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Minimal Prime Ideals of 2-Primal Rings and their Extensions

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Abstract. In this article, we discuss minimal prime ideals of skew polynomial rings over 2-primal Noetherian rings. Recall that a ring R is 2-primal if and only if $N(R) = P(R)$, i.e. if the prime radical is a completely semiprime.
Let now R be a 2-primal Noetherian, which is also an algebra over $\mathbb{Q}$. Let U be a minimal prime ideal of R and σ an automorphism of R such that $\sigma(U) = U$. Let δ be a σ-derivation of R such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. Then we show that $\delta(U) \subseteq U$ and $U[x; \sigma, \delta]$ is a minimal prime ideal of $R[x; \sigma, \delta]$.

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1. Introduction

We follow notation as in [5], but to make the paper self contained, we have the following:

A ring R means an associative ring with identity $1 \neq 0$, and any R-module unitary. $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Z}$ denotes the ring of integers and $\mathbb{N}$ denotes the set of positive integers unless other wise stated. For a ring R, the set of prime ideals of R is denoted by $Spec(R)$, the set of associated prime ideals of R (viewed as a right R-module) is denoted by $Ass(R_R)$, the set of minimal prime ideals of R is denoted by $MinSpec(R)$ and the set of completely prime ideals of R is denoted by $C.Spec(R)$. Prime radical of R is denoted by $P(R)$) and the set of nilpotent elements of R is denoted by $N(R)$). For any two ideals I, J of R; $I \subset J$ means that I is strictly contained in J. Let K be an ideal of a ring R such that $\sigma^m(K) = K$ for some integer $m \geq 1$, we denote $\cap_{i=1}^{m}\sigma^i(K)$ by K^0.

Let R be a ring, σ an automorphisms of R and δ a σ-derivation of R; i.e. $\delta : R \to R$ is an additive mapping satisfying $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$.

For example let σ be an automorphism of a ring R and $\delta : R \to R$ any map. Let $\phi : R \to M_2(R)$ be a homomorphism defined by

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}, \text{ for all } r \in R.$$

Then δ is a σ-derivation of R.

We recall that the Ore extension

$$R[x;\sigma,\delta] = \{f = \textstyle\sum_{i=0}^{n} x^i a_i,\ a_i \in R,\ n \in \mathbb{N}\}$$

under usual addition of polynomials and multiplication subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We denote $R[x;\sigma,\delta]$ by $O(R)$. If I is an ideal of R such that I is σ-stable (i.e. $\sigma(I) = I$) and is also δ-invariant (i.e. $\delta(I) \subseteq I$), then clearly $I[x;\sigma,\delta]$ is an ideal of $O(R)$, and we denote it as usual by $O(I)$.

In case σ is the identity map, we denote the ring of differential operators $R[x;\delta]$ by $D(R)$. If J is an ideal of R such that J is δ-invariant (i.e. $\delta(J) \subseteq J$), then clearly $J[x;\delta]$ is an ideal of $D(R)$, and we denote it as usual by $D(J)$.

In case δ is the zero map, we denote $R[x;\sigma]$ by $S(R)$. If K is an ideal of R such that K is σ-stable (i.e. $\sigma(K) = K$), then clearly $K[x;\sigma]$ is an ideal of $S(R)$, and we denote it as usual by $S(K)$.

We recall that the skew Laurent polynomial ring

$$R[x,x^{-1};\sigma] = \{f = \textstyle\sum_{i=-n}^{m} x^i a_i,\ a_i \in R\};\ m,\ n \in \mathbb{N}$$

under usual addition of polynomials and multiplication subject to the relation $ax = x\sigma(a)$ for all $a \in R$. We denote $R[x,x^{-1};\sigma]$ by $L(R)$. If U is an ideal of R such that U is σ-stable (i.e. $\sigma(U) = U$), then clearly $U[x,x^{-1};\sigma]$ is an ideal of $L(R)$, and we denote it as usual by $L(U)$.

Prime ideals of Ore extensions

Goodearl and Warfield proved in (2ZA) of [8] that if R is a commutative Noetherian ring, and if σ is an automorphism of R, then an ideal I of R is of the form $P \cap R$ for some prime ideal P of $R[x,x^{-1};\sigma]$ if and only if there is a prime ideal S of R and a positive integer m with $\sigma^m(S) = S$, such that $I = \cap\sigma^i(S)$, $i = 1, 2, ..., m$. They proved in Theorem (2.22) of [8] that if δ is a derivation of a commutative Noetherian ring R which is also an algebra over $\mathbb{Q}$ and P is a prime ideal of $R[x;\delta]$, then $P \cap R$ is a prime ideal of R and if S is a prime ideal of R with $\delta(S) \subseteq S$, then $S[x;\delta]$ is a prime ideal of $R[x;\delta]$. Gabriel proved in [7] that if R is a right Noetherian ring which is also an algebra over

$\mathbb{Q}$ and P is a prime ideal of $R[x;\delta]$, then $P \cap R$ is a prime ideal of R.

We also have the following in this direction:

Lemma 1.1. *Let R be a ring. Let σ be a an automorphism of R.*

1. *If P is a prime ideal of $S(R)$ such that $x \notin P$, then $P \cap R$ is a prime ideal of R and $\sigma(P \cap R) = P \cap R$.*
2. *If U is a prime ideal of R such that $\sigma(U) = U$, then $S(U)$ is a prime ideal of $S(R)$ and $S(U) \cap R = U$.*

Proof. The proof follows on the same lines as in Lemma (10.6.4) of McConnell and Robson [12]. □

Lemma 1.2. *Let R be a commutative Noetherian $\mathbb{Q}$-algebra. Let δ be a derivation of R. Then:*

1. *If P is a prime ideal of $D(R)$, then $P \cap R$ is a prime ideal of R and $\delta(P \cap R) \subseteq P \cap R$.*
2. *If U is a prime ideal of R such that $\delta(U) \subseteq U$, then $D(U)$ is a prime ideal of $D(R)$ and $D(U) \cap R = U$.*

Proof. See Theorem (2.22) of Goodearl and Warfield [8]. □

Associated prime ideals of Ore extensions

Carl Faith proved in [6] that if R is a commutative ring, then the associated prime ideals of the usual polynomial ring $R[x]$ (viewed as a module over itself) are precisely the ideals of the form $P[x]$, where P is an associated prime ideal of R.

H. Nordstrom has proved the following result in [14]:

Theorem (1.2) of [14]: Let R be a ring with identity and σ be a surjective endomorphism of R. Then for any right R-module M, $Ass(M[x;\sigma]_{R[x;\sigma]}) = \{I[x;\sigma],\ I \in \sigma - Ass(M)\}$.

In Corollary (1.5) of [14] Nordstrom has been proved that if R is a Noetherian ring and σ is an automorphism of R, then $Ass(M[x;\sigma]_S) = \{P_\sigma[x;\sigma],\ P \in Ass(M_R)\}$, where $P_\sigma = \cap_{i \in N} \sigma^{-i}(P)$ and $S = R[x;\sigma]$.

Concerning associated prime ideals of full Ore extensions $R[x;\sigma,\delta]$, S. Annin generalizes the above in the following way:

Definition (2.1) of Annin [2]: Let R be a ring and M_R be a right R-module. Let σ be an endomorphism of R and δ be a σ-derivation of R. M_R is said to be σ-compatible if for each $m \in M$, $r \in R$, we have $mr = 0$ if and only if $m\sigma(r) = 0$. Moreover M_R is said to be δ-compatible if for each $m \in M$, $r \in R$, we have $mr = 0$ implies that $m\delta(r) = 0$. If M_R is both σ-compatible and δ-compatible, M_R is said to be $(\sigma - \delta)$-compatible.

Theorem (2.3) of Annin [2]: Let R be a ring. Let σ be an endomorphism of R and δ be a σ-derivation of R and M_R be a right R-module. If M_R is $(\sigma - \delta)$-compatible, then $Ass(M[x]_S) = \{P[x] \mid P \in Ass(M_R)\}$.

In [10] Leroy and Matczuk have investigated the relationship between the associated prime ideals of an R-module M_R and that of the induced S-module M_S, where $S = R[x; \sigma, \delta]$ (σ is an automorphism and δ is a σ-derivation of a ring R). They have proved the following:

Theorem (5.7) of [10]: Suppose M_R contains enough prime submodules and let for $Q \in Ass(M_S)$. If for every $P \in Ass(M_R)$, $\sigma(P) = P$, then $Q = PS$ for some $P \in Ass(M_R)$.

Let R be a right Noetherian ring. Then we know that $MinSpec(R)$ is finite by Theorem (2.4) of Goodearl and Warfield [8] and for any automorphism σ of R, $P \in MinSpec(R)$ implies that $\sigma^j(P) \in MinSpec(R)$ for all positive integers j. Therefore, there exists some $m \in \mathbb{N}$ such that $\sigma^m(P) = P$ for all $P \in MinSpec(R)$. We denote $\cap_{i=1}^{m} \sigma^i(P)$ by P^0 as mentioned in introduction. We have a similar statement and notation for associated prime ideals of a right Noetherian ring R.

In Theorem (2.4) of [4] it has been proved that if R is a Noetherian ring and σ an automorphism of R, then

1. $P \in Ass(S(R)_{S(R)})$ if and only if there exists $U \in Ass(R_R)$ such that $S(P \cap R) = P$ and $(P \cap R) = U^0$.
2. $P \in MinSpec(S(R))$ if and only if there exists $U \in MinSpec(R)$ Such that $S(P \cap R) = P$ and $P \cap R = U^0$.

In Theorem (3.7) of [4] it has been proved that if R is a Noetherian $\mathbb{Q}$-algebra and δ a derivation of R, then

1. $P \in Ass(D(R)_{D(R)})$ if and only if $P = D(P \cap R)$ and $P \cap R \in Ass(R_R)$.

2. $P \in MinSpec(D(R))$ if and only if $P = D(P \cap R)$ and $P \cap R \in MinSpec(R)$.

In this paper we discuss the minimal prime ideals of a 2-primal Noetherian ring R and its extensions and prove the following:

Theorem A: Let R be a 2-primal Noetherian $\mathbb{Q}$-algebra, σ an automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then $P_1 \in Min.Spec(R)$ with $\sigma(P_1) = P_1$ implies that $O(P_1) \in Min.Spec(O(R)) \cap C.Spec(O(R))$. (This is proved in Theorem 2.7).

For more details and some basic results for the rings $R[x; \sigma, \delta]$, $R[x; \sigma]$, and $R[x; \delta]$, the reader is referred to chapters (1) and (2) of Goodearl and Warfield [8].

2. 2-Primal rings and Completely prime ideals

2-Primal rings

Recall that a ring R is 2-primal if and only if $N(R) = P(R)$, i.e. if the prime radical is a completely semiprime. An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$ for $a \in R$. We note that a reduced ring is 2-primal and so is a commutative ring. Also let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then R is 2-primal.

2-primal rings have been studied in recent years and are being treated by authors for different structures. In [11], Greg Marks discusses the 2-primal property of $R[x; \sigma, \delta]$, where R is a local ring, σ an automorphism of R and δ a σ-derivation of R. In Greg Marks [11], it has been shown that for a local ring R with a nilpotent maximal ideal, the Ore extension $R[x; \sigma, \delta]$ will or will not be 2-primal depending on the δ-stability of the maximal ideal of R. In the case where $R[x; \sigma, \delta]$ is 2-primal, it will satisfy an even stronger condition; in the case where $R[x; \sigma, \delta]$ is not 2-primal, it will fail to satisfy an even weaker condition.

Minimal prime ideals of 2-primal rings have been discussed by Kim and Kwak in [9]. 2-primal near rings have been discussed by Argac and Groenewald in [1]. For further details on 2-primal rings, we refer the reader to [1, 3, 9, 11].

Completely prime ideals

We have discussed some known facts about the prime ideals of Ore extensions. We shall now discuss completely prime ideals.

Recall that an ideal P of a ring R is completely prime if R/P is a domain, i.e. $ab \in P$ implies $a \in P$ or $b \in P$ for a, $b \in R$ (McCoy [13]).

In commutative sense completely prime and prime have the same meaning. We also note that every completely prime ideal of a ring R is a prime ideal, but the converse need not be true.

The following example shows that a prime ideal need not be a completely prime ideal.

Example 2.1. *Let* $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} = M_2(\mathbb{Z})$. *If* p *is a prime number, then the ideal* $P = M_2(p\mathbb{Z})$ *is a prime ideal of* R, *but is not completely prime, since for* $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, *we have* $ab \in P$, *even though* $a \notin P$ *and* $b \notin P$.

Regarding the relation between the completely prime ideals of a ring R and those of $O(R)$ the following result has been proved by Bhat[5]:

Theorem 2.2. *(Theorem 2.4 of Bhat[5]:) Let R be a ring, σ an automorphism of R and δ a σ-derivation of R. Then:*

1. *For any completely prime ideal P of R with $\delta(P) \subseteq P$ and $\sigma(P) = P$, $O(P)$ is a completely prime ideal of $O(R)$.*
2. *For any completely prime ideal U of $O(R)$, $U \cap R$ is a completely prime ideal of R.*

Theorem 2.3. *Let R be a ring and σ an automorphism of R. Then:*

1. *For any completely prime ideal P of R with $\sigma(P) = P$, $L(P)$ is a completely prime ideal of $L(R)$.*
2. *For any completely prime ideal U of $L(R)$, $U \cap R$ is a completely prime ideal of R.*

Proof. The proof is on the same lines as in Theorem 2.4 of Bhat[5]. □

In [15] Shin has proved the following:

Proposition 2.4. *(Proposition 1.11 of [15]) For a ring R, the following are equivalent:*

1. *Prime radical coincides with the set of nilpotent elements of R.*
2. *Every minimal prime ideal of R is completely prime.*

With this we prove the following:

Proposition 2.5. *Let R be a 2-primal Noetherian ring which is also an algebra over $\mathbb{Q}$. Let $U \in Min.Spec(R)$ and σ an automorphism of R such that $\sigma(U) = U$. Let δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then $\delta(U) \subseteq U$.*

Proof. Let R be 2-primal implies that $N(R) = P(R)$ and $P(R)$ is completely semiprime.

Now by Proposition (2.4) U is a completely prime ideal of R. Now $\sigma(U) = U$

Let now $V = \{a \in U \mid$ such that $\delta^k(a) \in U$ for all integers $k \geq 1\}$.

First of all, we will show that V is an ideal of R. Let $a, b \in V$. Then $\delta^k(a) \in U$ and $\delta^k(b) \in U$ for all integers $k \geq 1\}$. Now $\delta^k(a-b) = \delta^k(a) - \delta^k(b) \in U$ for all $k \geq 1\}$. Therefore $a - b \in V$. Also it is easy to see that for any $a \in V$ and for any $r \in R$, $ar \in V$ and $ra \in V$. Therefore V is a δ-invariant ideal of R.

We will now show that $V \in Spec(R)$. Suppose $V \notin Spec(R)$. Let $a \notin V$, $b \notin V$ be such that $aRb \subseteq V$. Let t, s be least such that $\delta^t(a) \notin U$ and $\delta^s(b) \notin U$. Now there exists $c \in R$ such that $\delta^t(a)c\sigma^t(\delta^s(b)) \notin U$. Let $d = \sigma^{-t}(c)$. Now $\delta^{t+s}(adb) \in U$ as $aRb \subseteq V$. This implies on simplification that $\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) + u \in U$, where u is sum of terms involving $\delta^l(a)$ or $\delta^m(b)$, where $l < t$ and $m < s$. Therefore by assumption $u \in U$ which implies that $\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) \in U$. This is a contradiction. Therefore, our supposition must be wrong. Hence $V \in Spec(R)$. Now $V \subseteq U$, so $V = U$ as $U \in Min.Spec(R)$. Hence $\delta(U) \subseteq U$. □

In above proposition the condition that $\delta(\sigma(a)) = \sigma(\delta(a))$, for all $a \in R$ is necessary. For example if $s = t = 1$, then $a \in U$, $b \in U$ and therefore, $\sigma^i(a) \in U$, $\sigma^i(b) \in U$ for all integers $i \geq 1$ as $\sigma(U) = U$. Now $\delta^2(adb) \in U$ implies that

$$\delta(a)\sigma(d)\delta(\sigma(b)) + \delta(a)\sigma(d)\sigma(\delta(b)) + u \in U.$$

where u is sum of terms involving a or b, or $\sigma^i(b)$. Therefore by assumption $u \in U$. This implies that

$$\delta(a)\sigma(d)\delta(\sigma(b)) + \delta(a)\sigma(d)\sigma(\delta(b)) \in U.$$

If $\delta(\sigma(a)) \neq \sigma(\delta(a))$, for all $a \in R$, then we get nothing out of it and if $\delta(\sigma(a)) = \sigma(\delta(a))$, for all $a \in R$, we get $\delta(a)\sigma(d)\sigma(\delta(b)) \in U$ which gives a contradiction.

Now we have with the following:

Theorem 2.6. *(Hilbert Basis Theorem): Let R be a right/left Noetherian ring. Let α and ρ be as above. Then the ore extension $O(R) = R[x, \alpha, \rho]$ is right/left Noetherian. Also $R[x, x^{-1}, \alpha]$ is right/leftNoetherian.*

Proof. See Theorems (1.12) and (1.17) of Goodearl and Warfield [8]. □

We now state and prove the following:

Theorem 2.7. *Let R be a 2-primal Noetherian $\mathbb{Q}$-algebra, σ an automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then $P_1 \in Min.Spec(R)$ with $\sigma(P_1) = P_1$ implies that $O(P_1) \in Min.Spec(O(R)) \cap C.Spec(O(R))$.*

Proof. Let $P_1 \in Min.Spec(R)$. Then $\delta(P_1) \subseteq P_1$ by Proposition (2.4). Now it can be seen that that $O(P_1) \in Spec(O(R))$. Suppose $O(P_1) \notin MinSpec(O(R))$ and $P_2 \subset O(P_1)$ be a minimal prime ideal of $O(R)$. Then $P_2 = O(P_2 \cap R) \subset O(P_1) \subseteq Min.Spec(O(R))$. Therefore $P_2 \cap R \subset P_1$ which is a contradiction, as $P_2 \cap R \in Spec(R)$. Hence $O(P_1) \in Min.Spec(O(R))$.

Also by Proposition (2.4) P_1 is a completely prime ideal of R, therefore, Theorem (2.2) implies that $O(P_1) \in C.Spec(O(R))$. Hence $O(P_1) \in Min.Spec(O(R)) \cap C.Spec(O(R))$.

□

Corollary 2.8. *Let R be a 2-primal Noetherian ring and σ an automorphism of R. Then $P_1 \in Min.Spec(R)$ with $\sigma(P_1) = P_1$ implies that $L(P_1) \in Min.Spec(L(R)) \cap C.Spec(L(R))$.*

Proof. Use 2.3 and 2.7. □

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