Generalization of $\oplus - \delta$–Supplemented Modules

Figen YÜZBAŞI ERYILMAZ and Şenol EREN

Ondokuz Mayıs University, Faculty of Sciences and Arts
Department of Mathematics, 55139, Samsun, Turkey
figenyuzbasi@gmail.com, seren@omu.edu.tr

Abstract

Let $R$ be a ring and $M$ be a left $R$–module. We say that an $R$–module $M$ is a generalized $\oplus - \delta$–supplemented module if every submodule of $M$ has a generalized $\delta$–supplement which is a direct summand of $M$. In this paper, several properties of these modules are given. We showed that any finite direct sum of generalized $\oplus - \delta$–supplemented modules is a generalized $\oplus - \delta$–supplemented module and every direct summand of a UC-extending generalized $\oplus - \delta$–supplemented module is a generalized $\oplus - \delta$–supplemented.

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1 Introduction & Preliminaries

Throughout this paper $R$ will be an associative ring with identity and all modules will be unital left $R$–modules. Let $M$ be an $R$–module. The notation $N \leq M$ means that $N$ is a submodule of $M$. $\text{Rad}(M)$ will indicate Jacobson radical of $M$. A submodule $N$ of an $R$–module $M$ is called small in $M$ and written $N \ll M$, if $M \neq N + L$ for every proper submodule $L$ of $M$. By a supplement of $N$ in $M$ we mean a submodule $K$ which is minimal in the collection of submodules $L$ of $M$ such that $M = N + L$. The module $M$ is called supplemented if every submodule has a supplement in $M$. For more detailed discussion on supplemented modules we refer to [6, 15].

Following [7], a module $M$ is called $\oplus$–supplemented if every submodule of $M$ has a supplement that is a direct summand of $M$. $\oplus$–supplemented modules are studied in [2, 3, 4].

Let $N$ and $K$ be any submodules of $M$ with $M = N + K$. If $N \cap K \leq \text{Rad}(K)$, then $K$ is called generalized supplement (according to [1], radical
supplement or briefly Rad-supplement) of \( N \) in \( M \) [13]. The notion of generalized supplemented modules was introduced by Xue in [16]. An \( R \)-module \( M \) is called generalized supplemented module or briefly \( GS \)-module (in [1] Rad-supplemented) if every submodule of \( M \) has a generalized supplement in \( M \). On the other hand, the module \( M \) is called a \( Rad \oplus - s \)-module, and denoted by \( Rad \oplus - s \)-module, if every submodule of \( M \) has a Rad-supplement that is a direct summand of \( M \). These modules were studied in [10, 11].

In [17], Zhou introduced the concept of \( \delta \)-small submodules as a generalization of small submodules. Recall that a submodule \( N \) of a module \( M \) is said to be \( \delta \)-small in \( M \) and denoted by \( N \ll_{\delta} M \) if whenever \( M = N + K \) and \( \frac{M}{K} \) singular, we have \( M = K \). The sum of all \( \delta \)-small submodules of a module \( M \) is denoted by \( \delta (M) \), which defines a preradical on the category of \( R \)-modules.

Let \( N \) be a submodule of a module \( M \). A submodule \( K \) of \( M \) is called a \( \delta \)-supplement of \( N \) in \( M \) provided that \( M = N + K \) and \( M \neq N + X \) for any proper submodule \( X \) of \( K \) with \( \frac{K}{X} \) singular; or equivalently, \( M = N + K \) and \( N \cap K \ll_{\delta} K \). The module \( M \) is called \( \delta \)-supplemented if every submodule of \( M \) has a \( \delta \)-supplement in \( M \) [5, 14]. Also, \( M \) is called \( \oplus - \delta \)-supplemented if every submodule of \( M \) has a \( \delta \)-supplement which is a direct summand of \( M \) [12]. Clearly \( \oplus - \delta \)-supplemented modules are \( \delta \)-supplemented and \( \oplus \)-supplemented modules are \( \oplus - \delta \)-supplemented.

According to [9], for two submodules \( N \) and \( K \) of \( M \), \( N \) is called generalized \( \delta \)-supplement of \( K \) in \( M \) if \( M = N + K \) and \( N \cap K \leq \delta (N) \). The module \( M \) is called generalized \( \delta \)-supplemented or briefly \( GS \) if every submodule \( N \) of \( M \) has a generalized \( \delta \)-supplement in \( M \).

In this note, we introduce generalized \( \oplus - \delta \)-supplemented modules. We answer the following natural question: is any factor module of a generalized \( \oplus - \delta \)-supplemented module generalized \( \oplus - \delta \)-supplemented? In addition, we investigate direct summand of these modules.

## 2 Main Results

**Definition 1** A module \( M \) is called generalized \( \oplus - \delta \)-supplemented module if every submodule of \( M \) has a generalized \( \delta \)-supplement which is a direct summand of \( M \).

An \( R \)-module \( M \) is said to have property \((D_3)\), if \( M_1 \) and \( M_2 \) are direct summand of \( M \) with \( M = M_1 + M_2 \), then \( M_1 \cap M_2 \) is also a direct summand of \( M \) [15].

**Proposition 2** Let \( M \) be a generalized \( \oplus - \delta \)-supplemented module with \((D_3)\). Then every direct summand of \( M \) is a generalized \( \oplus - \delta \)-supplemented module.
Proof. Let $U$ be a direct summand of $M$ and $N$ be a submodule of $U$. Then there exists a direct summand $V$ of $M$ such that $M = N + V$ and $N \cap V \leq \delta (V)$. By modularity, we have $U = N + (V \cap U)$. Since $M$ has $(D_3)$ property, $(U \cap V)$ is a direct summand of $M$ and so it is also a direct summand of $U$. Note that $N \cap (U \cap V) = N \cap V \leq \delta (V)$. By Lemma 2.2 in [9], $N \cap V \leq \delta (U \cap V)$. Therefore $U$ is a generalized $\oplus - \delta$--supplemented module.

Let $M$ be a module. A submodule $N$ of $M$ is closed in $M$ if $N$ has not a proper essential extension in $M$. In [8], he called a module $M$ is a $UC$--module if every submodule of $M$ has a unique closure in $M$. $M$ is called extending module if every closed submodule of $M$ is a direct summand of $M$.

Corollary 3 Let $M$ be a $UC$--extending module. If $M$ is a generalized $\oplus - \delta$--supplemented module, then every direct summand of $M$ is a generalized $\oplus - \delta$--supplemented module.

Proof. Since $M$ is a $UC$--extending module, $M$ has $(D_3)$ by Lemma 2.4 in [2]. Therefore the result follows from Proposition 2.

Theorem 4 Let $M_1$ and $M_2$ be generalized $\oplus - \delta$--supplemented modules. If $M = M_1 \oplus M_2$, then $M$ is a generalized $\oplus - \delta$--supplemented module.

Proof. Let $K$ be any submodule of $M$. Then $M = M_1 + M_2 + K$ and so $M_1 + M_2 + K$ has a generalized $\delta$--supplement 0 in $M$. Since $M_1$ is a generalized $\oplus - \delta$--supplemented module, $M_1 \cap (M_2 + K)$ has a generalized $\delta$--supplement $X$ in $M_1$ such that $X$ is direct summand of $M_1$. By Proposition 2.7 in [9], $X$ is a generalized supplement of $M_2 + K$ in $M$. Since $M_2$ is a generalized $\oplus - \delta$--supplemented module, $M_2 \cap (K + X)$ has a generalized $\delta$--supplement $Y$ in $M_2$ such that $Y$ is direct summand of $M_2$. Again applying Proposition 2.7 in [9], we get $X + Y$ is a generalized $\delta$--supplement of $K$ in $M$. Clearly, $X \oplus Y$ is a direct summand of $M$. Thus $M_1 \oplus M_2$ is a generalized $\oplus - \delta$--supplemented module.

Corollary 5 Any finite direct sum of generalized $\oplus - \delta$--supplemented modules is a generalized $\oplus - \delta$--supplemented module.

Let $M$ be a module with $S = \text{End}_R (M)$. A submodule $N$ is called fully invariant if for each $f \in S$, $f (N) \leq N$. Also $M$ is called duo provided, every submodule of $M$ is fully invariant [7].

Proposition 6 Let $M$ be a nonzero generalized $\oplus - \delta$--supplemented module and $U$ be a fully invariant submodule of $M$. Then the factor module $M / U$ is a generalized $\oplus - \delta$--supplemented module.
Proof. For any submodule $L$ of $M$ containing $U$, let $\frac{L}{U}$ be any submodule of $\frac{M}{U}$. Since $M$ is a generalized $\oplus - \delta$-supplemented module, there exist submodules $N$ and $N'$ of $M$ such that $M = L + N$, $L \cap N \leq \delta(N)$ and $M = N \oplus N'$. By Proposition 2.9 in [9], $\frac{(N+U)}{U}$ is a generalized $\delta$-supplement of $\frac{L}{U}$ in $\frac{M}{U}$. If we use Lemma 2.4 in [3], then we get $U = (U \cap N) \oplus (U \cap N')$. It follows that $(N + U) \cap (N' + U) \leq U$ and so $\frac{M}{U} = \frac{(N+U)}{U} \oplus \frac{(N'+U)}{U}$. Then $\frac{(N+U)}{U}$ is a generalized $\delta$-supplement of $\frac{L}{U}$ such that $\frac{(N+U)}{U}$ is a direct summand of $\frac{M}{U}$. Consequently, $\frac{M}{U}$ is a generalized $\oplus - \delta$-supplemented module. ■

Corollary 7 Let $M$ be a generalized $\oplus - \delta$-supplemented and duo module. Then every factor module of $M$ is a generalized $\oplus - \delta$-supplemented module.

Proposition 8 Let $M$ be a generalized $\oplus - \delta$-supplemented module and $U$ be a fully invariant submodule of $M$. If $U$ is a direct summand of $M$, then $U$ is a generalized $\oplus - \delta$-supplemented module.

Proof. Let $U$ be a direct summand of $M$ and $N$ be a submodule of $U$. Since $M$ is a generalized $\oplus - \delta$-supplemented, there exist $L$ and $L'$ of $M$, such that $M = N + L$, $N \cap L \leq \delta(L)$ and $M = L \oplus L'$. By Lemma 2.4 in [3], we have $U = (U \cap L) \oplus (U \cap L')$. If we show that $N \cap (U \cap L) = N \cap L \leq \delta(U \cap L)$, then the proof is complete. Since $M = N + L$, we have $U = N + (U \cap L)$, $N \cap L \leq \delta(M)$. Due to $U \cap L$ is a direct summand of $M$, we obtain $N \cap L \leq \delta(U \cap L)$ by Lemma 2.2 in [9]. Hence $U \cap L$ is a generalized $\delta$-supplement of $N$ in $U$ that is a direct summand of $U$. So it implies that $U$ is a generalized $\oplus - \delta$-supplemented module. ■

Theorem 9 Let $M$ be a module such that $M = M_1 \oplus M_2$ is a direct sum of submodules $M_1$ and $M_2$. Then $M_2$ is a generalized $\oplus - \delta$-supplemented module if and only if there exists a direct summand $K$ of $M$ such that $K \leq M_2$, $M = N + K$ and $N \cap K \leq \delta(K)$ for every submodule $\frac{N}{M_1}$ of $\frac{M}{M_1}$.

Proof. Let $\frac{N}{M_1}$ be any submodule of $\frac{M}{M_1}$. By hypothesis, there exist $K$, $K'$ submodules of $M_2$ such that $M_2 = (N \cap M_2) + K$, $(N \cap M_2) \cap K = N \cap K \leq \delta(K)$ and $M_2 = K \oplus K'$. Note that $M = M_1 + M_2 = N + K$. Since $K$ is a direct summand of $M_2$, we have $K$ is a direct summand of $M$.

Conversely, suppose that $\frac{M}{M_1}$ has the stated property. Let $H$ be a submodule of $M_2$. Consider the submodule $\frac{(H \oplus M_1)}{M_1} \leq \frac{M}{M_1}$. By hypothesis, there exists a direct summand $K$ of $M$ such that $K \leq M_2$, $M = (H + K) + M_1$ and $K \cap (H + M_1) \leq \delta(K)$. Then $M_2 = H + K$ and $H \cap K \leq \delta(K)$. Thus $K$ is a generalized $\delta$-supplemented of $H$ in $M_2$ and it is a direct summand of $M_2$. Therefore $M_2$ is generalized $\oplus - \delta$-supplemented. ■
**Theorem 10** Let $M_i$ ($1 \leq i \leq n$) be any finite collection of relatively projective modules. The module $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is a generalized $\oplus - \delta$-supplemented module if and only if $M_i$ is a generalized $\oplus - \delta$-supplemented module for each $1 \leq i \leq n$.

**Proof.** The necessity part is proved in Theorem 4. Conversely, it is sufficient to prove that $M_1$ is generalized $\oplus - \delta$-supplemented. Let $N$ be any submodule of $M_1$. Then there exist submodules $K$ and $K'$ of $M$ such that $M = N + K = K \oplus K'$ and $N \cap K \leq \delta (K)$. Note that $M = N + K = M_1 + K$. By Lemma 4.47 in [6], there exists a submodule $K_1$ of $K$ such that $M = M_1 \oplus K_1$. If we intersect the last equation with $K_1$, then we get $K = (M_1 \cap K) \oplus K_1$. Since $M_1 = (M_1 \cap K) + N$ and $M_1 \cap K$ is a direct summand of $M_1$, we obtain $N \cap K \leq \delta (M_1 \cap K)$ if we use Lemma 2.2 in [9]. Therefore $M_1$ is a generalized $\oplus - \delta$-supplemented module. ■

**References**


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