Then the following are equivalent:

1. Any \( \llbracket [R^{S, \leq}] \rrbracket \)-monomorphism \( [M^{S, \leq}] \rightarrow [N^{S, \leq}] \) induced by multiplication by some \( f \in \llbracket [R^{S, \leq}] \rrbracket \) is an isomorphism.

2. Any \( R \)-monomorphism \( M \rightarrow N \) induced by multiplication by some \( r \in R \) is an isomorphism.

**Proof** It is similar to the proof of Theorem 3.4. \( \square \)

## 4 Corollaries

**Corollary 4.1** Let \( S \) be a torsion-free and cancellative monoid, \((S, \leq)\) be artinian and narrow, \( R \) a commutative ring and \( M \) an \( R \)-module. Then
(1) \([M^{S,\leq}]\) is a semi Hopfian \([R^{S,\leq}]\)-module if and only if \(M\) is a semi Hopfian \(R\)-module.
(2) \([M^{S,\leq}]\) is a semi co-Hopfian \([R^{S,\leq}]\)-module if and only if \(M\) is a semi co-Hopfian \(R\)-module.

**Proof** If \((S, \leq)\) is torsion-free and cancellative, then by [14, 3.3], there exists a compatible strict total order \(\leq'\) on \(S\), which is finer than \(\leq\), that is, for any \(s, t \in S, s \leq t\) implies \(s \leq' t\). Since \((S, \leq)\) is artinian and narrow, by [14, 2.5] it follows that \((S, \leq')\) is artinian and narrow. Thus, by Theorem 3.2, \([M^{S,\leq}]\) is a semi Hopfian \([R^{S,\leq}]\)-module if and only if \(M\) is a semi Hopfian \(R\)-module, and by Theorem 3.4, \([M^{S,\leq}]\) is a semi co-Hopfian \([R^{S,\leq}]\)-module if and only if \(M\) is a semi co-Hopfian \(R\)-module. On the other hand, since \((S, \leq)\) is narrow, by [14, 4.4], \([R^{S,\leq}] = [R^{S,\leq}]\). Clearly \([M^{S,\leq}] = [M^{S,\leq}]\) and \([M^{S,\leq}] = [M^{S,\leq}]\). Now the result follows. □

Any submonoid of the additive monoid \(\mathbb{N} \cup \{0\}\) is called a *numerical* monoid. We have

**Corollary 4.2** Let \(S\) be a numerical monoid and \(\leq\) the usual natural order of \(\mathbb{N} \cup \{0\}\), \(R\) a commutative ring and \(M\) an \(R\)-module. Then
(1) \([M^{S,\leq}]\) is a semi Hopfian \([R^{S,\leq}]\)-module if and only if \(M\) is a semi Hopfian \(R\)-module.
(2) \([M^{S,\leq}]\) is a semi co-Hopfian \([R^{S,\leq}]\)-module if and only if \(M\) is a semi co-Hopfian \(R\)-module.

If \(S\) is the multiplicative monoid \((\mathbb{N}, \cdot)\), endowed with the usual order \(\leq\), then \([R^{(\mathbb{N},\cdot),\leq}]\) is the ring of arithmetical functions with values in \(R\), endowed with the Dirichlet convolution:

\[
(fg)(n) = \sum_{d|n} f(d)g(n/d), \quad \text{for each } n \geq 1.
\]

If \(M\) is a left \(R\)-module, then \([M^{(\mathbb{N},\cdot),\leq}]\) is a left \([R^{(\mathbb{N},\cdot),\leq}]\)-module with scalar multiplication:

\[
(f\phi)(n) = \sum_{d|n} f(d)\phi(n/d), \quad \text{for each } n \geq 1,
\]

where \(f \in \([R^{(\mathbb{N},\cdot),\leq}]\) and \(\phi \in \([M^{(\mathbb{N},\cdot),\leq}]\). The left \([R^{(\mathbb{N},\cdot),\leq}]\)-module \([M^{(\mathbb{N},\cdot),\leq}]\) is the set \(\{ \sum_{i=1}^{n} m_{i}x^{-i} \mid m_{i} \in M, i = 1, 2, \ldots, n, n \in \mathbb{N} \}\) with scalar multiplication as below:

\[
\left(\sum_{i \geq 1} r_{i}x^{i}\right) \left(\sum_{j \geq 1} m_{j}x^{-j}\right) = \sum_{j \geq 1} \left(\sum_{i \geq 1} r_{i}m_{i,j}\right) x^{-j}
\]

where \(\sum_{i \geq 1} r_{i}x^{i} \in \([R^{(\mathbb{N},\cdot),\leq}]\) and \(\sum_{j \geq 1} m_{j}x^{-j} \in \([M^{(\mathbb{N},\cdot),\leq}]\).
Corollary 4.3 Let $R$ be a commutative ring and $M$ an $R$-module. Then

(1) $[[M^{(R)}, \leq]]$ is a semi Hopfian $[[R^{(R)}, \leq]]$-module if and only if $M$ is a semi Hopfian $R$-module.

(2) $[M^{(R)}, \leq]$ is a semi co-Hopfian $[[R^{(R)}, \leq]]$-module if and only if $M$ is a semi co-Hopfian $R$-module.

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