Minimal Prime Ideals of 2-Primal Rings and their Extensions

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Abstract. In this article, we discuss minimal prime ideals of skew polynomial rings over 2-primal Noetherian rings. Recall that a ring \( R \) is 2-primal if and only if \( N(R) = P(R) \), i.e. if the prime radical is a completely semiprime.

Let now \( R \) be a 2-primal Noetherian, which is also an algebra over \( \mathbb{Q} \). Let \( U \) be a minimal prime ideal of \( R \) and \( \sigma \) an automorphism of \( R \) such that \( \sigma(U) = U \). Let \( \delta \) be a \( \sigma \)-derivation of \( R \) such that \( \delta(\sigma(a)) = \sigma(\delta(a)) \) for all \( a \in R \). Then we show that \( \delta(U) \subseteq U \) and \( U[x; \sigma, \delta] \) is a minimal prime ideal of \( R[x; \sigma, \delta] \).

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1. Introduction

We follow notation as in [5], but to make the paper self contained, we have the following:

A ring \( R \) means an associative ring with identity \( 1 \neq 0 \), and any \( R \)-module unitary. \( \mathbb{Q} \) denotes the field of rational numbers, \( \mathbb{Z} \) denotes the ring of integers and \( \mathbb{N} \) denotes the set of positive integers unless other wise stated. For a ring \( R \), the set of prime ideals of \( R \) is denoted by \( \text{Spec}(R) \), the set of associated prime ideals of \( R \) (viewed as a right \( R \)-module) is denoted by \( \text{Ass}(R_R) \), the set of minimal prime ideals of \( R \) is denoted by \( \text{MinSpec}(R) \) and the set of completely prime ideals of \( R \) is denoted by \( \text{C.Spec}(R) \). Prime radical of \( R \) is denoted by \( P(R) \) and the set of nilpotent elements of \( R \) is denoted by \( N(R) \)).

For any two ideals \( I, J \) of \( R \); \( I \subset J \) means that \( I \) is strictly contained in \( J \). Let \( K \) be an ideal of a ring \( R \) such that \( \sigma^m(K) = K \) for some integer \( m \geq 1 \),
we denote $\cap_{i=1}^{m}\sigma^{i}(K)$ by $K^0$.

Let $R$ be a ring, $\sigma$ an automorphisms of $R$ and $\delta$ a $\sigma$-derivation of $R$; i.e. $\delta: R \rightarrow R$ is an additive mapping satisfying $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$.

For example let $\sigma$ be an automorphism of a ring $R$ and $\delta: R \rightarrow R$ any map. Let $\phi: R \rightarrow M_{2}(R)$ be a homomorphism defined by

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}, \text{ for all } r \in R.$$  

Then $\delta$ is a $\sigma$-derivation of $R$.

We recall that the Ore extension

$$R[x; \sigma, \delta] = \{ f = \sum_{i=0}^{n} x^{i}a_{i}, a_{i} \in R, n \in \mathbb{N} \}$$

under usual addition of polynomials and multiplication subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We denote $R[x; \sigma, \delta]$ by $O(R)$. If $I$ is an ideal of $R$ such that $I$ is $\sigma$-stable (i.e. $\sigma(I) = I$) and is also $\delta$-invariant (i.e. $\delta(I) \subseteq I$), then clearly $I[x; \sigma, \delta]$ is an ideal of $O(R)$, and we denote it as usual by $O(I)$.

In case $\sigma$ is the identity map, we denote the ring of differential operators $R[x; \delta]$ by $D(R)$. If $J$ is an ideal of $R$ such that $J$ is $\delta$-invariant (i.e. $\delta(J) \subseteq J$), then clearly $J[x; \delta]$ is an ideal of $D(R)$, and we denote it as usual by $D(J)$.

In case $\delta$ is the zero map, we denote $R[x; \sigma]$ by $S(R)$. If $K$ is an ideal of $R$ such that $K$ is $\sigma$-stable (i.e. $\sigma(K) = K$), then clearly $K[x; \sigma]$ is an ideal of $S(R)$, and we denote it as usual by $S(K)$.

We recall that the skew Laurent polynomial ring

$$R[x, x^{-1}; \sigma] = \{ f = \sum_{i=-n}^{m} x^{i}a_{i}, a_{i} \in R \}; m, n \in \mathbb{N}$$

under usual addition of polynomials and multiplication subject to the relation $ax = x\sigma(a)$ for all $a \in R$. We denote $R[x, x^{-1}; \sigma]$ by $L(R)$. If $U$ is an ideal of $R$ such that $U$ is $\sigma$-stable (i.e. $\sigma(U) = U$), then clearly $U[x, x^{-1}; \sigma]$ is an ideal of $L(R)$, and we denote it as usual by $L(U)$.

**Prime ideals of Ore extensions**

Goodearl and Warfield proved in (2ZA) of [8] that if $R$ is a commutative Noetherian ring, and if $\sigma$ is an automorphism of $R$, then an ideal $I$ of $R$ is of the form $P \cap R$ for some prime ideal $P$ of $R[x, x^{-1}; \sigma]$ if and only if there is a prime ideal $S$ of $R$ and a positive integer $m$ with $\sigma^{m}(S) = S$, such that $I = \cap_{i=1}^{m}\sigma^{i}(S)$, $i = 1, 2, ..., m$. They proved in Theorem (2.22) of [8] that if $\delta$ is a derivation of a commutative Noetherian ring $R$ which is also an algebra over $\mathbb{Q}$ and $P$ is a prime ideal of $R[x; \delta]$, then $P \cap R$ is a prime ideal of $R$ and if $S$ is a prime ideal of $R$ with $\delta(S) \subseteq S$, then $S[x; \delta]$ is a prime ideal of $R[x; \delta]$. Gabriel proved in [7] that if $R$ is a right Noetherian ring which is also an algebra over
Q and P is a prime ideal of $R[x; \delta]$, then $P \cap R$ is a prime ideal of $R$.

We also have the following in this direction:

**Lemma 1.1.** Let $R$ be a ring. Let $\sigma$ be a an automorphism of $R$.

1. If $P$ is a prime ideal of $S(R)$ such that $x \notin P$, then $P \cap R$ is a prime ideal of $R$ and $\sigma(P \cap R) = P \cap R$.

2. If $U$ is a prime ideal of $R$ such that $\sigma(U) = U$, then $S(U)$ is a prime ideal of $S(R)$ and $S(U) \cap R = U$.

**Proof.** The proof follows on the same lines as in Lemma (10.6.4) of McConnell and Robson [12].

**Lemma 1.2.** Let $R$ be a commutative Noetherian $\mathbb{Q}$-algebra. Let $\delta$ be a derivation of $R$. Then:

1. If $P$ is a prime ideal of $D(R)$, then $P \cap R$ is a prime ideal of $R$ and $\delta(P \cap R) \subseteq P \cap R$.

2. If $U$ is a prime ideal of $R$ such that $\delta(U) \subseteq U$, then $D(U)$ is a prime ideal of $D(R)$ and $D(U) \cap R = U$.

**Proof.** See Theorem (2.22) of Goodearl and Warfield [8].

**Associated prime ideals of Ore extensions**

Carl Faith proved in [6] that if $R$ is a commutative ring, then the associated prime ideals of the usual polynomial ring $R[x]$ (viewed as a module over itself) are precisely the ideals of the form $P[x]$, where $P$ is an associated prime ideal of $R$.

H. Nordstrom has proved the following result in [14]:

**Theorem (1.2) of [14]:** Let $R$ be a ring with identity and $\sigma$ be a surjective endomorphism of $R$. Then for any right $R$-module $M$, $\text{Ass}(M[x; \sigma]_{R[x; \sigma]}) = \{I[x; \sigma], I \in \sigma - \text{Ass}(M)\}$.

In Corollary (1.5) of [14] Nordstrom has been proved that if $R$ is a Noetherian ring and $\sigma$ is an automorphism of $R$, then $\text{Ass}(M[x; \sigma]_S) = \{P_\sigma[x; \sigma], P \in \text{Ass}(M_R)\}$, where $P_\sigma = \cap \sigma^{-i}(P)$ and $S = R[x; \sigma]$.

Concerning associated prime ideals of full Ore extensions $R[x; \sigma, \delta]$, S. Annin generalizes the above in the following way:

**Definition (2.1) of Annin [2]:** Let $R$ be a ring and $M_R$ be a right $R$-module. Let $\sigma$ be an endomorphism of $R$ and $\delta$ be a $\sigma$-derivation of $R$. $M_R$ is said to be $\sigma$-compatible if for each $m \in M$, $r \in R$, we have $mr = 0$ if and only if $m\sigma(r) = 0$. Moreover $M_R$ is said to be $\delta$-compatible if for each $m \in M$, $r \in R$, we have $mr = 0$ implies that $m\delta(r) = 0$. If $M_R$ is both $\sigma$-compatible and $\delta$-compatible, $M_R$ is said to be $(\sigma - \delta)$-compatible.
**Theorem (2.3) of Annin** [2]: Let $R$ be a ring. Let $\sigma$ be an endomorphism of $R$ and $\delta$ be a $\sigma$-derivation of $R$ and $M_R$ be a right $R$-module. If $M_R$ is $(\sigma - \delta)$-compatible, then $\text{Ass}(M[x]_S) = \{P[x] \mid P \in \text{Ass}(M_R)\}$.

In [10] Leroy and Matczuk have investigated the relationship between the associated prime ideals of an $R$-module $M_R$ and that of the induced $S$-module $M_S$, where $S = R[x; \sigma, \delta]$ ($\sigma$ is an automorphism and $\delta$ is a $\sigma$-derivation of a ring $R$). They have proved the following:

**Theorem (5.7) of [10]**: Suppose $M_R$ contains enough prime submodules and let for $Q \in \text{Ass}(M_S)$. If for every $P \in \text{Ass}(M_R)$, $\sigma(P) = P$, then $Q = PS$ for some $P \in \text{Ass}(M_R)$.

Let $R$ be a right Noetherian ring. Then we know that $\text{MinSpec}(R)$ is finite by Theorem (2.4) of Goodearl and Warfield [8] and for any automorphism $\sigma$ of $R$, $P \in \text{MinSpec}(R)$ implies that $\sigma^j(P) \in \text{MinSpec}(R)$ for all positive integers $j$. Therefore, there exists some $m \in \mathbb{N}$ such that $\sigma^m(P) = P$ for all $P \in \text{MinSpec}(R)$. We denote $\cap_{i=1}^m \sigma^i(P)$ by $P^0$ as mentioned in introduction. We have a similar statement and notation for associated prime ideals of a right Noetherian ring $R$.

In Theorem (2.4) of [4] it has been proved that if $R$ is a Noetherian ring and $\sigma$ an automorphism of $R$, then

1. $P \in \text{Ass}(S(R)_S(R))$ if and only if there exists $U \in \text{Ass}(R_R)$ such that $S(P \cap R) = P$ and $P \cap R = U^0$.
2. $P \in \text{MinSpec}(S(R))$ if and only if there exists $U \in \text{MinSpec}(R)$ such that $S(P \cap R) = P$ and $P \cap R = U^0$.

In Theorem (3.7) of [4] it has been proved that if $R$ is a Noetherian $\mathbb{Q}$-algebra and $\delta$ a derivation of $R$, then

1. $P \in \text{Ass}(D(R)_D(R))$ if and only if $P = D(P \cap R)$ and $P \cap R \in \text{Ass}(R_R)$.
2. $P \in \text{MinSpec}(D(R))$ if and only if $P = D(P \cap R)$ and $P \cap R \in \text{MinSpec}(R)$.

In this paper we discuss the minimal prime ideals of a 2-primal Noetherian ring $R$ and its extensions and prove the following:

**Theorem A**: Let $R$ be a 2-primal Noetherian $\mathbb{Q}$-algebra, $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then $P_1 \in \text{MinSpec}(R)$ with $\sigma(P_1) = P_1$ implies that $O(P_1) \in \text{MinSpec}(O(R)) \cap C.Spec(O(R))$. (This is proved in Theorem 2.7).

For more details and some basic results for the rings $R[x; \sigma, \delta]$, $R[x; \sigma]$, and $R[x; \delta]$, the reader is referred to chapters (1) and (2) of Goodearl and Warfield [8].


2. 2-PRIMAL RINGS AND COMPLETELY PRIME IDEALS

2-Primal rings

Recall that a ring $R$ is 2-primal if and only if $N(R) = P(R)$, i.e. if the prime radical is a completely semiprime. An ideal $I$ of a ring $R$ is called completely semiprime if $a^2 \in I$ implies $a \in I$ for $a \in R$. We note that a reduced ring is 2-primal and so is a commutative ring. Also let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where $F$ is a field. Then $R$ is 2-primal.

2-primal rings have been studied in recent years and are being treated by authors for different structures. In [11], Greg Marks discusses the 2-primal property of $R[x; \sigma, \delta]$, where $R$ is a local ring, $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. In Greg Marks [11], it has been shown that for a local ring $R$ with a nilpotent maximal ideal, the Ore extension $R[x; \sigma, \delta]$ will or will not be 2-primal depending on the $\delta$-stability of the maximal ideal of $R$. In the case where $R[x; \sigma, \delta]$ is 2-primal, it will satisfy an even stronger condition; in the case where $R[x; \sigma, \delta]$ is not 2-primal, it will fail to satisfy an even weaker condition.

Minimal prime ideals of 2-primal rings have been discussed by Kim and Kwak in [9]. 2-primal near rings have been discussed by Argac and Groenewald in [1]. For further details on 2-primal rings, we refer the reader to [1, 3, 9, 11].

Completely prime ideals

We have discussed some known facts about the prime ideals of Ore extensions. We shall now discuss completely prime ideals.

Recall that an ideal $P$ of a ring $R$ is completely prime if $R/P$ is a domain, i.e. $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$ (McCoy [13]).

In commutative sense completely prime and prime have the same meaning. We also note that every completely prime ideal of a ring $R$ is a prime ideal, but the converse need not be true.

The following example shows that a prime ideal need not be a completely prime ideal.

Example 2.1. Let $R = \begin{pmatrix} Z & Z \\ Z & Z \end{pmatrix} = M_2(Z)$. If $p$ is a prime number, then the ideal $P = M_2(pZ)$ is a prime ideal of $R$, but is not completely prime, since for $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we have $ab \in P$, even though $a \notin P$ and $b \notin P$.

Regarding the relation between the completely prime ideals of a ring $R$ and those of $O(R)$ the following result has been proved by Bhat[5]:
Theorem 2.2. (Theorem 2.4 of Bhat[5]): Let $R$ be a ring, $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. Then:

1. For any completely prime ideal $P$ of $R$ with $\delta(P) \subseteq P$ and $\sigma(P) = P$, $O(P)$ is a completely prime ideal of $O(R)$.
2. For any completely prime ideal $U$ of $O(R)$, $U \cap R$ is a completely prime ideal of $R$.

Proof. The proof is on the same lines as in Theorem 2.4 of Bhat[5].

In [15] Shin has proved the following:

Proposition 2.4. (Proposition 1.11 of [15]) For a ring $R$, the following are equivalent:

1. Prime radical coincides with the set of nilpotent elements of $R$.
2. Every minimal prime ideal of $R$ is completely prime.

With this we prove the following:

Proposition 2.5. Let $R$ be a 2-primal Noetherian ring which is also an algebra over $\mathbb{Q}$. Let $U \in \text{Min.}\text{Spec}(R)$ and $\sigma$ an automorphism of $R$ such that $\sigma(U) = U$. Let $\delta$ a $\sigma$-derivation of $R$ such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then $\delta(U) \subseteq U$.

Proof. Let $R$ be 2-primal implies that $N(R) = P(R)$ and $P(R)$ is completely semiprime.

Now by Proposition (2.4) $U$ is a completely prime ideal of $R$. Now $\sigma(U) = U$ and let $V = \{a \in U \mid \delta^k(a) \in U \text{ for all integers } k \geq 1\}$.

First of all, we will show that $V$ is an ideal of $R$. Let $a, b \in V$. Then $\delta^k(a) \in U$ and $\delta^k(b) \in U$ for all integers $k \geq 1$. Now $\delta^k(a - b) = \delta^k(a) - \delta^k(b) \in U$ for all $k \geq 1$. Therefore $a - b \in V$. Also it is easy to see that for any $a \in V$ and for any $r \in R$, $ra \in V$. Therefore $V$ is a $\delta$-invariant ideal of $R$.

We will now show that $V \in \text{Spec}(R)$. Suppose $V \notin \text{Spec}(R)$. Let $a \notin V$, $b \notin V$ be such that $arb \subseteq V$. Let $t, s$ be least such that $\delta^t(a) \notin U$ and $\delta^s(b) \notin U$. Now there exists $c \in R$ such that $\delta^t(a)c\sigma^t(\delta^s(b)) \notin U$. Let $d = \sigma^{-t}(c)$. Now $\delta^{t+s}(c) \in U$ as $arb \subseteq V$. This implies on simplification that $\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) + u \in U$, where $u$ is sum of terms involving $\delta^l(a)$ or $\delta^m(b)$, where $l < t$ and $m < s$. Therefore by assumption $u \in U$ which implies that $\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) \in U$. This is a contradiction. Therefore, our supposition must be wrong. Hence $V \in \text{Spec}(R)$. Now $V \subseteq U$, so $V = U$ as $U \in \text{Min.}\text{Spec}(R)$. Hence $\delta(U) \subseteq U$. 

\[\blacksquare\]
In above proposition the condition that \( \delta(\sigma(a)) = \sigma(\delta(a)) \), for all \( a \in R \) is necessary. For example if \( s = t = 1 \), then \( a \in U \), \( b \in U \) and therefore, \( \sigma^i(a) \in U \), \( \sigma^i(b) \in U \) for all integers \( i \geq 1 \) as \( \sigma(U) = U \). Now \( \delta^2(ab) \in U \) implies that
\[
\delta(a)\delta(d)\delta(\sigma(b)) + \delta(a)\delta(d)\delta(\sigma(b)) + u \in U.
\]
where \( u \) is sum of terms involving \( a \) or \( b \), or \( \sigma^i(b) \). Therefore by assumption \( u \in U \). This implies that
\[
\delta(a)\delta(d)\delta(\sigma(b)) + \delta(a)\delta(d)\delta(\sigma(b)) \in U.
\]
If \( \delta(\sigma(a)) \neq \sigma(\delta(a)) \), for all \( a \in R \), then we get nothing out of it and if \( \delta(\sigma(a)) = \sigma(\delta(a)) \), for all \( a \in R \), we get \( \delta(a)\delta(d)\sigma(\delta(b)) \in U \) which gives a contradiction.

Now we have with the following:

**Theorem 2.6.** (Hilbert Basis Theorem): Let \( R \) be a right/left Noetherian ring. Let \( \alpha \) and \( \rho \) be as above. Then the ore extension \( O(R) = R[x, \alpha, \rho] \) is right/left Noetherian. Also \( R[x, x^{-1}, \alpha] \) is right/left Noetherian.

*Proof.* See Theorems (1.12) and (1.17) of Goodearl and Warfield [8].

We now state and prove the following:

**Theorem 2.7.** Let \( R \) be a 2-primal Noetherian \( \mathbb{Q} \)-algebra, \( \sigma \) an automorphism of \( R \) and \( \delta \) a \( \sigma \)-derivation of \( R \) such that \( \sigma(\delta(a)) = \delta(\sigma(a)) \) for all \( a \in R \). Then \( P_1 \in \text{Min.Spec}(R) \) with \( \sigma(P_1) = P_1 \) implies that \( O(P_1) \in \text{Min.Spec}(O(R)) \cap \text{C.Spec}(O(R)) \).

*Proof.* Let \( P_1 \in \text{Min.Spec}(R) \). Then \( \delta(P_1) \subseteq P_1 \) by Proposition (2.4). Now it can be seen that that \( O(P_1) \in \text{Spec}(O(R)) \). Suppose \( O(P_1) \notin \text{MinSpec}(O(R)) \) and \( P_2 \subseteq O(P_1) \) be a minimal prime ideal of \( O(R) \). Then \( P_2 = O(P_2 \cap R) \subseteq O(P_1) \subseteq \text{Min.Spec}(O(R)) \). Therefore \( P_2 \cap R \subseteq P_1 \) which is a contradiction, as \( P_2 \cap R \in \text{Spec}(R) \). Hence \( O(P_1) \in \text{Min.Spec}(O(R)) \).

Also by Proposition (2.4) \( P_1 \) is a completely prime ideal of \( R \), therefore, Theorem (2.2) implies that \( O(P_1) \in \text{C.Spec}(O(R)) \). Hence \( O(P_1) \in \text{Min.Spec}(O(R)) \cap \text{C.Spec}(O(R)) \).

**Corollary 2.8.** Let \( R \) be a 2-primal Noetherian ring and \( \sigma \) an automorphism of \( R \). Then \( P_1 \in \text{Min.Spec}(R) \) with \( \sigma(P_1) = P_1 \) implies that \( L(P_1) \in \text{Min.Spec}(L(R)) \cap \text{C.Spec}(L(R)) \).

*Proof.* Use 2.3 and 2.7.

**References**


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