Generalization of Generalized Supplemented Module

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Abstract. In this paper we present the second generalization of generalized supplemented module. This generalization is done through two stages namely, the transition from supplemented into $\text{Rad}$-supplemented followed by transition from $\text{Rad}$-supplemented into weakly $\text{Rad}$-supplemented. If $R$ be a Bass ring and $M$ be a injective module then $M$ is weakly $\text{Rad}$-supplemented module. Also we will generalize the $\bigoplus$-supplemented module into weak-$\bigoplus$-supplemented. We prove that every lifting module is a weak-$\bigoplus$-supplemented module. These stages depend on the conditions of the modules and rings.

Mathematics Subject Classification: 54C05, 54C08, 54C10

Keywords: $\text{Rad}$-supplemented, local module, hollow module, Bass ring, lifting module

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper all rings have the identity and modules are considered to be right modules. From [3], a submodule $N$ of $M$ is called small in $M$ ($N \ll M$) if for every submodule of $M$, $N + L = M$ straight $L = M$. The dual of small is big (essential). This means any submodule $N$ of $M$ is big in $M$ if the intersection of $N$ with $L$ is not equal to zero, where $L$ is submodule of $M$. A submodule $N$ of $M$ is called a supplement of $K$ in $M$ if $N + K = M$ and $N$ is minimal with respect to this property [15]. A module $M$ is called supplemented if any submodule $N$ of $M$ has a supplement in $M$ and $M$ is called amply supplemented module if for any two submodules $H$ and $G$ with $H + G = M$, $G$ contains a supplement of $H$ in $M$ [14]. Therefore if any module $M$ has no maximal submodule this means $M = \text{Rad}(M)$ such that $\text{Rad}(M)$ is
intersection of all maximal submodules of $M$. A module $M$ is called lifting if for all $N$ submodule of $M$, there is a decomposition $M = H \oplus K$ such that $H$ submodule of $N$ and $N \cap H$ is small in $M$ [13]. Any ring $R$ is called a left Bass ring if, $\text{Rad}(M)$ is small in $M$ such that $M \neq 0$ [7]. A module $M$ is called semi-local if $(M/\text{Rad}(M))$ is semi-simple [15]. From [11] A module $M$ is called hollow if every proper submodule $N$ of $M$ that $(M/N)$ is hollow has a coessential submodule that is direct summand of $M$. A module $M$ is called coatomic if every proper submodule of $M$ is contained in a maximal submodule of $M$[12]. A module $M$ is called local if $M$ has largest submodule (i.e. a proper submodule which contains all other proper submodules [15]. A ring $R$ is local if and only if $(R_R)$ is local module. Let $M$ be an $R$-module. Any submodules $N, K$ of $M$ we say that $N$ is called a weak supplement of $K$ in $M$ if $N + K = M$ and $N \cap K$ small in $M$. A module $M$ is called weakly supplemented if every submodule of $M$ has a weak supplement in $M$. Let $N$ and $K$ are submodules of $M$ then $N$ is called weak Rad-supplement of $K$ in $M$ if, $N + K = M$ and $N \cap K \subseteq \text{Rad}(M)$. A module $M$ is called weakly Rad-supplemented if every submodule of $M$ has a weak Rad-supplemented in $M$ [7]. In this work we are going to get a new conditions to generalize supplemented module into weakly Rad-supplemented and $\bigoplus$-supplemented into weak $\bigoplus$-supplemented module. Also, we used many type of modules to satisfying this objective as local, hollow, semi-local module.

2. Weakly Rad-supplemented module

Firstly, we begin the first stage in order to get a generalization for supplemented module. Every hollow module is Rad-supplemented and a module $M$ is hollow if and only if it local module therefore any module $M$ is local this means $M$ is Rad-supplement.

**Definition 2.1.** Let $M$ be an $R$-module and $N$, $L$ are submodules of $M$. Then $L$ is a radical supplement (Rad-supplement) of $N$ in $M$ if $N + L = M$ and $N \cap L \subseteq \text{Rad}(L)$.

Therefore, by this definition we can say that $M$ is Rad-supplemented if every submodule in $M$ has a Rad-supplement.

**Lemma 2.2.** Let $R$ be a ring and let $M$ be an injective module. Then $M$ has no maximal submodule.

**Lemma 2.3.** Let $R$ be a ring and $M$ be an injective module. Then $\text{Rad}(M) = M$. [6].

**Remark 2.4.** If $\text{Rad}(M) = M$ then $M$ is Rad-supplemented module.

**Theorem 2.5.** Let $R$ be any ring and let $M$ be an injective module with every $N$, $L$ are submodules of $M$ and $M = L + N$, $N \cap L \ll N$. Then $M$ is a Rad-supplemented module.
Proof. We have $L, N$ are submodules of $M$ such that $M = L + N$ and $L \cap N \ll N$. Then $N$ is supplement of $L$ in $M$. If every submodule of $M$ has supplement then $M$ is supplemented. Now since $M$ is injective module then by (Lemma 2.3 ) $M=\text{Rad}(M)$ and by ( Remark 2.4 ) we get $M$ is a Rad-supplemented module.

Remark 2.6. The transition from supplemented into Rad-supplemented it called generalization of supplemented module.

Definition 2.7. Let $M$ be an $R$-module. Then $M$ is called a projective cover of $N$ if $M$ is a projective and there exists an epimorphism $f$ from $M$ into $N$ such that kernel of $f$ is small in $M$.

Lemma 2.8. Let $N \to M$ be a projective cover, and let $N$ be a hollow module. Then $M$ is a hollow module [14].

Theorem 2.9. Let $M$ be an $R$-module. If $M$ is a hollow module, then $M$ is Rad-supplemented module.

Proof. Since $M$ is hollow module then every proper submodule of $M$ is small in $M$. Therefore every submodule of $M$ is supplement in $M$. Hence $M$ is Rad-supplemented module.

Lemma 2.10. Let $M$ be an $R$-module. If $M$ is local module then $M$ is a hollow module [16].

Remark 2.11. We recall that a supplemented module is Rad-supplemented.

In order to obtain a supplemented module for Rad-supplemented we need the following condition : (A ring $R$ is called left Bass if $\text{Rad}(M)$ is small in $M$ such that $M \neq 0$).

Theorem 2.12. Let $R$ be a left Bass ring. Then every Rad-supplemented module is supplemented [7].

Theorem 2.13. Let $R$ be a Bass ring and let $M$ be an injective module. Then $M$ is supplemented module.

Proof. Since $M$ is an injective module then $M$ has no maximal submodule. Therefore $\text{Rad}(M) = M$, and then $M$ is Rad-supplemented module. Also, every Rad-supplemented module over Bass ring is supplemented module (Theorem 2.12).

The next stage is a generalization of Rad-supplemented module into weakly Rad-supplemented module in several ways:

Remark 2.14. Weakly supplemented module is weakly Rad-supplemented module

Now from [4] we have the following implication
There is a natural question to determine the conditions of module to transition by direct from supplemented to weakly \( R \)-ad-supplemented module. We conclude:

**Theorem 2.15.** Let \( M \) be a nonzero \( R \)-module. If \( M \) is local module then \( M \) is weakly \( R \)-ad-supplemented module.

**Proof.** Since \( M \) is local module then by (Lemma 2.10) \( M \) is hollow module and from (Theorem 2.9) \( M \) is \( R \)-ad-supplemented module. Therefore \( M \) is weakly \( R \)-ad-supplemented module. \( \square \)

**Proposition 2.16.** Let \( M \) be an \( R \)-module and \( M \) be a coatomic with radical equal zero. If \( M \) is supplemented module then \( M \) is weakly \( R \)-ad-supplemented module.

**Proof.** Since \( M \) is a supplemented module then every submodule \( N \) of \( M \) is supplement of \( K \) in \( M \) such that \( K \) submodule of \( M \). Let \( (M/N) \) be hollow. Now we have \( \text{Rad}(M)=0 \), then \( M = K + N \), therefore \( M \) is hollow-lifting [see 13] and hence is hollow with \( \text{Rad}(M) \neq M \) this means \( M \) is hollow and finitely generated (cyclic). If \( M \) is hollow with cyclic then \( M \) is local, therefore \( M \) is weakly \( R \)-ad-supplemented module (Theorem 2.15). \( \square \)

Now we can return to (Theorem 2.13) to obtain weakly \( R \)-ad-supplemented module:

**Theorem 2.17.** Let \( R \) be a Bass ring and \( M \) be injective module. If every submodule \( N \) of \( M \) such that \( \text{Rad}(N) \subseteq \text{Rad}(M) \) then \( M \) is weakly \( R \)-ad-supplemented module.

**Proof.** By (Theorem 2.13) \( M \) is supplemented module. Then every supplemented module is \( R \)-ad-supplemented module this means \( L + N = M \) and \( N \cap L \subseteq \text{Rad}(N) \), but \( \text{Rad}(N) \subseteq \text{Rad}(M) \) then \( N \cap L \subseteq \text{Rad}(M) \) therefore \( N \) is weak \( R \)-ad-supplemented and hence \( M \) is weakly \( R \)-ad-supplemented module. \( \square \)
Lemma 2.18. Let $M$ be an $R$-module. Then $M$ is finitely generated supplemented module if and only if $M = L_1 + L_2 + ... + L_n$, for some local module $L_n$.

Proof. See [16].

Theorem 2.19. Let $M$ be an $R$-module. If $M$ is finitely generated and supplemented, then $M$ is a weakly $\text{Rad}$-supplemented module.

Proof. Since $R$ is semi-perfect then if every finitely generated $R$-module has a projective cover. From [15], every finitely generated left (respectively, right) $R$-module is supplemented and hence $M$ is a weakly $\text{Rad}$-supplemented.

Definition 2.20. Let $N$ submodule of $M$. Then $N$ lies over a summand of $M$ if there is a direct decomposition $M = R \oplus S$ with $R \subseteq N$, $S \cap N$ submodule of $M$.

Proposition 2.21. Let $M$ be a nonzero module and let $\psi$ be homomorphism from $M$ into $(M/(\text{Rad}(M)))$, and every submodule of $M$ lies over a summand of $M$. Then $M$ is weakly $\text{Rad}$-supplemented module.

Proof. Let $N$ be a small in $(M/(\text{Rad}(M)))$, then there is a $K$ submodule of $M$ such that $\psi(K) = N$. Since every submodule of $M$ lies over a summand of $M$, then there exists submodules $Q,R$ of $M$ such that $M = Q \oplus R, Q \subseteq K$ and $K \cap R$ is small in $M$. Then $\psi(Q) = \psi(K) = N$ and $(M/(\text{Rad}(M)))\psi = (Q) \oplus \psi(R) = N \oplus \psi(R)$. $N$ is direct summand of $(M/(\text{Rad}(M)))$. Then $(M/(\text{Rad}(M)))$ is semi-simple. If $(M/(\text{Rad}(M)))$ is semi-simple then $M$ is semi-local and so $R$ semi-simple which means $R$ semi-perfect and by (Theorem 2.20) $M$ is a weakly $\text{Rad}$-supplemented module.

Theorem 2.22. Let $M$ be an $R$-module such that $M$ not equal $\text{Rad}(M)$. Suppose that for every $N$ submodule of $M$, there is a direct decomposition $M = A \oplus B$ with $A \subseteq N, B \cap N$ submodule of $M$ and indecomposable. Then $M$ is a weakly $\text{Rad}$-supplemented module.

Proof. Since $M$ is indecomposable then $M$ is local module [10] and by (Theorem 2.15) $M$ is weakly $\text{Rad}$-supplemented module.

Theorem 2.23. Let $N$ be submodule of a module $M$. If $M$ is indecomposable and $\text{Rad}(M) \ll M$ and if $M$ is $N$-semi-potent, then $M$ is a weakly $\text{Rad}$-supplemented module.

Proof. By [1] $M$ is local module and by (Theorem 2.15) $M$ is weakly $\text{Rad}$-supplemented module.

Proposition 2.24. Let $M$ be an $R$-module. If there exists a maximal submodule $H$ of $M$ such that $H$ is small in $M$, then $M$ is local and then weakly $\text{Rad}$-supplemented module.
Proof. Let \( K \) be a proper submodule of \( M \). Suppose that \( K \) not equal \( M \). Then \( H \subseteq H + K \subseteq M \). Therefore \( H = H + K \) or \( H + K = M \). Let \( H + K = M \), which means \( K = M \). This contradiction with \( H \) small in \( M \), therefore, \( H = H + K \), then \( K \subseteq H \) and hence \( H \) is largest proper submodule in \( M \) (\( M \) is local). Then \( M \) is weakly supplemented, and hence \( M \) is weakly Rad-supplemented module.

The most important results can be summarized in the following diagrams:

Diagram 2

Diagram 3

Diagram 4
3. **Weak-$\bigoplus$-Supplemented Module**

In this section we are studying some properties of $\bigoplus$-supplemented module and make a generalization of $\bigoplus$-supplemented into weak $\bigoplus$-supplemented by new conditions. If $M$ lifting module then $M$ is a $\bigoplus$-supplemented module. Also if $R$ be a semi-simple ring and $M$ be a $\bigoplus$-supplemented module then $M$ is injective module. Let $R$ be a ring then if every $\bigoplus$-supplemented $R$-module is injective this imply $R$ is a left Noetherian V-ring and semi-simple [2].

**Definition 3.1.** Let $M$ be an $R$-module. If any submodule of $M$ has a supplement that is a direct summand of $M$ and $M$ is supplemented, then $M$ is called a $\bigoplus$-supplemented module.

**Definition 3.2.** A module $M$ is called amply supplemented if for any two submodules $H$ and $K$ with $H + K = M$, $K$ contains a supplement of $H$ in $M$.

Now we can rewrite above definition by other way in order to obtain $\bigoplus$-supplemented module. We explore many new properties for $\bigoplus$-supplemented module. If $M$ is amply supplemented and any supplement submodule of $M$ is a direct summand of $M$ then $M$ has lifting property. Therefore we can present the following lemmas:

**Lemma 3.3.** Let $M$ be an $R$-module. If $M$ is lifting module then $M$ is a $\bigoplus$-supplemented module.

**Lemma 3.4.** Let $N$ be submodule of $M$, then there exists a direct summand $L$ of $M$ such that $L$ submodule of $N$ and $(N/L)$ submodule of $\text{Rad}((M)/(L))$. Then $M$ is hollow and $\bigoplus$-supplemented module.

**Definition 3.5.** Let $M$ be an $R$-module and $N$ submodule of $M$. Then $N$ lies above a direct summand of $M$ if there exists $H$ and $G$ are submodules of $N$ such that $H \bigoplus G = M$ and $N \cap G \ll G$.

**Definition 3.6.** Any module $M$ is called $(D_1)$-module if every $N$ submodule of $M$ is lies above a direct summand of $M$.

**Lemma 3.7.** Let $M$ be an $R$-module. Then $M$ is $(D_1)$-module if and only if $M$ is lifting module.

**Theorem 3.8.** Let $M$ be an $R$-module. If $M$ is $(D_1)$-module, then $M$ is a $\bigoplus$-supplemented module.

**Proof.** Since $M$ is $(D_1)$-module, then $N$ is lies above a direct summand of $M$. Therefore there exists $H$, $G$ are submodules of $N$, such that $H \bigoplus G = M$ and $N \cap G \ll G$. Then $M$ is a lifting module and by ( Lemma 3.3 ) $M$ is a $\bigoplus$-supplemented module.

**Corollary 3.9.** Let $M$ be an amply supplemented module such that every submodule of $M$ lies above a direct summand of $M$. Then $M$ is a $\bigoplus$-supplemented module.
Proof. Suppose $M$ is an amply supplemented module. Then there are submodules $L, K$ of $M$ such that $L + K = M$ and $K$ contains supplement of $L$. Since every submodule of $M$ lies above a direct summand of $M$ then $M$ is $(D_1)$-module therefore $M$ is a lifting module and hence by (Lemma 3.3) $M$ is a $\bigoplus$-supplemented module.

**Corollary 3.10.** If $M$ is a $(D_1)$-module, then $(M/\text{Rad}(M))$ is a $\bigoplus$-supplemented module.

Proof. Let $M$ be a $(D_1)$-module. Then every submodule of $M$ lies above a direct summand of $M$ this means $M$ is a lifting module. Then $M$ is a $\bigoplus$-supplemented module and hence $(M/(\text{Rad}(M)))$ is a $\bigoplus$-supplemented.

**Proposition 3.11.** Let $M$ be an $R$-module and let $N$ submodule of $M$ such that $N$ supplement in $M$. Then $M$ is a $\bigoplus$-supplemented module.

Proof. Firstly, we must prove that $\text{Rad}(M)=0$. Suppose $\text{Rad}(M)$ is not equal zero, there exists nonzero element $r$ belong to $\text{Rad}(M)$. We have $Rr$ supplement, then $Rr + H = M$ and $Rr \cap H \ll Rr$ such that $H$ submodule of $M$. Since $r$ belong to $\text{Rad}(M)$, then $Rr \ll M$ and $H = M$ and hence $Rr \ll Rr$ which is impossible. Then $\text{Rad}(M)=0$. Now we have $N$ is a supplement, then $M = N + K$ and $N \cap K \ll K$ such that $K$ submodule of $M$ therefore $N \cap K \subseteq \text{Rad}(M) = 0$, then $N \cap K = 0$ and hence $M = N \bigoplus K$, then $M$ is semi simple and hence is a $\bigoplus$-supplemented module.

**Lemma 3.12.** Let $R$ be a semi-simple ring and $M$ be a $\bigoplus$-supplemented module. Then $M$ has no maximal submodule.

**Definition 3.13.** A ring $R$ is called a left $V$-ring if every simple left $R$-module is injective.

**Theorem 3.14.** Let $R$ be $V$-ring and $M$ be an $R$-module. If $M$ is the sum of its simple submodules, then $M$ is a $\bigoplus$-supplemented module.

Proof. Since $M$ is the sum of simple submodules then $M$ is semi-simple. Now $M$ is semi-simple and $R$ is $V$-ring. Then $M$ is a $\bigoplus$-supplemented module.

**Proposition 3.15.** Let $M$ be an $R$-module. If $N$ submodule of $M$ is linearly compact and lies above a direct summand of $M$. Then $M$ is a $\bigoplus$-supplemented module.

Proof. Since $N$ is linearly compact then $N$ has ample supplements in $M$. Therefore if $K$ submodule of $M$ and $L$ submodule of $N$ such that $N + K = M$, there is supplement $L$ of $N$. Then by (Definition 3.2) $M$ is amply supplemented module. Now $M$ is amply supplemented and $N$ is lies above a direct summand of $M$ then by (Corollary 3.9) $M$ is a $\bigoplus$-supplemented module.
**Definition 3.16.** Let $M$ be an $R$-module. Then $M$ is weak-$\bigoplus$-supplemented module if for each semi-simple submodule $N$ of $M$ there exists a direct summand $K$ of $M$ such that $M = N + K$ and $N \cap K$ is small in $K$.

**Remark 3.17.** Note that $w$-$\bigoplus$-supplemented module not imply $\bigoplus$-supplemented but the converse is true.

**Example 3.18.** Let $R$ be a local Artinian ring with radical $W$ such that $W^2 = 0$, $Q = R = W$ is commutative, $\dim(QW) = 2$ and $\dim(WQ) = 1$. Then the indecomposable injective right $R$-module $U = [(R \bigoplus R)/D]$ with $D = (ur; -v\ r): r \in R$ is a $w$-$\bigoplus$-supplemented module, but is not $\bigoplus$-supplemented [13].

**Theorem 3.19.** Let $M$ be an $R$-module. If $M$ is a lifting module then $M$ is $w$-$\bigoplus$-supplemented module.

**Proof.** Since $M$ is a lifting module then $M$ is amply supplemented and any supplement submodule of $M$ is direct summand of $M$. Therefore by (Corollary 3.9) $M$ is a $\bigoplus$-supplemented module and by (Remark 3.17) $M$ is a $w$-$\bigoplus$-supplemented module.

**Corollary 3.20.** If $M$ is $(D_1)$-module then $M$ is a $w$-$\bigoplus$-supplemented module.

**Proof.** Since $M$ is $(D_1)$-module then $M$ is lifting module and by above theorem $M$ is a $w$-$\bigoplus$-supplemented module.

**Corollary 3.21.** Let $N$ be submodule of $M$. If $N$ lies above a direct summand of $M$ then $M$ is a $w$-$\bigoplus$-supplemented module.

**Proof.** By (Definition 3.6) and (Corollary 3.21).

**Corollary 3.22.** If $M$ is Noetherian $R$-module and $(D_1)$-module then $M$ is a $w$-$\bigoplus$-supplemented module.

**Proof.** Since $M$ is Noetherian module then every submodule of $M$ is finitely generated. Now $M$ is $(D_1)$-module that is mean $M$ is lifting module therefore $M$ is finitely lifting. Then $M$ is lifting module [18] and hence $M$ is a $w$-$\bigoplus$-supplemented module (Theorem 3.19.)

Therefore we get the following implications:

\[ N \text{ lies above a direct summand} \quad \Rightarrow \quad M \text{ (D_1)-module} \quad \Rightarrow \quad M \text{ lifting module} \]

\[ M \text{ W-\bigoplus-supplemented} \quad \Rightarrow \quad M \text{ strongly \bigoplus-supplemented} \]

Diagram 5
4. Acknowledgement

The authors would like to acknowledge the financial support received from Universiti Kebangsaan Malaysia under the research grant UKM-ST-06-FRGS0146-2010. The authors also wish to gratefully acknowledge all those who have generously given of their time to referee our paper.

References


Received: September, 2012