On the Vanishing and Finiteness Properties of Generalized Local Cohomology Modules

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Abstract

Let $R$ be a commutative noetherian ring, $a$ an ideal of $R$ and $M, N$ finite $R$–modules. We prove that the following statements are equivalent.

(i) $H^i_a(M, N)$ is finite for all $i < n$.
(ii) $\text{Coass}_R(H^i_a(M, N)) \subseteq V(a)$ for all $i < n$.
(iii) $H^i_a(M, N)$ is coatomic for all $i < n$.

If $\text{pd} M$ is finite and $r$ be a non-negative integer such that $r > \text{pd} M$ and $H^i_a(M, N)$ is finite (resp. minimax) for all $i \geq r$, then $H^i_a(M, N)$ is zero (resp. artinian) for all $i \geq r$.

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1 Introduction

Throughout $R$ is a commutative noetherian ring. Generalized local cohomology was given in the local case by J. Herzog [5] and in the more general case by M.H Bijan-Zadeh [2]. Let $a$ denote an ideal of a ring $R$. The generalized local cohomology defined by

$$H^i_a(M, N) \cong \lim_{\rightarrow n} \text{Ext}^i_R(M/a^n M, N).$$

This concept was studied in the articles [8], [5] and [9]. Note that this is in fact a generalization of the usual local cohomology, because if $M = R$, 
then $H^i_a(R, N) = H^i_a(N)$. Important problems concerning local cohomology are vanishing, finiteness and artinianness results (see [6]).

In Section 2 we show in 2.1 that if $M$ is finite and all generalized local cohomology modules $H^i_a(M, N)$ are coatomic for all $i < n$, then they are finite for all $i < n$. In fact this is another condition equivalent to Falting’s Local-global Principle for the finiteness of generalized local cohomology modules (see [1, Theorem 2.9]). In Theorem 2.2 we generalize Yoshida’s theorem ([10, Theorem 3.1]).

In Section 3, we prove in 3.2, that when $M$ is a finite $R$–module of finite projective dimension such that the generalized local cohomology modules $H^i_a(M, N)$ are minimax modules for all $i \geq r$, (where $r > \text{pd} M$) then they must be artinian.

For unexplained terminology we refer to [3] and [4].

2 Finiteness and vanishing

An $R$–module $M$ is called coatomic when each proper submodule $N$ of $M$ is contained in a maximal submodule $N'$ of $M$ (i.e. such that $M/N' \cong R/m$ for some $m \in \text{Max} R$). This property can also be expressed by $\text{Coass}_R(M) \subset \text{Max} R$ or equivalently that any artinian homomorphic image of $M$ must have finite length. In particular all finite modules are coatomic. Coatomic modules have been studied by Zöschinger [12].

**Theorem 2.1** Let $R$ be a noetherian ring, $a$ an ideal of $R$ and $M, N$ finite $R$–modules. The following statements are equivalent:

(i) $H^i_a(M, N)$ is coatomic for all $i < n$.

(ii) $\text{Coass}_R(H^i_a(M, N)) \subset V(a)$ for all $i < n$.

(iii) $H^i_a(M, N)$ is finite for all $i < n$.

**proof:** By [1, Theorem 2.9] and[12, 1.1, Folgerung] we may assume that $(R, m)$ is a local ring.

(i) $\Rightarrow$ (ii) It is trivial by the definition of coatomic modules.

(ii) $\Rightarrow$ (iii) By [15, Satz 1.2] there is $t \geq 1$ such that $a^t H^i_a(M, N)$ is finite for all $i < n$. Therefore there is $s \geq t$ such that $a^s H^i_a(M, N) = 0$ for all $i < n$, and apply [1, Theorem 2.9].

(iii) $\Rightarrow$ (i) Any finite $R$–module is coatomic.

The following results are generalizations of [10, Proposition 3.1].
Theorem 2.2 Let \((R, \mathfrak{m})\) be a local ring, \(a\) be an ideal of \(R\) and \(M\) be a finite module of finite projective dimension. Let \(N\) be a finite module and \(r > \text{pd} M\). If \(H^i_a(M, N)\) is finite for all \(i \geq r\), then \(H^i_a(M, N) = 0\) for all \(i \geq r\).

**proof:** We prove by induction on \(d = \dim N\). If \(d = 0\), By [9, Theorem 3.7], it follows that \(H^i_a(M, N) = 0\) for all \(i > \text{pd} M + \dim(M \otimes_R N)\) and so the claim clearly holds for \(n = 0\). Now suppose \(d > 0\) and \(H^i_a(M, N) = 0\) for all \(i > r\). It is enough to show \(H^r_a(M, N) = 0\). First suppose \(\text{depth}_R N > 0\). Take \(x \in m\) which is \(N\)–regular. Then \(\dim N/xN = d - 1\). The exact sequence

\[
0 \rightarrow N \rightarrow N 
\]

induces the exact sequence

\[
H^r_a(M, N) \rightarrow H^r_a(M, N) \rightarrow H^r_a(M, N/xN) \rightarrow H^{r+1}_a(M, N) = 0
\]

It yields that \(H^i_a(M, N/xN) = 0\) for all \(i > r\). Hence by induction hypothesis we get \(H^r_a(M, N/xN) = 0\). Thus we have \(H^r_a(M, N) = 0\) by Nakayama’s lemma. Next suppose \(\text{depth}_R N = 0\). Put \(L = \Gamma_m(N)\). Since \(L\) have finite length, so we have \(\dim L = 0\) and therefore \(H^i_a(M, L) = 0\) for all \(i > \text{pd} M\). But from the exact sequence

\[
0 \rightarrow L \rightarrow N \rightarrow N/L \rightarrow 0
\]

we get the exact sequence

\[
\ldots \rightarrow H^r_a(M, L) \rightarrow H^r_a(M, N) \rightarrow H^r_a(M, N/L) \rightarrow H^{r+1}_a(M, L) \rightarrow \ldots
\]

hence we have \(H^i_a(M, N) \cong H^i_a(M, N/L)\) for all \(i > \text{pd} M\), and we get the required assertion from the first step.

Theorem 2.3 Let \(a\) be an ideal of \(R\) and \(M\) a finite \(R\)–module of finite projective dimension. Let \(N\) be a finite \(R\)–module and \(r > \text{pd} M\). The following statements are equivalent:

(i) \(H^i_a(M, N) = 0\) for all \(i \geq r\).

(ii) \(H^i_a(M, N)\) is finite for all \(i \geq r\).

(iii) \(H^i_a(M, N)\) is coatomic for all \(i \geq r\).

**proof:** (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii) Trivial. (iii) \(\Rightarrow\) (i) By use of theorem 2.2 and [12, 1.1, Folgerung] we may assume that \((R, \mathfrak{m})\) is a local ring. Note that coatomic modules satisfy Nakayama’s lemma. So the proof is the same as in theorem 2.2.

In the following corollary \(\text{cd}_a(M, N)\) denote the supremum of \(i\)’s such that \(H^i_a(M, N) \neq 0\).

**Corollary 2.4** Let \(a\) an ideal of \(R\), \(M\) a finite \(R\)–module of finite projective dimension and \(N\) a finite \(R\)–module. If \(c := \text{cd}_a(M, N) > \text{pd} M\), then \(H^c_a(M, N)\) is not coatomic in particular is not finite.
3 Artinianness

Recall that a module $M$ is a minimax module if there is a finite (i.e. finitely generated) submodule $N$ of $M$ such that the quotient module $M/N$ is artinian. Thus the class of minimax modules includes all finite and all artinian modules. Moreover, it is closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of $R$–modules. Minimax modules have been studied by Zink in [11] and Zöschinger in [13, 14]. See also [7].

**Lemma 3.1** Let $M$ and $N$ be two $R$–module. If $f : R \to S$ is a flat ring homomorphism, then

$$H^i_a(M, N) \otimes_R S \cong H^i_a(M \otimes_R S, N \otimes_R S).$$

**proof:** It is easy and we lift it to the reader.

**Theorem 3.2** Let $a$ an ideal of $R$ and $M$ a finite $R$–module of finite projective dimension. Let $N$ be a finite $R$–module and $r > \text{pd } M$. If $H^i_a(M, N)$ is a minimax module for all $i \geq r$, then $H^i_a(M, N)$ is an artinian module for all $i \geq r$.

**proof:** Let $p$ be a non-maximal prime ideal of $R$. Then by the definition of minimax module and lemma 3.1 $H^i_a(M, N)_p \cong H^i_{a|p}(M_p, N_p)$ is a finite $R_p$–module for all $i \geq r$. By theorem 2.2, $H^i_a(M, N)_p = 0$ for all $i \geq r$, thus $\text{Supp}_R(H^i_a(M, N)) \subseteq \text{Max } R$ for all $i \geq r$. By [7, Theorem 2.1], $H^i_a(M, N)$ is artinian for all $i \geq r$.

Let $q_a(M, N)$ denote the supremum of the $i$’s such that $H^i_a(M, N)$ is not artinian with the usual convention that the supremum of the empty set of integers is interpreted as $-\infty$.

**Corollary 3.3** Let $a$ an ideal of $R$, $M$ a finite $R$–module of finite projective dimension and $N$ a finite $R$–module. If $q := q_a(M, N) > \text{pd } M$, then $H^i_a(M, N)$ is not minimax in particular is not finite.

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**References**

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