Semigroups of Quasi-Open Mappings
and Lattice-Equivalence

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Abstract

In this paper we consider the semigroups of quasi-open functions. A mapping \( f \) between topological spaces \( X \) and \( Y \) is quasi-open if for any non-empty open set \( U \subset X \), the interior of \( f(U) \) in \( Y \) is non-empty. We give an abstract characterization of semigroups of quasi-open mappings defined on a certain class of topological spaces.

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1 Introduction

In [5] Thron introduced a concept of lattice-equivalence of topological spaces. Let \( X \) be a topological space and let \( C(X) \) be the lattice of closed sets of \( X \). Two topological spaces \( X \) and \( Y \) are said to be lattice equivalent if there is a bijective map from \( C(X) \) onto \( C(Y) \) which together with its inverse is order preserving. Thron proved among others that for \( T_D \)-spaces \( X \) and \( Y \), any lattice-isomorphism \( \phi : C(Y) \rightarrow C(X) \) can be induced by a homomorphism \( f : X \rightarrow Y \). It is worth noting that several researchers dealt with the concept of lattice equivalent topological spaces and representations of an abstract lattice as the family of closed sets on a topological space [1], [6]. Several researchers focused their efforts on the characterization of topological spaces by semigroups of continuous, open, and closed mappings [4], [7]. A map \( f \) between topological spaces \( X \) and \( Y \) is quasi-open if for any non-empty open set \( U \subset X \), the interior of \( f(U) \) in \( Y \) is non-empty. The quasi-open maps have the properties similar to those of the continuous maps. But the quasi-open maps and the continuous maps are not related. Some characterizations of \( M_1 \)-spaces, in terms
of quasi-open maps given by Kao in [2]. If \( f \) and \( g \) are both quasi-open, then the function composition is also quasi-open. Let \( Q(X) \) denote the semigroup of quasi-open maps from a topological space \( X \) into itself with composition of functions as multiplication. The purpose of this paper is to investigate semigroups of quasi-open maps in light of lattice-equivalence. It is obvious that if \( X \) and \( Y \) are homeomorphic then the semigroups \( Q(X) \) and \( Q(Y) \) are isomorphic. If \( Q(X) \) and \( Q(Y) \) are isomorphic, must \( X \) and \( Y \) be homeomorphic? In general, the answer is no. Let \( X \) denote any set with more than two elements containing the elements \( \eta, \xi \). Consider the topological spaces \( Y = (X, \tau_1) \) and \( Z = (X, \tau_2) \) with \( \tau_1 = \{\emptyset, \{\eta\}, X\} \) and \( \tau_2 = \{\emptyset, \{\eta\}, X\setminus\{\xi\}, X\} \). Evidently \( Q(Y) \) and \( Q(Z) \) are isomorphic but \( Y \) and \( Z \) are not homeomorphic. In this paper, we give an abstract characterization of semigroups of quasi-open maps for a certain class of topological spaces.

2 An Abstract Characterization of Semigroups of Quasi-Open Maps

A topological space \( X \) is said to be a \( T_D \)-space if for every point \( \xi \) in \( X \) the set \( \{\xi\} \setminus \{\xi\} \) is closed [5]. We denote the set \( \{\xi\} \setminus \{\xi\} \) by \( \{\xi\}' \). Obviously, each \( T_D \)-space is \( T_0 \)-space and each \( T_1 \)-space is \( T_D \)-space. We call a topological space \( X \) a \( T_D^+ \)-space if it is a \( T_D \)-space with no one-point open sets and if for every point \( \xi \) in \( X \) and for every open set \( U \) containing \( \xi \) the set \( U \cap (X \setminus \{\xi\}) \) is not empty. Note that each \( T_1 \)-space without isolated points is \( T_D^+ \)-space.

Lemma 1 Let \( X \) be a \( T_D^+ \)-space and let \( \xi \in X \) and let \( a, b \) be arbitrary elements of \( Q(X) \). The condition

\[
\forall f, g \in Q(X), \ f a = g a \rightarrow f b = g b
\]  

is necessary and sufficient for \( b(X) \subseteq a(X) \).

Proof. If the condition \( b(X) \subseteq a(X) \) is satisfied, then for every \( x \in X \) there exists a point \( \xi \in X \) such that \( b(x) = a(\xi) \). Then

\[
fb(x) = f(b(x)) = f(a(\xi)) = fa(\xi) = ga(\xi) = g(a(\xi)) = g(b(x)) = gb(x).
\]

So, condition (1) holds.

Now let condition (1) hold for some \( a, b \in Q(X) \). Suppose that the set \( b(X)\setminus a(X) \) is not empty. For any point \( \xi = b(x) \) in \( b(X)\setminus a(X) \) there exist \( f, g \in Q(x) \), such that \( f(\xi) \neq g(\xi) \) but \( f(x) = g(x) \) for all \( x \in X \setminus \{\xi\} \). Indeed, select a point \( \xi \in X \) and consider the map \( f : X \to X \) defined by

\[
f(x) = \begin{cases}
\eta_1 & \text{if } x = \xi \\
x & \text{if } x \neq \xi
\end{cases}
\]
and the map $g : X \to X$ defined by
\[
g(x) = \begin{cases} 
\eta_2 & \text{if } x = \xi \\
x & \text{if } x \neq \xi 
\end{cases}
\]
where $\eta_1 \neq \eta_2$ are any fixed points in $X \setminus \{\xi\}$. The maps $f$ and $g$ are quasi-open and we have $f(\xi) \neq g(\xi)$ but $f(x) = g(x)$ for all $x \in X \setminus \{\xi\}$. Then for every $x \in X$ the point $a(x)$ is in $X \setminus \{\xi\}$ and therefore $fa(x) = f(a(x)) = g(a(x)) = ga(x)$. But for $\xi = b(x) \in b(X)a(X)$ we have $fb(x) = f(b(x)) = f(\xi) \neq g(\xi) = g(b(x))$ which contradicts to (1).  

**Lemma 2** Let $X$ and $Y$ be $T^+_\sigma$-spaces and let $\varphi : Q(X) \to Q(Y)$ be an isomorphism between semigroups $Q(X)$ and $Q(Y)$. If $a(X) \subseteq b(X)$ for some $a, b \in Q(X)$ then $(\varphi a)(Y) \subseteq (\varphi b)(Y)$. Hence if $a(X) = b(X)$ for some $a, b \in Q(X)$ then $(\varphi a)(Y) = (\varphi b)(Y)$.

**Proof.** Suppose that $b(X) \subseteq a(X)$. If $f(\varphi a) = g(\varphi a)$ for some elements $f, g \in Q(Y)$ then there exist $f, g \in Q(X)$ such that $f \neq \varphi f$ and $g \neq \varphi g$. Then $(\varphi f)(\varphi a) = (\varphi g)(\varphi a)$ and since $\varphi$ is an isomorphism, $\varphi(fa) = \varphi(ga)$ and $fa = ga$. We have $fb = gb$, by Lemma 1. Again, since $\varphi$ is an isomorphism, then $(\varphi f)(\varphi b) = (\varphi g)(\varphi b)$ and therefore $f(\varphi b) = g(\varphi b)$. Because $f(\varphi b) = g(\varphi b)$ is true for every $f, g \in Q(Y)$ satisfying the condition $f(\varphi a) = g(\varphi a)$ it follows from Lemma 1 that $(\varphi b)(Y) \subseteq (\varphi a)(Y)$. In the same way, we could show that if $a(X) \subseteq b(X)$ then $(\varphi a)(Y) \subseteq (\varphi b)(Y)$.  

Let $X$ be a $T^+_\sigma$-space that has an open base, each element of which is an image of $X$ under a quasi-open mapping and let $\Lambda$ be a class of all such spaces. For instance, the open subsets of the $\alpha$-cube $I^\alpha$, $\alpha \geq 1$, the set $\mathbb{R}$ of real numbers with Zariski topology and any topological space $X$, $|X| \geq \aleph_0$, with cofinite topology belong to the class $\Lambda$.

**Lemma 3** Let $X \in \Lambda$ and let $U$ be any open subset of $X$. Then there exists a quasi-open mapping $a \in Q(X)$ such that $a(X) = U$.

**Proof.** Let $X \in \Lambda$ and $\mathcal{B}$ is an open base of $X$. Suppose that $U$ is an open subset of $X$ and $i : U \to X$ is the inclusion map, which is open map. Let $V_1 \in \mathcal{B}$ and $V_1 \subseteq U$, then there exists a quasi-open mapping $f$ from $X$ onto $V_1$. Consider the restriction of $f$ to $X \setminus \overline{U}$. Since restriction of a quasi-open map to an open set is quasi-open, this map is quasi-open. Denote by $g$ the extension of this mapping to $X \setminus \overline{U}$ obtained by assigning all boundary points of $U$ to any fixed point in $U$. The mapping $a : X \to U$ defined by
\[
a(x) = \begin{cases} 
i(x), & \text{if } x \in U \\
g(x), & \text{if } x \in X \setminus U
\end{cases}
\]
is a quasi-open map and \( a(X) = U \).

Let \( X \) be a topological space. The family \( O(X) \) of all open sets of \( X \) is a complete distributive lattice if set inclusion is taken as the ordering. By the duality principle for ordered sets, two topological spaces \( X \) and \( Y \) are homeomorphic if and only if lattices \( O(X) \) and \( O(Y) \) are isomorphic [5].

**Theorem 4** Let \( X, Y \in \Lambda \). If the semigroups \( Q(X) \) and \( Q(Y) \) are isomorphic then the lattices \( O(X) \) and \( O(Y) \) are lattice-isomorphic.

**Proof.** Let \( U \) be any open subset of \( X \). By Lemma 3 there exists a quasi-open function \( a \in Q(X) \) such that \( a(X) = U \). Since the semigroups \( Q(X) \) and \( Q(Y) \) are isomorphic there exists a quasi-open function \( a' \in Q(Y) \) such that \( \varphi a = a' \). Let \( a'(Y) = U' \). We define a map \( \theta \) from \( O(X) \) to \( O(Y) \) by assigning to each open set \( U \subset X \) the set \( U' \subset Y \). The map \( \theta \) does not depend on the choice of \( a \in Q(X) \). Indeed, if \( a(X) = U \) and \( b(X) = V \) then Lemma 2 says that \( (\varphi a)(Y) = (\varphi b)(Y) = U' \). Let \( U \) and \( V \) be any two different open subsets of \( X \). By Lemma 3 there exist two quasi-open functions \( a, b \in Q(X) \) such that \( a(X) = U \) and \( b(X) = V \). Since the semigroups \( Q(X) \) and \( Q(Y) \) are isomorphic it follows from Lemma 2 that \( (\varphi a)Y \neq (\varphi b)Y \). Hence \( \theta \) is bijective. Now suppose that \( U' \) is an arbitrary open set in \( Y \). Since the semigroups \( Q(X) \) and \( Q(Y) \) are isomorphic it follows from Lemma 2 that there exists an open set \( U \subset X \) such that \( \theta(U) = U' \). Again it follows from Lemma 2 that if \( U \subseteq V \) then \( \theta(U) \subseteq \theta(V) \). From Theorem 2.1 of [5] it follows that the topological spaces \( X \) and \( Y \) are homeomorphic. ■

**Theorem 5** Let \( X, Y \in \Lambda \). The semigroups \( Q(X) \) and \( Q(Y) \) are isomorphic if and only if the spaces \( X \) and \( Y \) are homeomorphic.

**Proof.** It is obvious that if \( X \) and \( Y \) are homeomorphic then \( Q(X) \) and \( Q(Y) \) are isomorphic. Specifically, if \( h \) is a homeomorphism from \( X \) onto \( Y \), then \( f \rightarrow h \circ f \circ h^{-1} \) is an isomorphism from \( Q(X) \) onto \( Q(Y) \). The proof of the necessary condition follows from Theorem 4. ■

**References**


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