Some Remarks on Derivations of Leibniz Algebras

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Abstract

The present paper concerns solvability of finite dimensional Leibniz algebras. As one of the main results of this paper, if Leibniz algebra $L$ has a derivation $d : L \to L$, such that $L^n \subset d(L)$, for some $n > 1$, then $L$ is solvable.

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1 Introduction

In [3] F. Ladish proved that a finite group $G$, admitting an element $a$ with the property $G' = [a, G]$ is solvable. Using this result M. shahryari in [4] proved a similar theorem for Lie algebras in more general framework, he showed that a finite dimensional Lie algebra $L$ over a field of characteristic zero admitting an abelian algebra of derivations $D \leq Der(L)$, with the following property

$$L^n \subseteq \sum_{d \in D} d(L)$$

for some $n > 1$, is necessarily solvable.

In this work we consider some general properties of Leibniz algebra and its derivation. We extend some results obtained for derivations of Lie algebras in [4] to the case of Leibniz algebras.

It is worth noting that in 1955, Jacobson [2] proved essential theorem in which every Lie algebra over a field of characteristic zero admitting a nonsingular derivation is nilpotent.
2 PRELIMINARIES

In this section we give necessary definitions and preliminary concepts.

Definition 2.1 An algebra \((L, [-, -])\) over a field \(F\) is said to be a Leibniz algebra if for any \(x, y, z \in L\) the so-called Leibniz identity

\[
[x, [y, z]] = [[x, y], z] - [[x, z], y]
\]

holds.

A subalgebra \(H\) of a Leibniz algebra is said to be a two-sided ideal if \([L, H] \subseteq L\) and \([H, L] \subseteq L\). Let \(H\) and \(K\) be two sided ideals of \(L\). The commutator ideal of \(H\) and \(K\), denoted by \([H, K]\) is the two-sided ideal of \(L\) spanned by the brackets \([h, k]\), \([k, h]\), \(h \in H, k \in K\).

For a given Leibniz algebra \(L\), we define the lower central and derived series to the sequences of two-sided ideals defined recursively as follows:

\[
L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \geq 1; \quad L^{[1]} = L, \quad L^{[s+1]} = [L^s, L^s], \quad s \geq 1.
\]

Definition 2.2 Leibniz algebra \(L\) is said to be solvable (nilpotent) if there exists \(n \in \mathbb{N}\) \((m \in \mathbb{N})\) such that \(L^{[n]} = 0\) \((L^m = 0)\).

Definition 2.3 For a Leibniz algebra \(L\), a Linear map \(d : L \rightarrow L\) is said to be a derivation if

\[
d[x, y] = [d(x), y] + [x, d(y)]
\]

for all \(x, y \in L\).

Definition 2.4 Inner derivations are a kind of derivations in which we have \(R_x : L \rightarrow L\) (for a fixed \(x \in L\)) such that \(R(x)(y) = [y, x]\) is a derivation.

Definition 2.5 The form \(f(a, b) = \text{tr}(R_a, R_b)\) for \(a, b \in L\) is called the killing form of the Leibniz algebra \(L\). A biLinear form \((a, b)\) on \(L\) satisfying the condition

\[
f([a, c], b) + f(a, [b, c]) = 0
\]

is called an invariant form on \(L\).

Theorem 2.6 Let \(L\) be a Leibniz algebra over an algebraically closed field of zero characteristic. Then \(L\) is solvable if and only if \(\text{tr}(R_a, R_b) = 0\) for any \(a \in L^2\).

proof. [1]
3 Main theorems and Results

In this section \( L \) is finite dimensional Leibniz algebra over a field \( K \) of characteristic zero. \( L^n \) and \( L^{[n]} \) represent the \( n \)-th terms of lower central series and derived series of \( L \), respectively. Also we denote by \( \text{Der}(L) \) the algebra of derivation of \( L \).

**Definition 3.1** A subspace \( h \subset L \) is called left (resp. right) ideal if for any \( a \in h \) and \( x \in L \) one has \([x,a] \in h\) (resp. \([a,x] \in h\)), if \( h \) is both left and right ideal, then \( h \) is called two-sided ideal.

**Theorem 3.2** Let \( L \) be a finite dimensional Leibniz algebra over \( K \) and suppose that \( U \) is a two-sided ideal. Suppose that there is an abelian subalgebra \( A \) such that \( U \subseteq [A,U] \), for some \( n > 1 \). Then \( U \) is solvable.

**proof.** We assume that \( K \) is algebraically closed, so we use Cartan’s criterion. Let \( S = U^n \) and define a bilinear form on \( L \) by

\[
f(\alpha, \beta) = \text{tr}(R_\alpha, R_\beta) \quad \text{for} \ \alpha, \beta \in L
\]

\( f \) is killing form of \( L \). Since \( S \) is an ideal of \( L \), \( f \) is associative,

\[
f([a,b], c) = f(a, [b,c]).
\]

Now we restrict \( f \) on \( S \), so we have

\[
f_S(a,b) = \text{tr}(R_a, R_b) \quad \text{for} \ \alpha, \beta \in S
\]

We apply Cartan’s criterion for algebra \( R(S) \) and consider as \( R(S) \)-module. If we consider \( R(S) = \frac{S}{Z(S)} \) where \( Z(S) \) is the center of \( S \). Obviously \( f(S,S') = 0 \) implies \( f(S',S') = 0 \), that is \( \text{tr}(R_a, R_b) = 0 \) for any \( a \in S \). Thus \( R(S) \) is solvable and solvability of \( R(S) \) is equivalent to solvability of \( S \). Hence \( U \) is solvable.

We now suppose that \( K \) is not necessarily algebraically closed, so we use the algebraic closure of \( K \). Let \( \bar{K} \) be algebraic closure of \( K \) and \( \bar{L} = \bar{K} \otimes_K A \). Now \( \bar{L} \) is a finite dimensional Leibniz algebra over \( \bar{K} \) in which \( \bar{K} \otimes_K U \) is an ideal and \( \bar{K} \otimes_u U \) is an abelian subalgebra. Further, we have

\[
(\bar{K} \otimes_K U)^n = \bar{K} \otimes_K U^n \subseteq [\bar{K} \otimes_K A, \bar{K} \otimes_K U].
\]

So \( \bar{K} \otimes_K U \) is solvable, that is, there exist a number \( m \) such that \( (\bar{K} \otimes_K U)^{(m)} = 0 \). On the other hand,

\[
(\bar{K} \otimes_K U)^{(m)} = \bar{K} \otimes_K U^{(m)},
\]

therefore \( U^{(m)} = 0 \).
Corollary 3.3 Suppose that there exist an abelian subalgebra $A \leq L$ and an integer $n > 1$, such that $L^n \subseteq [A, L]$. Then $L$ is solvable.

Finally we apply Theorem 3.2 to derivations of Leibniz algebra.

Corollary 3.4 Suppose that there exist an abelian subalgebra $D \leq \text{Der}(L)$ and an integer $n > 1$, such that

$$L^n \subseteq \sum_{d \in D} d(L).$$

Then $L$ is solvable.

**Proof.** Let $\hat{L} = D \ltimes L$, the natural semidirect product. We assume that $L$ is two-sided ideal. $D$ is an abelian subalgebra in $\hat{L}$. Note that in the semidirect product,

$$[D, L] = \sum_{d \in D} d(L),$$

hence the assumption is just $L^n \subseteq [D, L]$. So $L$ is solvable by Theorem 3.2.

References


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